My scientific life with Antonio

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- A quick summary of what we did with Antonio during more than 20 years,
- and a very personal tribute to the influence of Antonio on my work.

Interruptions and objections, especially by Antonio, are most welcome !

When did I meet Antonio for the first time ?

Don't remember exactly, it was more than 35 years ago. . .

In 1987 or 1988 probably, at the ENS.

At that time I was doing categorical semantics and Antonio was working on some surprising extensions of PCF and their semantics.

The only denotational models I knew were Girard's coherence spaces (thanks to his 1986 DEA lecture) and ω-sets (realizability model), used in my PhD.

At some point, I don't remember why, Antonio explained me sequentiality, a fascinating idea !

Sequentiality (Vuillemin, Milner, Sazonov, at least)

Imagine a function $f : \mathbf{Bool}_\perp \times \cdots \times \mathbf{Bool}_\perp \to \mathbf{Bool}_\perp$ which is computable deterministically.

Then it is monotone (Scott), but has a much stronger property.

If $f(\vec{x}) = \perp$, there are two possibilities

- the situation is hopeless : $\vec{y} \geq \vec{x} \Rightarrow f(\vec{y}) = \perp$
- or there is i such that $x_i = \perp$ and if $\vec{y} \geq \vec{x}$ satisfies $f(\vec{y}) \neq \perp$ then $y_i \neq \perp$.

i is a *sequentiality index* of f (at $\vec{x}$) : *a* place where the computation of $f(\vec{x})$ is stuck by lack of information about the argument.

Are there sequentiality indexes for function types ?

No notion of index on the space of sequential functions for which evaluation si sequential.

Berry-Curien solution : sequential *algorithms* instead of sequential *functions*.

All types, including function types, become *data types* which can be presented by Concrete Data Structures. In a CDS :

Configuration = set of cells filled by values.

Sequentiality index = cell.

Intuitively : sequential algo = sequential function + a Skolem function for indexes.

Our first attempt at reformalizing SA

Use coherence spaces.

Define an exponential where $|!E|$ is a set of sequences of elements of $|E|$ (instead of sets or multisets).

Very nice... but does not work : no Seely isos.

Other (less obvious and deeper) issue : as observed by Girard, already for MALL, sequentiality fails in coherence spaces.

Sequential structures : our first new CCC, 1990

An abstraction of CDS : $X = (X_*, X^*, \vdash_X)$ where X_* is a (dI-)domain, X^* as set of linear maps (the "cells") $X_* \to \{\bot < \top\}$, $\vdash_X \subseteq X_* \times X^*$ as an accessibility relation.

+ some technical conditions.

$\leadsto$ a CCC where $(f, \varphi) : X \to Y$ is $f : X_* \to Y_*$ Scott continuous and for each $x \in X_*$,

$$\varphi_x : Y^*_{f(x)} \to X^*_x = \{\text{cells accessible at } x\}$$

satisfying sequentiality + other technical conditions.

Strong stability : our next new CCC, 1990-91

If $\vec{x}(1), \ldots, \vec{x}(k) \in \mathbf{Bool}_{\perp}^{n}$, the standard way of saying that they are coherent is : there is $\vec{x} \in \mathbf{Bool}_{\perp}^{n}$ such that $\vec{x}(i) \leq \vec{x}$ for each i.

We introduced a much more liberal notion :
$\vec{x}(1), \ldots, \vec{x}(k) \in \mathbf{Bool}_{\perp}^{n}$ are linearly coherent if, for all index $j \in \{1, \ldots, n\}$, if $x(i)_j \neq \perp$ for all i then $x(1)_j = \cdots = x(k)_j$.

Basic observation

A monotone $f : \mathbf{Bool}_{\perp}^{n} \to \mathbf{Bool}_{\perp}^{p}$ is sequential iff it maps any linearly coherent set to a linearly coherent set, and commutes with the infima of linearly coherent sets.

Cells cannot be extended to sets of sequential functions, but the notion of coherence can $\rightsquigarrow$ qualitative domains with coherence (qDC).

Qualitative domains are simplicial complexes : similar to coherence spaces, but we can have points a_1, a_2, a_3 such that each $\{a_i, a_j\}$ is a clique but $\{a_1, a_2, a_3\}$ is not a clique (triangle without its face), etc.

This is an essential feature of sequential *functions* : Gustave function.

We built a CCC of qDC's and strongly stable functions, model of PCF etc : a purely functional alternative to SA !

Failed attempt at building FA models based on qDC's

Just as Berry and Curien did with SA, we refined our qDC model with "extensionality" restrictions, this led to new models but none of them was fully abstract (FA).

Thanks to the work of Loader (published 1998) we understood later that such attemps should necessarily fail : operational equivalence on finitary PCF is undecidable whereas the semantics in such models is computable.

First divergence with Antonio

I pursued my personal obsession : understand the connection between SA and strongly stable functions $\rightsquigarrow$ hypercoherences, an LL refinement of our qDC model (1992). And then I moved to Marseille (1994).

Antonio started to work on other topics : bidomains, intersection types, degrees of parallelism (somehow connected to sequentiality), logical relations etc.

Second main collaboration with Antonio

In 1998, Antonio spent one month in Marseille.

At that time I was stuck with sequentiality, hypercoherences etc.

We decided to try and develop some logical relations for LL.

Problem : in coherence spaces, hypercoherences etc, the dualizing object $\perp$ is too poor (one point space). Already observed by Girard $\rightsquigarrow$ *On denotational completeness.*

On denotational semantics and phase semantics

We developed a *phase space valued logical relations* on coherence spaces.

Then we understood that the coherence relation was redundant.

$\rightsquigarrow$ phase-space valued denotational semantics of LL on top of the Relational Model **Rel** of LL.

In the late 90's **Rel** was not as popular as today, it was rather a kind of piece of folklore universally considered as badly behaved.

Main outcomes :

- Indexed Linear Logic, a *Linear Logic of coherence*.
- Phase model of this ILL $\rightsquigarrow$ denotational model of LL.
- All these models have the same webs (coming from **Rel**).
- $\rightsquigarrow$ completely new concrete models, which, I think, couldn't have been discovered otherwise, for instance *non-uniform coherence spaces*.

Example : non-uniform coherence spaces

$X = (|X|, \frown_X, \smile_X)$, a web with two disjoint binary symmetric relations : strict coherence and strict incoherence.

Warning : $a \frown_X a$ and $a \smile_X a$ are possible !

Involutive linear negation : $X^\perp = (|X|, \smile_X, \frown_X)$

A suprising exponential in NUCS (extracted from the general ILL phase space semantics)

$|!X| = \mathcal{M}_{\mathrm{fin}}(|X|)$ and

- $[a_1, \ldots, a_n] \smile_{!X} [a_{n+1}, \ldots, a_k]$ if $\exists i \neq j \; a_i \smile_X a_j$
- $[a_1, \ldots, a_n] \frown_{!X} [a_{n+1}, \ldots, a_k]$ if $\neg([a_1, \ldots, a_n] \smile_{!X} [a_{n+1}, \ldots, a_k])$ and $\exists i \forall j \neq i \; a_i \frown_X a_j$.

Who could pull this definition out of the hat ?

Self-incoherence : $[\mathbf{t}, \mathbf{f}] \smile_{!(\mathbf{Bool})} [\mathbf{t}, \mathbf{f}]$ where $\mathbf{Bool} = 1 \oplus 1$.

This is crucial because it is easy to define a PCF term of type $\mathbf{Bool} \to \mathbf{Bool}$ whose interpretation in **Rel** contains $([\mathbf{t}, \mathbf{f}], \mathbf{t})$ and $([\mathbf{t}, \mathbf{f}], \mathbf{f})$.

This second main interaction with Antonio completely changed my point of view on denotational semantics !

- The most basic model of LL is **Rel**, and not **Scott** or **Coh** ;
- the most basic exponential is non-idemptent (multisets) and non-uniform, for instance $[\mathbf{t}, \mathbf{f}] \in !\mathbf{Bool}$;
- "coherence" is a superstructure on top of **Rel** which does not change essentially the interpretation of proofs ;
- there can be a lot of different notions of "coherence" ;

Rel is a skeleton on which you can add flesh in many different ways.

$\leadsto$ Köthe spaces, finiteness spaces, probabilistic coherence spaces, differential λ-calculus and differential LL.

Third main collaboration with Antonio

In 2005 I come back to Paris, PPS.

With Antonio, we immediately resume our collaboration, now with new partners : Nino Salibra and Giulio Manzonetto, and later Alberto Carraro.

$\rightsquigarrow$ revisting the semantics of the λ-calculus and intersection types in the relational setting.

13 joint papers with Antonio, from 1991 to 2012.

Looking forward to the next one... In the meantime

Thank You Antonio !