A modern approach of Indexed Linear Logic

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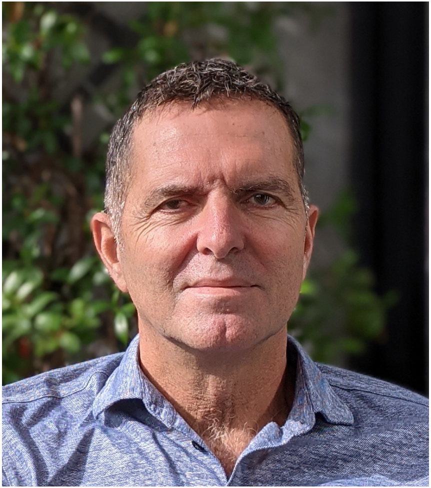
Indexed linear logic, An awesome discovery from this guy :



(Maybe with somone else... but that's not important)

Why was IndLL cool at that time ?

Extracted from the phase model of linear logic

In can be seen as the internal logic of the phase model

Can be use to build non-uniform coherent models

Gave birth to hypercoherent models

(please don't ask question, I don't fully understand this part)

Generalises non-idempotent intersection types to the whole LL

Why is IndLL even cooler now ?

Gives intersection types for any calculus encoded in μLL^2

Simply typed CbN λ-calculus, simply typed CbV λ-calculus, Bang-calculus, System T, ADT,...

A bit more complex for unrestricted fixedpoints or untyped calculi.

Can be modularised to represent other IT but also many more !

Non-idempotent IT, PCoh IT, graded types, BLL...

Between semantics and syntax an alternative point of view over calculi !

Between pure syntax (LL) and pure semantics (intersection types), a whole range of systems with different levels of expresivity and "inferenceability".

What is in this talk ?

My sweet dream : an algebraic theory of decorations

Types can be endowed with semantic "decoration" precising behaviors of the typed terms.
We aim at specifying an algebraic universe of possible decoration systems.

The "non-idempotent" Indexed linear logic

We will present $\mu\mathsf{Ind}_{\texttt{Set}*\texttt{Part}}\mathsf{LL}$ the modularisation of indexed linear logic (with fixedpoints) that extends non-idempotent intersection types.

Refining/decorating type systems : Church and Curry knitted together

À la Church decoration on Curry typed terms

A typed term $\vdash t : A$ may be "decorated" by a more precise type

$$\vdash_{I} t : A^{❀} \lhd A$$

I is a decoration context, later called "locus"

Same proof if any

$$\frac{\pi^{❀}}{\Gamma^{❀} \vdash_{I} t : A^{❀}} \implies \frac{\pi}{\Gamma \vdash t : A}$$

Example : size types

$$\vdash_{i,j} concat : [A]_i * [A]_j \to [A]_{i+j} \lhd [A] * [A] \to [A]$$

A logical counterpart : step-preserving forgetfulness

Every decorated formula $A^{❀}$ refines a unique formula A

Same goes for proofs.

This preserves cut-elimination

if $\dfrac{\pi^{❀}}{\Gamma^{❀} \vdash_{I} A^{❀}}$ and if $\pi \leadsto \rho$ then there is a decoration $\pi^{❀} \leadsto \rho^{❀}$

Curry and Howard's blind-spot

$\dfrac{\pi}{\Gamma \vdash A} \quad \Longrightarrow$ for any given $\Gamma^{❀}$, $A^{❀}$ and I, at most one proof $\dfrac{\pi^{❀}}{\Gamma^{❀} \vdash_{I} A^{❀}}$

Difficult... at least make it true in your models...

Sub-decoration : more or less refined information

Two proofs with different decorations are OK and can even represent level of precision

$$\dfrac{\pi^{\text{✿}}}{\Gamma^{\text{✿}} \vdash_I A^{\text{✿}}} \lhd \dfrac{\pi^{\text{❀}}}{\Gamma^{\text{❀}} \vdash_J A^{\text{❀}}} \lhd \dfrac{\pi}{\Gamma \vdash A}$$

where '$\lhd$' means "is more precise than"

Precision can be internalised as subtyping

$$\dfrac{\dfrac{\pi^{\text{✿}}}{\Gamma^{\text{✿}} \vdash_I A^{\text{✿}}} \qquad \text{✿} \subseteq \text{❀}}{\Gamma^{\text{✿}} \vdash_I A^{\text{❀}}}$$

using a meta rule without CH content or integrated into other rules

My long term objectives

Transform models into decorated logics

I am convinced that many models of LL can be fully characterised by a well suited decoration-system of LL.

Should extends outside of LL.

Study continuum between syntax and model

Somewhere between the model decoration and the absence of decoration should be a precise but “inferenceable” one !

Inferenceability : qualitative vs quantitative property

We are not looking for a fully inferenceable type system,
but to an “approximable inference” with nice properties

Formulae and locci

Indexed formulae

$$X \in \bigcup_{I \in \texttt{Set}} \texttt{variable}(I)$$

$$u \in \text{quasi-injective functions}$$

$$f, i, j \in \text{functions}$$

$$A, B ::= f(X) \mid f(X)^{\perp} \mid \mathbf{1} \mid \perp \mid \mathbf{0} \mid \top \mid A \otimes B \mid A \parr B$$
$$\mid A\,{}_i{\oplus}_j B \mid A\,{}_i{\&}_j B \mid !_u A \mid ?_u A \mid \mu_f X.A \mid \nu_f X.A$$

Formulae are defined over locci

$$\frac{}{\emptyset \Vdash \mathbf{0}} \qquad \frac{I \Vdash A \qquad J \Vdash B}{I \uplus J \Vdash A\,{}_{inj_1}{\oplus}_{inj_2} B} \qquad \frac{J \Vdash A \quad u : I \leftarrow J}{I \Vdash !_u A}$$

Base change : a recurrent pattern

Proofs can even be transported by rewriting / base-change

$$f\left(\frac{\pi^{❀}}{\Gamma^{❀} \vdash_J A^{❀}}\right) = \frac{f\left(\pi^{❀}\right)}{f\left(\Gamma^{❀}\right) \vdash_I f\left(A^{❀}\right)} = \frac{\pi^{✿}}{\Gamma^{✿} \vdash_I A^{✿}}$$

f traverses the term (like LL's negation), only modifying decoration.

Base-changes are crucial in indexed LL, in bounded LL and some other related systems.

$$f(\mathbf{1}) := \mathbf{1} \qquad f(A \otimes B) := f(A) \otimes f(B)$$

$$f(\mathbf{0}) := \mathbf{0} \qquad f\left(A_i \oplus_j B\right) := f_{|i}(A)_{i_{|f}} \oplus_{j_{|f}} f_{|j}(B)$$

$$f(!_u A) := !_{u_{|f}} f_{|u}(A) \qquad f\left(\mu_g X.A\right) := \mu_{f;g} X.A$$

Multiplicatives : Nothing new happen

$$\frac{\Gamma \vdash_I A, \Delta}{\Gamma, A^\perp \vdash_I \Delta} \qquad \frac{\Gamma, A \vdash_I \Delta}{\Gamma \vdash_I A^\perp, \Delta} \qquad \frac{\Gamma \vdash_I A, B, \Delta}{\Gamma \vdash_I B, A, \Delta} \qquad \frac{\Gamma \vdash_I A \qquad A \vdash_I \Delta}{\Gamma \vdash_I \Delta}$$

$$\frac{}{\vdash_I \mathbf{1}} \qquad \frac{\Gamma \vdash_I A \qquad \Delta \vdash_I B}{\Gamma, \Delta \vdash_I A \otimes B} \qquad \frac{\Gamma, A \vdash_I B}{\Gamma \vdash_I A \multimap B}$$

Intuition : Linear λ-calculus's IT has no intersection

The loci I represents a set of “names” for intersection types that we are composing pointwise

Additives are transversal operators

$$\frac{0 \Vdash \Gamma}{\mathbb{0} \vdash_{0} \Gamma} \qquad \frac{A \vdash_{I} \mathtt{inj}_1(\Gamma) \qquad B \vdash_{J} \mathtt{inj}_2(\Gamma)}{A_{\mathtt{inj}_1} \oplus_{\mathtt{inj}_2} B \vdash_{I \uplus J} \Gamma} \qquad \frac{\Gamma \vdash_{I} A \qquad 0 \Vdash B}{\Gamma \vdash_{I} A_{\mathtt{id}} \oplus_{\mathtt{init}} B}$$

The base-change is contravariant

Even if Γ is defined over $I \uplus J$, $\mathtt{inj}_1(\Gamma)$ is defined over I.

Additives are transversal operators who lives in the (op)fibration

$$\frac{0 \Vdash \Gamma}{\mathbb{0} \vdash_{0} \Gamma} \qquad \frac{A \vdash_{I} \mathtt{inj}_1(\Gamma) \qquad B \vdash_{J} \mathtt{inj}_2(\Gamma)}{A_{\mathtt{inj}_1}\oplus_{\mathtt{inj}_2} B \vdash_{I \uplus J} \Gamma} \qquad \frac{\Gamma \vdash_{I} A \qquad 0 \Vdash B}{\Gamma \vdash_{I} A_{\mathtt{id}}\oplus_{\mathtt{init}} B}$$

The base-change is contravariant

Even if Γ is defined over $I \uplus J$, $\mathtt{inj}_1(\Gamma)$ is defined over I.

The model is an indexation functor and the coproduct lives in the fibration

In $\int$Model we have

$$(I, A) \oplus (J, B) \quad := \quad (I + J, A \;{}_{I}\!\oplus_{J}\; B)$$

that is a Cartesian co-product !

The product lives, similarly, in the op-fibration.

Exponentials : non-idempotency in action

$$\frac{\Gamma \vdash_I ?_{f_1;u} f_1(A), ?_{f_2;u} f_2(A)}{\Gamma \vdash_I ?_u A} \quad \text{where} \quad \begin{array}{l} f_1 := (J \subseteq J \cup K) \\ f_2 := (K \subseteq J \cup K) \end{array}$$

$$\frac{\Gamma \vdash_I}{\Gamma \vdash_I ?_u B} \qquad \frac{\Gamma \vdash_I f(B) \qquad \forall x, f(x) \in u^{-1}(x)}{\Gamma \vdash_I ?_u B}$$

$$\frac{!_{w_{1|v}} v_{|w_1}(A_1), \ldots, !_{w_{n|v}} v_{|w_n}(A_n) \vdash_J B \qquad v : I \leftarrow J}{!_{w_1} A_1, \ldots, !_{w_n} A_n \vdash_I !_v B}$$

Very different from original IndLL

- The weakening and the dereliction are more often allowed
  $\rightsquigarrow$ subtyping
- The base change are not just injections and can merge elements
  $\rightsquigarrow$ set vs multiset

Remark : in Indexed linear logics, one can have set-like exponential without subtyping due to the orientation of the idempotency rule.

Fixed point : a new operator

$$\frac{\Gamma \vdash_{I} f(A[\mu_{\mathtt{id}} X.A/X])}{\Gamma \vdash_{I} \mu_f X.A} \qquad \frac{\Gamma \vdash_{I} f(A[B/X]))}{B \vdash_{I} \nu_f X.A}$$

where $f(X)[B/X] := f(B)$

The f in $\mu_f X.A$ and $f(X)$ are just "explicit base change"

as in "explicit substitution"

Vertical equivalence

Recall : we want Curry and Howard's blind-spot

$$\frac{\pi}{\Gamma \vdash A} \quad \Longrightarrow \quad \begin{array}{l}\text{for any given } \Gamma^{❀}, A^{❀} \text{ and } I, \\ \text{at most one proof } \dfrac{\pi^{❀}}{\Gamma^{❀} \vdash_I A^{❀}}\end{array}$$

Wrong but obtained by equating the "equality" proofs :

$A \equiv B$ if there are equiprovable with opposite proofs

$$\frac{\pi_1}{A \vdash_I B} \text{ and } \frac{\pi_2}{B \vdash_I A}$$

collapsing into the equality :

$$\mathtt{init}(\pi_1) = \mathtt{init}(\pi_2) = \mathtt{id}_{\mathtt{init}(A)}$$

Results

Non-idempotent intersection types

Up-to $\equiv$, the formulae $A, B ::= X \mid !_u A \multimap B$ of locci $\{*\}$ correspond exactly to intersection types.

In addition, if $\dfrac{\pi_1}{\vdash_{\{*\}} A}$ then as a λ-term π_1 is of intersection type A

Also true for CbV intersection type (using the boring translation)

Abstract data types

Up-to $\equiv$, the formulae $A = \mu_f X.(\mathbf{1}_i \oplus_j g(X))$ of locci $\{*\}$ correspond exactly to natural numbers $[n] := \mu_{* \mapsto n} X.(\mathbf{1}_{\mathtt{zero}} \oplus_{\mathtt{incr}} \mathtt{id}_{\mathbb{N}}(X))$

This result should be extendable to any abstract data-type

To represent GADTs, we need the second order.

Non-results

Untyped version has too much types

Contrary to original IndLL, $\{\mu_f X.!_u X \multimap X \mid u, f\}/_{\equiv}$ contains too many types.
For examples, it embeds Park's filter model with the intersection type $\omega = \{\omega\} \to \omega$.

Infinitary calculi

We could expect to get infinitary calculi using the greatest fixpoint operator ν, but it is not exactly what we want as proofs remains finite.
To be investigated...