1 Preliminaries

1.1 Term Rewriting Systems (TRS)

1.2 Randomized Strategy (CbR)

Definition 1. *A TRS $\mathcal{S}$ is* CbR-friendly *if weakly normalisable terms are CbR-normalisable with probability 1:*

$$\forall M \in \mathcal{S}, \quad M\Downarrow_w \quad \Rightarrow \quad M\Downarrow_{CbR} = 1.$$

2 Confluent TRS

Let M a TRS:

- which signature has a unic constant $\mathbf{0}$ and 3 unary functions $\boldsymbol{S}$, $\boldsymbol{dup}$ and $\boldsymbol{tower}$,
- which fragment $(\mathbf{0}, \boldsymbol{S}, \boldsymbol{dup})$ are the encoding of peano integer and dupplication:

$$\boldsymbol{dup}\,\mathbf{0} \to \mathbf{0} \qquad\qquad \boldsymbol{dup}\,\boldsymbol{S}\ x \to \boldsymbol{SS}\ \boldsymbol{dup}\ x$$

- the function $\boldsymbol{tower}$ is the key feature that enable non determinism and potential divergence:

$$\boldsymbol{tower}\,\mathbf{0} \to \mathbf{0} \qquad \boldsymbol{tower}\,\boldsymbol{S}\ x \to \boldsymbol{tower}\ x \qquad \boldsymbol{tower}\ x \to \boldsymbol{tower}\ \boldsymbol{dup}\ x.$$

The system is linear, trivialy confluent and every term is weakly normalising.
However, the randomized strategy over the term $\boldsymbol{tower}\,\mathbf{0}$ is not almost surely terminating.

Too dificult to prove directly. Look for an easier example.

3 Orthogonal TRS

With orthogonal TRS, the standardisation theorem allows us to refrase our question using the leftmost randomness:

Definition 2. *An ortogonal TRS is* leftmost random *if the leftmost redex will be evantually reduced by the randomised strategy with probability* 1.

Proposition 1. *An orthogonal TRS which is leftmost random is CbR-friendly.*

Proof. Let M be a term in a leftmost random system. In a sequence of random reduction, the leftmost redex will be eventually reduced, by leftmost randomness. And by easy induction, we can see that for any n, eventually the leftmost redexes will be reduced n times successivelly. Since a converging term accept only a bounded number of leftmost reduction, the sequence in necessarily finite. □

Notice that the converse is not true. For example, a TRS which is not leftmost random but has no weakly converging term is CbR-friendly. In fact, leftmost randomness is a much more natural property for orthogonal TRS that allows more generalisations.

However, from a combinatorial point of view, the probability of reducing one element is difficult to comput, and we prefer considering the size of the terms. This leads to the notion of following notion of mean-explosiveness:

Definition 3. *A term M is mean-explosive if its size (in terms of redexes) is growing so fast (in expected value) that the following infinite sum converges:*

$$\sum_{n\geq 0} \frac{1}{\mathbb{E}^n_\#(M)} \quad < \quad \infty.$$

Proposition 2. *An orthogonal TRS is leftmost random, and* a fortiori *CbR-friendly, if none of its terms are mean-explisive.*

Proof. The probability of a term to reduce at least once its leftmost redex is given by:

$$1 - \prod_{n\geq 0}(1 - \mathbb{E}^n_{\#^{-1}}(M)).$$

This term has value 1 whenever $\sum_n \mathbb{E}^n_{\#^{-1}}(M)$ diverges. To conclud it is sufficient to prove that $\mathbb{E}^n_{\#^{-1}}(M) \geq \frac{1}{\mathbb{E}^n_{\#}(M)}$. Indeed, for any $(\alpha_i)_{i\leq m}$ we have:

$$\begin{aligned} & \forall i < j \leq n,\ \frac{(\alpha_i - \alpha_j)^2}{\alpha_i\alpha_j} \geq 0 \\ \text{thus} \quad & \sum_i\sum_{j>i}(\frac{\alpha_i}{\alpha_j} + \frac{\alpha_j}{\alpha_i} - 2) \geq 0 \\ & \sum_i\sum_j(\frac{\alpha_i}{\alpha_j} - 1) \geq 0 \\ & (\sum_{i\leq n}\frac{1}{\alpha_i})(\sum_{i\leq n}\alpha_i) \geq n^2 \\ & \frac{\sum_i \frac{1}{\alpha_i}}{n} \geq \frac{n}{\sum_i \alpha_i} \end{aligned}$$

□

3.1 The key cases

We consider a TRS $\mathbb{C}\mathtt{opy}$ with two symbols □ : 0 and ***copy*** : 2 and with a unic rule:

$$\boldsymbol{copy}(x, y) \to \boldsymbol{copy}(\boldsymbol{copy}(x, y), \boldsymbol{copy}(x, y)).$$

The randomized strategy applied to this TRS corresponds to a variant of Remy's algorithm.Combinatoricians have studied this variation, which study leads to the following conjecture: Ref

Conjecture 1 (Devroye)**.** *The serie given by the expected value of the size the n^{th} reduction of a term $M \in \mathbb{C}\mathtt{opy}$ is quasilinear:*

$$\mathbb{E}^n_{\#}(M) \ \sim\ n\log(n).$$

Corollary 4. *Assuming Devroye's conjecture, $\mathbb{C}\mathtt{opy}$ is leftmost random.*

We now consider the TRS $\mathbb{C}\mathtt{opy}_p$ with two symbols □ : 0 and $\boldsymbol{copy}_p$: 2 and with a unic rule:

$$\boldsymbol{copy}_p(x, y) \to \boldsymbol{copy}_p(\underbrace{\boldsymbol{copy}_p(x, y), \cdots, \boldsymbol{copy}_p(x, y)}_{p \text{ copies}}).$$

We can now extand Devroye's conjecture to this p-ary version:

Conjecture 2 (p-Devroye)**.** *The serie given by the expected value of the size the n^{th} reduction of a term $M \in \mathbb{C}\mathtt{opy}_p$ is quasilinear:*

$$\mathbb{E}^n_{\#}(M) \ \sim\ n\log(n).$$

We still atribut this conjecture to Devroye as this is morally the exact same conjecture. Indeed, the expected value the size the n^{th} reduction can be given looking at the expected value of the height of the n^{th} reduction:

$$\mathbb{E}^n_{\#}(M) \ \sim \sum_{i\leq n} p.\mathbb{E}^i_H(M).$$

And the Devroye's conjecture can be refrase as:

$$\mathbb{E}^n_H(M) \ \sim\ \log(n).$$

Now one can see that the expected height will just be reduced by the considered degree p and that the p-version of Devroye's conjecture is not realy stronger than the original version.

4 λ-calculus

5 Devroye's conjecture

We consider a TRS $\mathbb{C}\mathtt{opy}$ with two symbols $\square : 0$ and $\boldsymbol{copy} : 2$ and with a unic rule:

$$\boldsymbol{copy}(x, y) \to \boldsymbol{copy}(\boldsymbol{copy}(x, y), \boldsymbol{copy}(x, y)).$$

The randomized strategy applied to this TRS corresponds to a variant of Remy's algorithm.Combinatoricians have studied this variation, which study leads to the following conjecture: Ref

Conjecture 3 (Devroye)**.** *The serie given by the expected value of the size the n^{th} reduction of a term $M \in \mathbb{C}\mathtt{opy}$ is quasilinear:*

$$\mathbb{E}^n_{\#}(M) \;\sim\; n\log(n).$$

Given a tree t, we denote:

- a node $x \in t$ is an occurence of $\boldsymbol{copy}$ in t,
- the depth of a node $x \in t$ is 0 if x is the root and $n+1$ is x's father hase depth n,
- the size $\#t$ of t is the number of nodes in t, which is also the number of t' such that $t \to t'$,
- the strata $\#_n t$ of t is the number of nodes of depth t, so that

$$\#t \quad = \quad \sum_{n\geq 0} \#_n t \tag{1}$$

- the mean depth $d(t)$ is given by

$$d(t) \quad := \quad \frac{\sum_{n\geq 0} n.\#_n t}{\#t}\,, \tag{2}$$

- the hight $H(t)$ is the maximal depth:

$$H(t) \quad := \quad \max\{n \mid \#_n t \neq 0\}\,, \tag{3}$$

- the expected value of $u(t)$, for $u \in \{\#, \#_n, d, H\}$, after one reduction is denoted:

$$\mathbb{E}_u(t) \quad := \quad \frac{\sum_{t'\leftarrow t} u(t)}{\#t} \tag{4}$$

- the expected growth of $u(t)$, for $u \in \{\#, \#_n, d, H\}$, after one reduction is denoted:

$$\mathbb{\Delta}_u(t) \quad := \quad \mathbb{E}_u(t) - u(t)\,, \tag{5}$$

- we also denote $\mathbb{E}^k_u(t)$ and $\mathbb{\Delta}^k_u(t)$ the expexted and Growth values after k steps.

Lemma 5. *For all $n \geq 1$:*

$$\sum_{t\to t'} \#_n t' \quad = \quad \#t.\#_n t + (n-1)\Big(2\#_{n-1}t - \#_n t\Big) \tag{6}$$

so that:

$$\mathbb{\Delta}_{\#_n}(t) \quad = \quad \frac{2n.\#_{n-1}t - n\#_n t}{\#t} \tag{7}$$

$$\mathbb{\Delta}_{\#}(t) \quad = \quad 2 + d(t) \tag{8}$$