A modern approach of Indexed Linear Logic

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(part is joint work with T.Ehrhard and F. Olimpieri)

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Indexed linear logic,
An awesome discovery from this guy:

(Maybe with someone else… but that’s not important)
Why was IndLL cool at that time?

- Extracted from the phase model of linear logic
  - In can be seen as the internal logic of the phase model

- Can be used to build non-uniform coherent models
  - Gave birth to hypercoherent models

- Generalises non-idempotent intersection types to the whole LL

(please don’t ask question, I don’t fully understand this part)
Why is IndLL even cooler now?

Gives intersection types for any calculus encoded in $\mu LL^2$
Simply typed CbN $\lambda$-calculus, simply typed CbV $\lambda$-calculus,
Bang-calculus, System T, ADT,...

A bit more complex for unrestricted fixedpoints or untyped calculi.

Can be modularised to represent other IT but also many more!
Non-idempotent IT, PCoh IT, graded types, BLL...

Between semantics and syntax
an alternative point of view over calculi!

Between pure syntax (LL) and pure semantics (intersection types), a
whole range of systems with different levels of expressivity and
“inferenceability”.

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What is in this talk?

My sweet dream: an algebraic theory of decorations

Types can be endowed with semantic “decoration” precising behaviors of the typed terms. We aim at specifying an algebraic universe of possible decoration systems.

The “non-idempotent” Indexed linear logic

We will present $\mu \text{Ind}_{\text{Set}^* \text{Part}} \text{LL}$ the modularisation of indexed linear logic (with fixedpoints) that extends non-idempotent intersection types.
Refining/decorating type systems: Church and Curry knitted together

À la Church decoration on Curry typed terms

A typed term $\vdash t : A$ may be “decorated” by a more precise type

$$\vdash I t : A \ll A$$

$I$ is a decoration context, later called “locus”

Same proof if any

$$\frac{\pi}{\Gamma \vdash t : A} \quad \Rightarrow \quad \frac{\pi}{\Gamma \vdash t : A}$$

Example: size types

$$\vdash_{i,j} concat : [A]_i \times [A]_j \rightarrow [A]_{i+j} \ll [A] \times [A] \rightarrow [A]$$
A logical counterpart: step-preserving forgetfulness

Every decorated formula $A^\star$ refines a unique formula $A$

Same goes for proofs.

This preserves cut-elimination

If $\frac{\pi}{\Gamma^\star \vdash A^\star}$ and if $\pi \rightsquigarrow \rho$ then there is a decoration $\pi^\star \rightsquigarrow \rho^\star$

Curry and Howard’s blind-spot

For any given $\Gamma^\star$, $A^\star$ and $I$,

at most one proof $\frac{\pi^\star}{\Gamma^\star \vdash A^\star}$

Difficult... at least make it true in your models...
Sub-decoration:
more or less refined information

Two proofs with different decorations are OK and can even represent level of precision

\[ \pi \vdash A \psi \ll 
\pi \vdash A \phi \ll 
\pi \vdash A \]

where ‘\ll’ means “is more precise than”

Precision can be internalised as subtyping

\[ \pi \vdash A \phi \ll 
\phi \subseteq \phi \]

using a meta rule without CH content or integrated into other rules
My long term objectives

### Transform models into decorated logics
I am convinced that many models of LL can be fully characterised by a well suited decoration-system of LL.

Should extends outside of LL.

### Study continuum between syntax and model
Somewhere between the model decoration and the absence of decoration should be a precise but “inferenceable” one!

### Inferenceability : qualitative vs quantitative property
We are not looking for a fully inferenceable type system, but to an “approximable inference” with nice properties
Indexed formulae

\[ X \in \bigcup_{I \in \text{Set}} \text{variable}(I) \]

\[ u \in \text{quasi-injective functions} \]

\[ f, i, j \in \text{functions} \]

\[ A, B ::= f(X) \mid f(X)^\perp \mid 1 \mid \bot \mid 0 \mid \top \mid A \otimes B \mid A \bowtie B \]

\[ A_i \oplus_j B \mid A_i \&_j B \mid !_u A \mid ?_u A \mid \mu_f X.A \mid \nu_f X.A \]

Formulae are defined over locci

\[ \emptyset \vdash 0 \]

\[ I \vdash A \quad J \vdash B \]

\[ I \uplus J \vdash A_{\text{inj}_1} \oplus_{\text{inj}_2} B \]

\[ J \vdash A \quad u : I \leftarrow J \]

\[ I \vdash !_u A \]
Base change: a recurrent pattern

Proofs can even be transported by rewriting / base-change

\[ f \left( \frac{\pi \ast}{\Gamma \ast \downarrow f \ A \ast} \right) = f \left( \pi \ast \right) \frac{f \left( \Gamma \ast \right) \downarrow f \left( A \ast \right)}{f \left( \Gamma \ast \right) \downarrow f \left( A \ast \right)} = \frac{\pi \ast}{\Gamma \ast \downarrow f \ A \ast} \]

\( f \) traverses the term (like LL’s negation), only modifying decoration.

Base-changes are crucial in indexed LL, in bounded LL and some other related systems.

\[
\begin{align*}
f(1) & := 1 \\
f(0) & := 0 \\
f(!u A) & := !_{u|f} f_{|u}(A) \\
f(A \otimes B) & := f(A) \otimes f(B) \\
f(A_i \oplus_j B) & := f_i(A)_{|f} \oplus_j f_j(B) \\
f(\mu_g X.A) & := \mu_{f;g} X.A
\end{align*}
\]
Multiplicatives:
Nothing new happen

\[
\begin{align*}
\Gamma \vdash A, \Delta & \quad \Gamma, A \vdash \Delta \\
\Gamma, A \perp \vdash \Delta & \quad \Gamma \vdash A, B, \Delta \\
\Gamma \vdash A, \Delta & \quad \Gamma \vdash B, A, \Delta \\
\Gamma \vdash A & \quad \Gamma \vdash A \perp, \Delta \\
\Gamma, A \perp, \Delta & \quad \Gamma, \Delta \vdash A \otimes B \\
\Gamma \vdash A & \quad \Gamma, A \vdash B \\
\Gamma, \Delta \vdash A \rightarrow B & \\
\Gamma, \Delta \vdash 1 & \\
\Gamma \vdash 1 & \\
\Gamma \vdash \Gamma & \\
\end{align*}
\]

Intuition: Linear λ-calculus’s IT has no intersection

The loci $I$ represents a set of “names” for intersection types that we are composing pointwise.
Additives are transversal operators

\[
\frac{0 \vdash \Gamma}{\emptyset \vdash \Gamma} \quad \frac{A \mid \text{inj}_1(\Gamma) \quad B \mid \text{inj}_2(\Gamma)}{A \text{inj}_1 \oplus \text{inj}_2 B \mid \biguplus J \Gamma} \quad \frac{\Gamma \mid A \quad 0 \vdash B}{\Gamma \mid A_{\text{id} \oplus \text{init}} B}
\]

The base-change is contravariant

Even if \( \Gamma \) is defined over \( I \uplus J \), \( \text{inj}_1(\Gamma) \) is defined over \( I \).
Additives are transversal operators who lives in the (op)fibration

\[
\begin{array}{c}
0 \vdash \Gamma \\
\hline
\Gamma \vdash 0 \\
A \vdash \text{inj}_1(\Gamma) & B \vdash \text{inj}_2(\Gamma)
\end{array}
\]

\[
\Gamma \vdash \text{id} \oplus \text{init} B
\]

The base-change is contravariant

Even if \( \Gamma \) is defined over \( I \uplus J \), \( \text{inj}_1(\Gamma) \) is defined over \( I \).

The model is an indexation functor and the coproduct lives in the fibration

In the model we have

\[
(I, A) \oplus (J, B) := (I + J, A \uplus J B)
\]

that is a Cartesian co-product!

The product lives, similarly, in the op-fibration.
Exponentials: non-idempotency in action

\[
\begin{align*}
\Gamma \vdash ?_{f_1;u} f_1(A), ?_{f_2;u} f_2(A) & \quad \text{where} \quad f_1 := (J \subseteq J \cup K) \\
\Gamma \vdash ?_{u} A \\
\Gamma \vdash f(B) & \quad \forall x, f(x) \in u^{-1}(x) \\
\Gamma \vdash ?_{u} B
\end{align*}
\]

Very different from original IndLL

- The weakening and the dereliction are more often allowed
  \(\rightsquigarrow\) subtyping

- The base change are not just injections and can merge elements
  \(\rightsquigarrow\) set vs multiset

Remark: in Indexed linear logics, one can have set-like exponential without subtyping due to the orientation of the idempotency rule.
Fixed point: a new operator

\[
\Gamma \vdash f(A[\mu_{\text{id}}X.A/X]) \\
\frac{\Gamma \vdash f(A[\mu_{\text{id}}X.A/X])}{\Gamma \vdash \mu_f X.A}
\]

\[
\Gamma \vdash f(A[B/X]) \\
\frac{\Gamma \vdash f(A[B/X])}{B \vdash \nu_f X.A}
\]

where \( f(X)[B/X] := f(B) \)

The \( f \) in \( \mu_f X.A \) and \( f(X) \) are just “explicit base change” as in “explicit substitution”
Recall : we want Curry and Howard’s blind-spot

\[ \frac{\pi}{\Gamma \vdash A} \implies \] 

for any given \( \Gamma^{\bullet}, A^{\bullet} \) and \( I \),

at most one proof \[ \frac{\pi^{\bullet}}{\Gamma^{\bullet} \vdash A^{\bullet}} \]

Wrong but obtained by equating the “equality” proofs :

\[ A \equiv B \text{ if there are equiprovable with opposite proofs} \]

\[ \frac{\pi_1}{A \vdash B} \text{ and } \frac{\pi_2}{B \vdash A} \]

collapsing into the equality :

\[ \text{init}(\pi_1) = \text{init}(\pi_2) = \text{id}_{\text{init}(A)} \]
Results

Non-idempotent intersection types

Up-to $\equiv$, the formulae $A, B ::= X | !_u A \circ B$ of locci $\{\ast\}$ correspond exactly to intersection types.

In addition, if $\pi_1 \vdash \{\ast\} A$ then as a $\lambda$-term $\pi_1$ is of intersection type $A$

Also true for CbV intersection type (using the boring translation)

Abstract data types

Up-to $\equiv$, the formulae $A = \mu_f X.(1_i \oplus_j g(X))$ of locci $\{\ast\}$ correspond exactly to natural numbers $[n] := \mu_{\ast \rightarrow n} X.(1_{\text{zero}} \oplus_{\text{incr}} id_{\mathbb{N}}(X))$

This result should be extendable to any abstract data-type

To represent GADTs, we need the second order.
Untyped version has too much types

Contrary to original IndLL, $\{\mu f X. !_u X \rightarrow X \mid u, f\}/\equiv$ contains too many types.
For examples, it embeds Park’s filter model with the intersection type $\omega = \{\omega\} \rightarrow \omega$.

Infinitary calculi

We could expect to get infinitary calculi using the greatest fixpoint operator $\nu$, but it is not exactly what we want as proofs remains finite. To be investigated...