A bridge between semirings

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Introduction

Semirings $\mathcal{S}, \mathcal{R}$ for resource management

Many papers are using **semirings** for modelizing **quantitative resources**.

Bounded logics $B_{\mathcal{S}}LL$ [Ghica&Al13,Brunel&Al13]

Generalisation of previous works that decompose LL exponential along semirings.
No generic concrete model.

Model $Rel^{\mathcal{R}}$ of LL [CarraroEhrhardSalibra09]

The relational model endowed with a **non free exponential** using semiring-indexed multisets.

General theorem

Every bounded logic $B_{\mathcal{S}}LL$ has as model $REL^{(\mathcal{R},\tilde{\in},\mathcal{S})}$.

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- Ghica&Smith vs Brunel&Al
- Refining models of LL
- Brunel&Al's

Concrete semantic

- Rel
- Our $\mathrm{Rel}^{(\mathcal{R},\tilde{\in},\mathcal{S})}$
- Carraro&All's $\mathrm{Rel}^{\mathcal{R}}$
- Universality

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Definition of ordered semirings

Ordered semiring $\mathcal{S}$

An ordered semiring is a structure $\mathcal{S} = (|\mathcal{S}|, +, 0, *, 1, \leq)$ where $(|\mathcal{S}|, \leq)$ is a poset, $0, 1 \in |\mathcal{S}|$ and where $+, \cdot : |\mathcal{S}|^2 \to \mathcal{S}$ such that:

- $+$ and $*$ are associative,
- $+$ is commutative and distribute over $*$,
- 0 is neutral for $+$ and 1 is neutral for $*$,
- if $J \leq J'$ and $K \leq K'$ then $J * J' \leq K * K'$ and $J + J' \leq K + K'$.

Examples of ordered semirings

- $\mathbb{1}$: the trivial Semiring with just one element.
- $\mathbb{P}oly$: the polynomials with their usual sum, product and order.
- $\mathbb{T}rop = (\underline{\mathbb{N}}, max, -\infty, +, 0, \leq)$ and $\mathbb{A}rt = (\bar{\mathbb{N}}, min, \infty, +, 0, \leq)$,
- $\mathbb{A}ut_{\mathbb{M}}$: for any commutative monoid $(\mathbb{M}, +, 0)$, the automorphisms $(\mathbf{Mon}(\mathbb{M}, \mathbb{M}), +, cst_0, \circ, id_{\mathbb{M}}, id_{\mathbf{Mon}(\mathbb{M},\mathbb{M})})$

Grammar of B$_\mathcal{S}$LL

Formulas

(formulas) $\quad A, B, C := \alpha \mid A \otimes B \mid A \multimap B \mid !_K A \qquad , \forall K \in \mathcal{S}$

This is only IMB$_\mathcal{R}$LL and can be generalised.

Sequents $\Gamma \vdash A$

IMLL plus:

$$\frac{\Gamma \vdash B}{\Gamma, !_0 A \vdash B}\,(!w) \qquad \frac{\Gamma, A \vdash B}{\Gamma, !_1 A \vdash B}\,(!d) \qquad \frac{\Gamma, !_J A, !_K A \vdash B}{\Gamma, !_{J+K} A \vdash B}\,(!c)$$

$$\frac{!_{J_1} A_1, \cdots, !_{J_n} A_n \vdash B}{!_{K * J_1} A_1, \cdots !_{K * J_n} A_n \vdash !_K B}\,(!s) \qquad \frac{\Gamma, !_J A \vdash B \qquad K \geq J}{\Gamma, !_K A \vdash B}\,(\mathcal{S}w)$$

Curry-Howard: $\Lambda_{\mathcal{S}}$

Grammar of $\Lambda_{\mathcal{S}}$

Given a semiring $\mathcal{S}$ we define typed lambda calculus:

$$\begin{array}{llllll} \text{(terms)} & \mathbf{\Lambda} & M, N & ::= & x \mid \lambda x.M \mid M\ N & \\ \text{(types)} & \mathbf{T}_{\ell}^{\mathcal{S}} & \theta & := & \alpha \mid K{\cdot}\theta \multimap \theta & , \forall K \in \mathcal{S} \end{array}$$

Some examples (with $\mathcal{S} = Hom(\mathbb{N})$)

$\lambda xy.y$: $0{\cdot}\theta \multimap 1{\cdot}\theta' \multimap \theta'$ $\quad$ $\lambda xy.x(xy)$: $4{\cdot}(3{\cdot}\theta \multimap \theta) \multimap 9{\cdot}\theta \multimap \theta$

$\lambda xy.xyy$: $1{\cdot}(2{\cdot}\theta \multimap 3{\cdot}\theta \multimap \theta') \multimap 5{\cdot}\theta \multimap \theta'$

$\lambda xyz.x(yz)$: $1{\cdot}(f{\cdot}\theta \multimap \theta) \multimap f{\cdot}(g{\cdot}\theta \multimap \theta) \multimap f \circ g{\cdot}\theta \multimap \theta$

$\lambda xyz.y(xz)$: $g{\cdot}(f{\cdot}\theta \multimap \theta) \multimap 1{\cdot}(g{\cdot}\theta \multimap \theta) \multimap g \circ f{\cdot}\theta \multimap \theta$

Typing judgements

Ghica's linear type system

$$\frac{}{x{:}1{\cdot}\theta \vdash x{:}\theta}\ \text{Id} \qquad \frac{\Gamma \vdash M{:}\theta}{\Gamma, x{:}0{\cdot}\theta' \vdash M{:}\theta}\ \text{Weak} \qquad \frac{\Gamma, x{:}K{\cdot}\theta \vdash M{:}\theta'}{\Gamma \vdash \lambda x.M{:}K{\cdot}\theta \multimap \theta'}\ \text{Abs}$$

$$\frac{\Gamma \vdash M{:}K{\cdot}\theta \multimap \theta' \qquad \Gamma' \vdash N{:}\theta \qquad |\Gamma| = |\Gamma'|}{\Gamma + K{*}\Gamma' \vdash M\ N : \theta'}\ \text{App}$$

where $|\Gamma| = |\Gamma'|$ means the equality of contexts except for the multiplicity of the types:

$$|(x_i{:}K_i{\cdot}\theta_i)_{i\leq k}| = |(y_i{:}L_i{\cdot}\theta'_i)_{i\leq k'}| \Leftrightarrow (k = k' \ \wedge\ x_i = y_i \ \wedge\ \theta_i = \theta'_i)$$

and where $\Gamma + K{\cdot}\Gamma'$ is the context obtain by applying addition and multiplication to the semiring:

$$(x_i{:}K_i{\cdot}\theta_i)_i + (x_i{:}L_i{\cdot}\theta_i)_i := (x_i{:}(K+L){\cdot}\theta_i)_i$$
$$K{*}(x_i{:}L{\cdot}\theta_i)_i := (x_i{:}K{*}L{\cdot}\theta_i)_i$$

Examples and interest

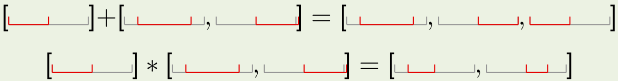
Counting resources

$\mathbb{N}$: if $M : n{\cdot}A \multimap U$ then the KAM on $M\ N$ will evaluate at most n times the N.

$\mathbb{P}oly$: if M is typable then $M\ N$ has a polynomial head reduction on the size on N.

$\mathbb{A}rt$: with streams of type Str and the operations $tl : 1{\cdot}Str \multimap Str$ and $hd : 1{\cdot}Str \multimap U$, if $M : n{\cdot}Str \multimap A$, its argument will be explored at depth at most n.

A more complicated example: Ghica&Smith's



This computes the sequentially of an execution.

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Categorical semantic

There is two choices of categorical semantic in the literature:

Brunel&al's categorical semantic

- − Difficult to understand
- − Over 20 commutative diagrams
- + Accept any model of LL as a degenerative model
- + The syntax itself is a model

GhicaSmith's categorical semantic

- + Quite simple to understand and realise
- − The only accepted model of LL is the singleton (up-to iso)
- − The syntax is not a model

Restricting models of LL into models of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

LL model $+\ \pi_J : !A \to !_J A\ +\ \iota_{,J} : !_J A \to !A\ +$ 5 diagrams $=\ B_{\mathcal{S}}$LL model

Brunel&al's categorical semantic

Ordered semirings as categories

Any ordered semiring can be seen as bimonoidal category whose objects are the elements of the semirings and whose homesets $\mathcal{S}(J, K)$ are either singleton (when $J \leq K$) or empty.

Bounded exponential situation

A bounded exponential situation is the given of:

- a symmetric monoidal category (model of MLL) $(\mathcal{A}, \otimes, \multimap, \mathbf{1})$,
- a bifunctor: $\_\bullet\_ : \mathcal{S} \times \mathcal{A} \to \mathcal{A}$
- 6 naturals transformations:

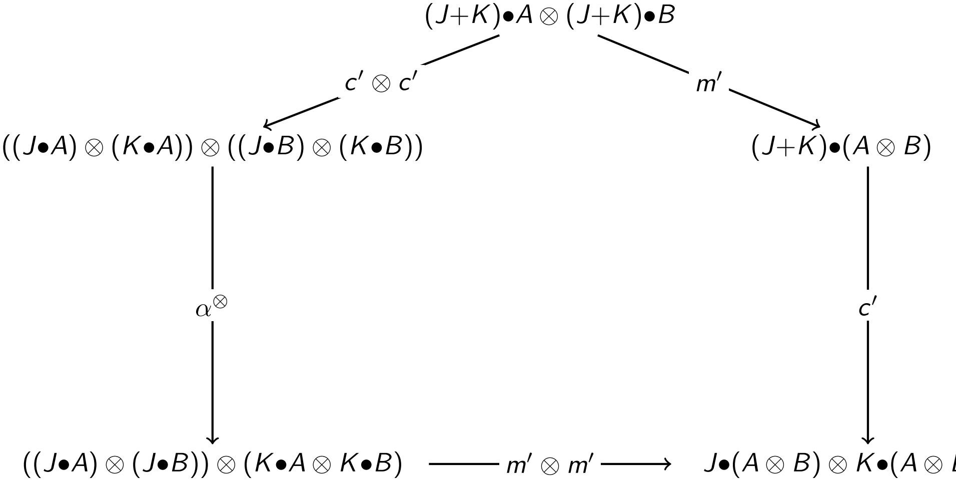
$$\delta' : (J{*}K){\bullet}A \Longrightarrow J{\bullet}K{\bullet}A \qquad \epsilon' : 1{\bullet}A \Longrightarrow A$$

$$c' : (J{+}K){\bullet}A \Longrightarrow J{\bullet}A \otimes K{\bullet}A \qquad w' : 0{\bullet}A \Longrightarrow \mathbf{1}$$

$$m' : J{\bullet}A \otimes J{\bullet}B \Longrightarrow J{\bullet}(A \otimes B) \qquad n' : \mathbf{1} \Longrightarrow J{\bullet}\mathbf{1}$$

- and more than 20 commutative diagrams.

An example of wanted diagram



Obtaining bounded exponential as retractions of usual exponential

Bounded exponential situation by retraction-embedding

A bounded exponential situation by retraction-embedding is the given of:

- a model of MELL $(\mathcal{A}, \otimes, !, \mathbf{1}, \multimap, \delta, \epsilon, c, w, m, n)$,
- a bifunctor: $\_\bullet\_ : \mathcal{S} \times \mathcal{A} \to \mathcal{A}$
- 2 naturals transformations:

$$\iota : J\bullet A \Longrightarrow !A \qquad \pi : !A \Longrightarrow J\bullet A$$

- and 5 commutative diagrams.

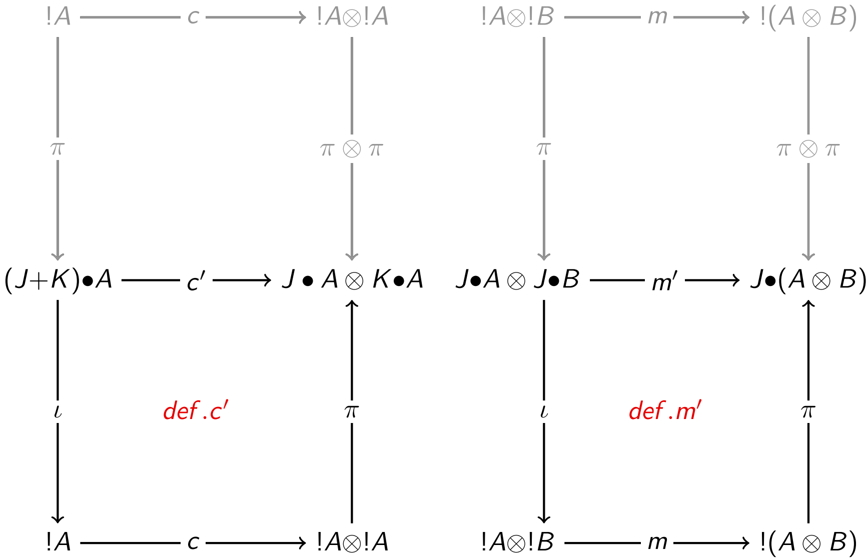
The retraction-embedding diagram

The equation $\iota_{J,A}; \pi_{J,A} = id_{J\bullet A}$ is not *a priori* needed, but is true in every interesting considered model.

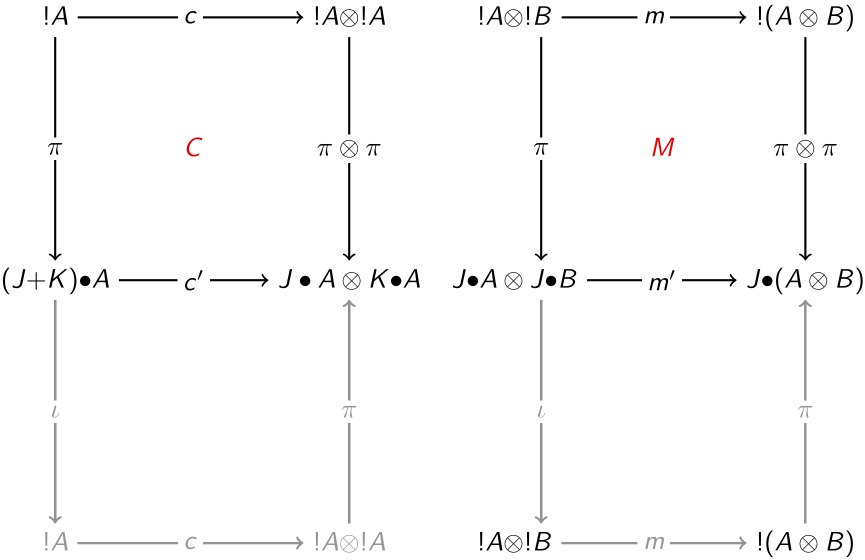
An examples and interest of wanted diagrams

$$
\begin{CD}
!A @>c>> !A\otimes !A \\
@V{\pi}VV @VV{\pi\otimes\pi}V \\
(J+K)\bullet A @. J\bullet A\otimes K\bullet A \\
@V{\iota}VV @AA{\pi}A \\
!A @>c>> !A\otimes !A
\end{CD}
\qquad
\begin{CD}
!A\otimes !B @>m>> !(A\otimes B) \\
@V{\pi}VV @VV{\pi\otimes\pi}V \\
J\bullet A\otimes J\bullet B @. J\bullet(A\otimes B) \\
@V{\iota}VV @AA{\pi}A \\
!A\otimes !B @>m>> !(A\otimes B)
\end{CD}
$$

An examples and interest of wanted diagrams



An examples and interest of wanted diagrams



Theorem

Theorem:

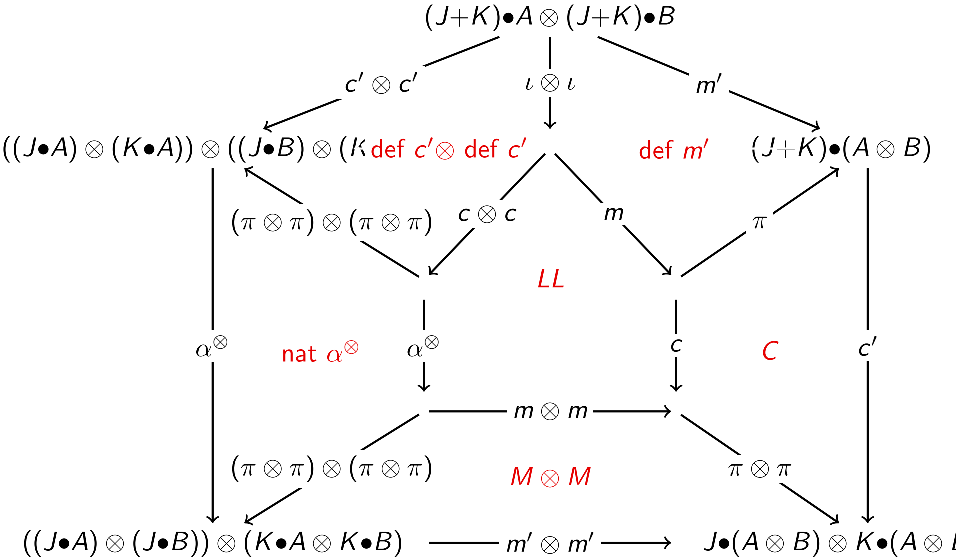
LL model $+\ \pi_J :!A \to J{\bullet}A + \iota_{,J} : J{\bullet}A \to !A\ \ +$ 5 diagrams $=\ B_{\mathcal{S}}$LL model

$$\delta'_{J,K,A} = \iota_{J*K,A}; \delta_A; !\pi_{K,A}; \pi_{J,K\bullet A} \qquad \epsilon'_A = \iota_{1,A}; \epsilon_A$$

$$c'_{J,K,A} = \iota_{J+K,A}; c_A; (\pi_{J,A} \otimes \pi_{K,A}) \qquad w'_A = \iota_{0,A}; w_A$$

$$m'_{J,A,B} = (\iota_{J,A} \otimes \iota_{J,B}); m_{A,B}; \pi_{J,A\otimes B} \qquad n'_J = n; \pi_{J,\mathbf{1}}$$

An example of wanted diagram



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The model Rel of LL [folklore]

The category *Rel* : a model of MLL

Objects: Sets **Morphisms:** Relations

Multiplicatifs: $A \otimes B = A \multimap B = A \times B$ and $1 = \{*\}$

$Rel^{\mathbb{N}}$: a model of MELL

Exponential: $!A = \mathcal{M}_f(A)$

$$\delta_A = \{(u, V) \mid V \in !!A, u = \Sigma V\} \qquad \epsilon_A = \{([a], a) \mid a \in A\}$$

$$c_A = \{(u, (v, w)) \mid u = v+w\} \quad w_A = \{([], *)\} \quad n = \{(*, [*, ..., *])\}$$

$$m_{A,B} = \{((u, v), w) \mid \forall a, u(a) = \Sigma_{b \in B} w(a, b), \forall b, v(b) = \Sigma_{a \in A} w(a, b)\}$$

Hypothesis: multiplicity [Carraro&Al]

A semiring $\mathcal{R}$ has multiplicities if

(MS1) it is positive: $m+n=0 \;\Rightarrow\; m=n=0$

(MS2) it is discreet: $m+n=1 \;\Rightarrow\; m=0$ or $n=0$

(MS3) it has additive splitting properties.

$$m_1+n_1=m_2+n_2 \;\Rightarrow\; \exists(p_{ij})_{1\leq i,j\leq 2}, m_i=p_{i1}+p_{i2},\;\; n_j=p_{1j}+p_{2j}.$$

(MS4) it has multiplicative k-ary splitting property:

$$n_1+n_2=pm \;\Rightarrow \exists k, p_1, p_2, m_{11}, m_{12}, m_{21}, m_{22}, p=\sum_{j\leq k} p_j,$$

$$n_i=\sum_{j\leq k} p_j m_{ji}, \forall j\leq k, m=m_{j1}+mj2$$

Examples

- $\mathbb{N}$
- $\bar{\mathbb{N}}$
- $\mathbb{N}\times\mathbb{N}$
- $\mathcal{M}_f(\mathbb{N})$

$\mathcal{R}$ morally behave like $\mathbb{N}$

- $\mathcal{R}$ contain $\mathbb{N}$ as sub semiring
- The sum in $\mathcal{R}$ is the one of $\mathbb{N}$

The model $\mathrm{Rel}^{\mathcal{R}}$ of LL [Carraro&Al]

Definition of $\mathcal{R}_f\langle A\rangle$

Given a semiring $\mathcal{R}$ and a set A, the space of functions $f : A \to \mathcal{R}$ of **finite support** is denoted $\mathcal{R}_f\langle A\rangle$. It is a semimodule over $\mathcal{R}$ with:

- A commutative sum: $(f+g)(x) = f(x)+g(x)$
- An external product: $(K\cdot f)(x) = K\cdot f(x)$

$Rel^{\mathcal{R}}$: a model of MELL

Exponential: $!A = \mathcal{R}_f\langle A\rangle$

$$\delta_A = \{(u, V) \mid V \in !!A, u = \Sigma_{v\in !A} V(v)\cdot v\} \qquad \epsilon_A = \{([1\cdot a], a) \mid a \in A\}$$

$$c_A = \{(u, (v, w) \mid u = v+w \in !A\} \quad w_A = \{([], *)\} \quad n = \{(*, [J\cdot *]) | J\in\mathcal{R}\}$$

$$m_{A,B} = \{((u, v), w) \mid \forall a, u(a) = \Sigma_{b\in B} w(a, b), \forall b, v(b) = \Sigma_{a\in A} w(a, b)\}$$

It is a generalisation of $\mathrm{Rel}^{\mathbb{N}}$

Indeed $\mathcal{M}_f(A)$ is an other notation for $\mathbb{N}\langle A\rangle$.

The model $\mathrm{Rel}^{(\mathcal{R},\tilde{\in},\mathcal{S})}$ of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

Semiring refinement $(\mathcal{R}, \tilde{\in}, \mathcal{S})$

- $\mathcal{R}$ is a semiring with multiplicities,
- $\mathcal{S}$ is an ordered semiring,
- $\tilde{\in}$ is a relation between those,
- $1_{\mathcal{R}} \tilde{\in} 1_{\mathcal{S}}$ and $0_{\mathcal{R}} \tilde{\in} 0_{\mathcal{S}}$,
- $n \tilde{\in} J$ and $m \tilde{\in} K$ imply $n+m \,\tilde{\in}\, J+K$ and $n*m \,\tilde{\in}\, J*K$,
- $J \leq K$ iff $J \tilde{\subseteq} K$.

$Rel^{(\mathcal{R},\tilde{\in},\mathcal{S})}$: a model of $\mathrm{B}_{\mathcal{S}}\mathrm{LL}$

Bounded exponential: $J\bullet A = \{u \in \mathcal{R}_f\langle A\rangle \mid \Sigma_{a\in A} u(a) \tilde{\in} J\}$

projection and injection:: $\pi_{J,A} = \iota_{J,A} = \{(u,u) \mid u \in J\bullet A\}$

Since $!A = \mathcal{R}\langle A\rangle$, we have that $i\bullet A \subseteq !A$

Some examples

$\mathcal{R}$	$\mathcal{S}$	$\tilde{\in}$	$0\bullet A$	$1\bullet A$
	$(\mathbb{B}ool, id)$	$\{(0,0)\} \cup \{(n+1,1)\}$	$\{[]\}$	$\mathcal{M}_f(A) - \{[]\}$
	$(\mathbb{B}ool, 0\leq 1)$	$\{(0,0)\} \cup \{(n,1)\}$	$\{[]\}$	$\mathcal{M}_f(A)$
	$(\mathbb{B}ool, 1\leq 0)$	$\{(n,0),(n+1,1)\}$	$\mathcal{M}_f(A)$	$\mathcal{M}_f(A) - \{[]\}$
	$(\mathbb{B}ool, id)$	$\mathcal{R} \times \mathbb{B}ool - \{(0,1),(1,0)\}$	$\mathcal{M}_f(A) - \{[a]\}$	$\mathcal{M}_f(A) - \{[]\}$
$\mathbb{N}$	$(\mathbb{Z}/2\mathbb{Z}, id)$	$\{(2n,0),(2n+1,1)\}$	even size	odd size
	$(\mathbb{N}, id)$	$\{(n,n) \mid n \in \mathbb{N}\}$	$n\bullet A = \{[a_1, ..., a_n]\}$	
	$(\mathbb{N}, \leq_{\mathbb{N}})$	$\{(m,n) \mid m \leq n\}$	$n\bullet A = \{[a_1, ..., a_m] \mid m \leq n\}$	
	$(\bar{\mathbb{N}}, \leq_{\bar{\mathbb{N}}})$	$\{(m,n) \mid m \leq n\}$	$n\bullet A$: idem	$\omega\bullet A = \mathcal{M}_f(A)$
	$(\mathcal{P}^*(\mathbb{N}), \subseteq)$	$\{(n,U) \mid n \in U\}$	$U\bullet A = \{[a_1, ..., a_n] \mid n \in U\}$	

The semiring $\mathcal{M}_f(\mathbb{M})$

$\mathcal{M}_f(\mathbb{M})$

Given a monoid $\mathbb{M}$, $\mathcal{M}_f(\mathbb{M})$ is a semiring with:

- A commutative sum: $(f+g)(x) = f(x)+g(x)$
- A product (Dirichlet convolution):

$$(f\cdot g)(x) = \sum_{\substack{y,z \\ x=y\cdot z}} f(y)\cdot g(z)$$

Example of such semirings

- $\mathbb{N} = \mathcal{M}_f(\mathbb{B}ool)$
- Ghica's $\mathcal{M}_f(Aff_1^c)$
- $\mathbb{P}oly = \mathcal{M}_f(\mathbb{N}_+)$

$\mathcal{M}_f(\mathbb{M})$ has multiplicities

$\mathcal{M}_f(\mathcal{S}_*)$ refines $\mathcal{S}$

$$[K_1, ..., K_n]\ \tilde{\in}_{\mathcal{S}}\ J \quad \text{iff} \quad \Sigma_i K_i \leq J$$

The two level of semirings

$\mathcal{R}$: semantic semiring

- $\mathcal{R}$ represent the actual resources.
- 0 : the absence of resources.
- 1 : the unitary amount of resources.
- Any bag of resources can be separated and reformed

$\mathcal{S}$: syntactical semiring

- $\mathcal{S}$ represent information on resources.
- 0: you may not have resources.
- Sames goes for 1.
- $\leq_{\mathcal{S}}$ is the accuracy of the information.

$\mathcal{S}$ as a quotient

With $\mathcal{R} = \mathcal{M}_f(\mathcal{S}_*)$ where you forgot "paralleled history".

$\mathcal{S}$ as non-determinism

$\mathcal{S} = (\mathcal{P}_f(\mathcal{R}), \subseteq)$ can encode non deterministic operators.

$\mathcal{S}$ as a limit

$\mathcal{S} = (\mathcal{P}(\mathcal{R}), \subseteq)$ or $\mathcal{S} = \overline{\mathcal{R}}$ can encode fixpoints.

Extensions

Does $\mathcal{S}$ have to be a semiring?

Remark: only $0\tilde{\in}J$ is necessary to perform weakening in J and $1\tilde{\in}J$ to perform dereliction.

One can replace 0 and 1 by sets Z and U such that:

- if $J \in Z$ then $J{+}K \geq K$
- if $J \in U$ then $J * K \geq K$
- if $J \in Z$ then $J * K \in Z$

Alex's intersection type

$$[\![J_1 \cdot A_1 \odot \cdots \odot J_n \cdot A_n]\!] = \{\sigma_i u_u \in ![\![A]\!] \mid \forall i \leq n, u_i \in [\![J_i \cdot A_i]\!]\}$$

Where A is the simple type obtained by forgetting resource annotation in any of the A_i.

Theorem and conclusion

General theorem

Any bounded logic $\mathsf{B}_{\mathcal{S}}\mathsf{LL}$ has as model $\mathsf{REL}^{\mathcal{M}_f(\mathcal{S}),\tilde{\in}_{\mathcal{S}},\mathcal{S}}$.

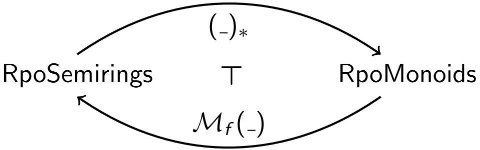
but it may has other more accurate relational models

Current works:

- Identifying the categorical sens of the relation $\tilde{\in}$.
- Investigate the Locals of 1.
- Identifying the sens of the natural transformations ι and π.
- Creating non degenerated models via double gluing.
- Find models when $\mathcal{S}$ is a bimonoidal category and not a semiring.
- Generalising all this to resource-dependants logics.

Forgetful adjunction

If we denotes RpoSemirings the category of ordered semirings and relation (preserving the structure) and RpoMonoids the same category for monoids:



Indeed $(\_)_*$ is the forgetful functor and $\mathcal{M}_f(\_)$ the free functor generating the free semimodule over the multiplicative monoid $\mathbb{M}$.
We have that $\tilde{\in}_{\mathcal{S}} = der(\mathcal{M}_f((\_)_*))_{\mathcal{S}} : (\mathcal{M}_f(()\mathcal{S}_*) \to \mathcal{S})$