ON THE CHARACTERIZATION OF MODELS OF $\mathcal{H}^*$: THE OPERATIONAL ASPECT

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Abstract. We give a characterization, with respect to a large class of models of untyped λ-calculus, of those models that are fully abstract for head-normalization, *i.e.*, whose equational theory is $\mathcal{H}^*$. An extensional K-model D is fully abstract if and only if it is hyperimmune, *i.e.*, non-well founded chains of elements of D cannot be captured by any recursive function.

This article, together with its companion paper [**?**] form the long version of [**?**]. It is a standalone paper that present a purely syntactical proof of the result as opposed to its companion paper that present an independent and purely semantical proof of the exact same result.

Introduction

The histories of full abstraction and denotational semantics of λ-calculi are both rooted in four fundamental articles published in the course of a year.

In 1976, Hyland [**?**] and Wadsworth [**?**] independently[1] proved the first full abstraction result of Scott's D_∞ for $\mathcal{H}^*$. The following year, Milner [**?**] and Plotkin [**?**] showed respectively that PCF (a Turing-complete extension of the simply typed λ-calculus) has a unique fully abstract model up to isomorphism and that this model is not in the category of Scott domains and continuous functions.

Later, various articles focused on circumventing Plotkin counter-example [**?**, **?**] or investigating full abstraction results for other calculi [**?**, **?**, **?**]. However, hardly anyone pointed out the fact that Milner's uniqueness theorem is specific to PCF, while $\mathcal{H}^*$ has various models that are fully abstract but not isomorphic.

The quest for a general characterization of the fully abstract models of head normalization started by successive refinements of a sufficient, but unnecessary condition [**?**, **?**, **?**], improving the proof techniques from 1976 [**?**, **?**]. x While these results shed some light on various fully abstract semantics for $\mathcal{H}^*$, none of them could reach a full characterization.

In this article, we give the first full characterization of the full abstraction of an observational semantics for a specific (but large) class of models. The class we choose is that of Krivine-models, or K-models [**?**, **?**]. This class, described in Section **??**, is essentially the subclass of Scott complete lattices (or filter models [**?**]) which are prime algebraic. We add two further conditions: extensionality and test-sensibility. Extensionality is a standard and perfectly understood notion that require the model to respect the η-equivalence, notice that it is a necessary condition for the full abstraction

[1] Notice, however, that the idea already appears in Wadsworth thesis 3 years earlier.

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if $\mathcal{H}^*$. On the other hand, test-sensibility is a new notion that we are introducing but which is equivalent to the more commune notion of approximability (by B/"om trees). Test-sensibility basically states that the model is sensible for an extension of the λ-calculus called tests.

The extensional and test-sensible K-models are the objects of our characterization and can be seen as a natural class of models obtained from models of linear logic [**?**]. Indeed, the extensional K-models correspond to the extensional reflexive objects of the co-Kleisli category associated with the exponential comonad of Ehrhard's ScottL category [**?**] (Prop. **??**).

We achieve the characterization of full abstraction for $\mathcal{H}^*$ in Theorem **??**: a model D is fully abstract for $\mathcal{H}^*$ iff D is *hyperimmune* (Def. **??**). Hyperimmunity is the key property our study introduces in denotational semantics. This property is reminiscent of the Post's notion of hyperimmune sets in recursion theory. Hyperimmunity in recursion theory is not only undecidable, but also surprisingly high in the hierarchy of undecidable properties (it cannot be decided by a machine with an oracle deciding the halting problem) [**?**].

Roughly speaking, a model D is hyperimmune whenever the λ-terms can have access to only well-founded chains of elements of D. In other words, D might have non-well-founded chains $d_0 \geq d_1 \geq \cdots$, but these chains "grow" so fast (for a suitable notion of growth), that they cannot be contained in the interpretation of any λ-term.

The intuition that full abstraction of $\mathcal{H}^*$ is related with a kind of well-foundation can be found in the literature (*e.g.*, Hyland's [**?**], Gouy's [**?**] or Manzonetto's [**?**]). Our contribution is to give, with hyperimmunity, a precise definition of this intuition, at least in the setting of K-models.

A finer intuition can be described in terms of game semantics. Informally, a game semantic for the untyped λ-calculus takes place in the arena interpreting the recursive type $o = o \to o$. This arena is infinitely wide (by developing the left o) and infinitely deep (by developing the right o). Moves therein can thus be characterized by their nature (question or answer) and by a word over natural numbers. For example, $q(2.3.1)$ represents a question in the underlined "o" in $o = o{\to}(o{\to}o{\to}(\underline{o}{\to}o){\to}o){\to}o$. Plays in this game are potentially infinite sequences of moves, where a question of the form $q(w)$ is followed by any number of deeper questions/answers, before an answer $a(w)$ is eventually provided, if any.

A play like $q(\epsilon), q(1)...a(1), q(2)...a(2), q(3)...$ is admissible: one player keeps asking questions and is infinitely delaying the answer to the initial question, but some answers are given so that the stream is productive. However, the full abstraction for $\mathcal{H}^*$ forbids non-productive infinite questioning like in $q(\epsilon), q(1), q(1.1), q(1.1.1)...$, in general. Nevertheless, disallowing *all* such strategies is sufficient, but not necessary to get full abstraction. The hyperimmunity condition is finer: non productive infinite questioning is allowed *as long as* the function that chooses the next question grows faster than any recursive function (notice that in the example above that choice is performed by the constant $(n \mapsto 1)$ function). For example, if $(u_i)_{i\geq 0}$ grows faster than any recursive function, the play $q(\epsilon), q(u_1), q(u_1.u_2), q(u_1.u_2.u_3)...$ is perfectly allowed.

Incidentally, we obtain a significant corollary (also expressed in Theorem **??**) stating that full abstraction coincides with inequational full abstraction for $\mathcal{H}^*$ (equivalence between observational and denotational orders). This is in contrast to what happens to other calculi [**?**, **?**].

In the literature, most of the proofs of full abstraction for $\mathcal{H}^*$ are based on Nakajima trees [**?**] or some other notion of quotient of the space of Böhm trees. The usual approach is too coarse because it considers arbitrary Böhm trees which are not necessarily images of actual λ-terms. To overcome this we propose two different techniques leading to two different proofs of the main result: one purely semantical and the other purely syntactical. In this article we only present the later, the former being the object of a companion paper [**?**].

The semantic proof approaches the problem from a novel angle that consists in the use of a new tool: the *calculi with tests* (Def. **??**). These are syntactic extensions of the λ-calculus with operators defining compact elements of the given models. Since the model appears in the syntax, we are able to perform inductions (and co-inductions) directly on the reduction steps of actual terms, rather than on the construction of Böhm trees.

The idea of test mechanisms as syntactic extensions of the λ-calculus was first used by Bucciarelli *et al.* [**?**]. Even though it was mixed with a resource-sensitive extension, the idea was already used to define morphisms of the model. Nonetheless, we can notice that older notions like Wadsworth's labeled $\lambda\bot$-calculus [**?**] seem related to calculi with tests. The calculi with tests are not *ad hoc* tricks, but powerful and general tools.

One of the purposes of this article is to demonstrate the interest of tests in the study of the relations between denotational and operational semantics. Calculi with tests are sort of a dual of Böhm trees. While the latter constitutes a syntactical model for the λ-calculus; a calculus with tests is a the semantical language for some K-model. While Böhm trees are built upon the λ-calculus and reduce the problem of full abstraction to the semantical level; a calculus with tests is built upon the model and reduces this problem to the syntactical level. We claim that, regarding relations between denotational and operational semantics, Böhm trees and λ-calculi with tests are equally powerful tools, but extend differently to other frameworks.

1. Preliminaries and result

1.1. **Preliminaries.**

1.1.1. *Preorders.*
Given two partially ordered sets $D = (|D|, \leq_D)$ and $E = (|E|, \leq_E)$, we denote:

- $D^{op} = (|D|, \geq_D)$ the reverse-ordered set.
- $D \times E = (|D| \times |E|, \leq_{D\times E})$ the Cartesian product endowed with the pointwise order:
$$(\delta, \epsilon) \leq_{D\times E} (\delta', \epsilon') \quad \text{if} \quad \delta \leq_D \delta' \quad \text{and} \quad \epsilon \leq_E \epsilon'.$$
- $\mathcal{A}_f(D) = (|\mathcal{A}_f(D)|, \leq_{\mathcal{A}_f(D)})$ the set of finite antichains of D (*i.e.*, finite subsets whose elements are pairwise incomparable) endowed with the order :
$$a \leq_{\mathcal{A}_f(D)} b \;\Leftrightarrow\; \forall \alpha \in a, \exists \beta \in b, \alpha \leq_D \beta$$

In the following will we use D for $|D|$ when there is no ambiguity. Initial Greek letters α, β, γ... will vary on elements of ordered sets. Capital initial Latin letters A, B, C... will vary over subsets of ordered sets. And finally, initial Latin letters a, b, c... will denote finite antichains.

An *order isomorphism* between D and E is a bijection $\phi : |D| \to |E|$ such that ϕ and ϕ^{-1} are monotone.

Given a subset $A \subseteq |D|$, we denote $\downarrow A = \{\alpha \mid \exists \beta \in A, \alpha \leq \beta\}$. We denote by $\mathcal{I}(D)$ the set of *initial segments of* D, that is $\mathcal{I}(D) = \{\downarrow A \mid A \subseteq |D|\}$. The set $\mathcal{I}(D)$ is a prime algebraic complete lattice with respect to the set-theoretical inclusion. The *sups* are given by the unions and the *prime elements* are the downward closure of the singletons. The *compact elements* are the downward closure of finite antichains.

The domain of a partial function f is denoted by $Dom(f)$. The *graph* of a Scott-continuous function $f : \mathcal{I}(D) \to \mathcal{I}(E)$ is
$$\text{graph}(f) = \{(a, \alpha) \in \mathcal{A}_f(D)^{op} \times E \mid \alpha \in f(\downarrow a)\} \tag{1.1}$$

Notice that elements of $\mathcal{I}(\mathcal{A}_f(D)^{op}\times E)$ are in one-to-one correspondence with the graphs of Scott-continuous functions from $\mathcal{I}(D)$ to $\mathcal{I}(E)$.

1.1.2. *λ-calculus.*
The λ-terms are defined up to α-equivalence by the following grammar using notation "*à la* Barendregt" [**?**] (where variables are denoted by final Latin letters x, y, z...):

$$(\lambda\text{-terms})\quad \Lambda \quad M,N \quad ::= \quad x \quad | \quad \lambda x.M \quad | \quad M\,N$$

We denote by $\mathrm{FV}(M)$ the set of free variables of a λ-term M. Moreover, we abbreviate a nested abstraction $\lambda x_1...x_k.M$ into $\lambda\vec{x}^k M$, or, when k is irrelevant, into $\lambda\vec{x}M$. We denote by $M[N/x]$ the capture-free substitution of x by N.
The λ-terms are subject to the β-reduction:

$$(\beta) \qquad (\lambda x.M)\,N \overset{\beta}{\to} M[N/x]$$

A context C is a λ-term with possibly some occurrences of a hole, *i.e.*:

$$(\text{contexts})\quad \Lambda^{(\!|.|\!)} \quad C ::= \quad (\!|.|\!) \quad | \quad x \quad | \quad \lambda x.C \quad | \quad C_1\,C_2$$

The writing $C(\!|M|\!)$ denotes the term obtained by filling the holes of C by M. The small step reduction $\to$ is the closure of (β) by any context, and $\to_h$ is the closure of (β) by the rules:

$$\frac{M \to_h M'}{\lambda x.M \to_h \lambda x.M'} \qquad \frac{M \to_h M' \qquad M \text{ is an application}}{M\,N \to_h M'\,N}$$

The transitive reduction $\to^*$ (resp $\to_h^*$) is the reflexive transitive closure of $\to$ (resp $\to_h$).
The big step head reduction, denoted $M{\Downarrow^{\mathrm{h}}}N$, is $M \to_h^* N$ for N in a *head-normal form*, *i.e.*, $N = \lambda x_1...x_k.y\; M_1 \cdots M_k$, for $M_1, ..., M_k$ any terms. We write $M{\Downarrow^{\mathrm{h}}}$ for the (*head*) *convergence*, *i.e.*, whenever there is N such that $M{\Downarrow^{\mathrm{h}}}N$.

Example 1.1.

- The *identity term* $\boldsymbol{I} := \lambda x.x$ is taking a term and return it as it is:
$$\boldsymbol{I}\,M \quad \to \quad M.$$
- The n^{th} Church numeral, denoted by $\underline{n}$, and the successor function, denoted by $\boldsymbol{S}$, are defined by
$$\underline{n} := \lambda f x.\underbrace{f\,(f \cdots f\,(f}_{n \text{ times}}\; x) \cdots), \qquad\qquad \boldsymbol{S} := \lambda u f x.u\; f\;(f\; x).$$
Together they provide a suitable encoding for natural numbers, with $\underline{n}$ representing the n^{th} iteration.
- The *looping term* $\boldsymbol{\Omega} := (\lambda x.xx)\,(\lambda x.xx)$ infinitely reduces into itself, notice that $\boldsymbol{\Omega}$ is an example of a diverging term:
$$\boldsymbol{\Omega} \quad \to \quad (x\,x)[\lambda y.y\,y/x] \quad = \quad \boldsymbol{\Omega} \quad \to \quad \boldsymbol{\Omega} \quad \to \quad \cdots.$$
- The *Turing fixpoint combinator* $\boldsymbol{\Theta} := (\lambda uv.v\,(u\,u\,v))\,(\lambda uv.v\,(u\,u\,v))$ is a term that computes the least fixpoint of its argument (if it exists):
$$\begin{aligned}\boldsymbol{\Theta}\,M &\to (\lambda v.v\,((\lambda uv.v\,(u\,u\,v))\,(\lambda uv.v\,(u\,u\,v))v))\,M\\ &= (\lambda v.v\,(\boldsymbol{\Theta}\,v))\,M\\ &\to M\,(\boldsymbol{\Theta}\,M).\end{aligned}$$

Other notions of convergence exsit (strong, lazy, call by value...), but our study focuses on head convergence, inducing the equational theory denoted by $\mathcal{H}^*$.

Definition 1.2. The *observational preorder* and *equivalence* denoted $\sqsubseteq_{\mathcal{H}^*}$ and $\equiv_{\mathcal{H}^*}$ are given by:

$$M \sqsubseteq_{\mathcal{H}^*} N \quad \text{if} \quad \forall C,\ C(\!|M|\!)\Downarrow^{\mathrm{h}} \ \Rightarrow\ C(\!|N|\!)\Downarrow^{\mathrm{h}},$$
$$M \equiv_{\mathcal{H}^*} N \quad \text{if} \quad M \sqsubseteq_{\mathcal{H}^*} N \text{ and } N \sqsubseteq_{\mathcal{H}^*} M.$$

The resulting (in)equational theory is called $\mathcal{H}^*$.

Henceforth, convergence of a λ-term means head convergence, and full abstraction for λ-calculus means full abstraction for $\mathcal{H}^*$.

Definition 1.3. A model of the untyped λ-calculus with an interpretation $[\![-]\!]$ is:

- fully abstract (for $\mathcal{H}^*$) if for all $M, N \in \Lambda$:
$$M \equiv_{\mathcal{H}^*} N \quad \text{if} \quad [\![M]\!] = [\![N]\!],$$
- inequationally fully abstract (for $\mathcal{H}^*$) if for all $M, N \in \Lambda$:[2]
$$M \sqsubseteq_{\mathcal{H}^*} N \quad \text{if} \quad [\![M]\!] \subseteq [\![N]\!].$$

Concerning recursive properties of λ-calculus, we will use the following one:

Proposition 1.4 ([**?**, Proposition 8.2.2] [3])**.**
Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of terms such that:

- $\forall n \in \mathbb{N}, M_n \in \Lambda^0$*,*
- $(n \mapsto M_n)$ *is recursive,*

then there exists F such that:

$$\forall n, F\ \underline{n}\ \rightarrow^* M_n.$$

1.2. **K-models.**

We introduce here the main semantical object of this article: extensional K-models [**?**][**?**]. This class of models of the untyped λ-calculus is a subclass of filter models [**?**] containing many extensional models from the continuous semantics, like Scott's D_∞ [**?**].

1.2.1. *The category* ScottL$_!$.

Extensional K-models correspond to the extensional reflexive Scott domains that are prime algebraic complete lattices and whose application embeds prime elements into prime elements [**?**, **?**]. However we prefer to exhibit K-models as the extensional reflexive objects of the category ScottL$_!$ which is itself the Kleisli category over the linear category ScottL [**?**].

Definition 1.5. We define the Cartesian closed category ScottL$_!$ [**?**, **?**, **?**]:

- *objects* are partially ordered sets.
- *morphism* from D to E are a Scott-continuous function between the complete lattices $\mathcal{I}(D)$ and $\mathcal{I}(E)$.

[2] It can be generalised by replacing $\subseteq$ by any order on the model.

[3] This is not the original statement. We remove the dependence on $\vec{x}$ that is empty in our case and we replace the β-equivalence by a reduction since the proof of Barendregt [**?**] works as well with this refinement.

The *Cartesian product* is the disjoint sum of posets. The *terminal object* $\top$ is the empty poset. The *exponential object* $D{\Rightarrow}E$ is $\mathcal{A}_f(D)^{op}{\times}E$. Notice that an element of $\mathcal{I}(D{\Rightarrow}E)$ is the graph of a morphism from D to E (see Equation (**??**)). This construction provides a natural isomorphism between $\mathcal{I}(D{\Rightarrow}E)$ and the corresponding homset. Notice that if $\simeq$ denotes the isomorphism in ScottL$_!$, then:

$$D \Rightarrow D \Rightarrow \cdots \Rightarrow D \simeq (\mathcal{A}_f(D)^{op})^n \times D. \tag{1.2}$$

For example $D \Rightarrow (D \Rightarrow D) \simeq \mathcal{A}_f(D)^{op} \times (\mathcal{A}_f(D)^{op} \times D) = (\mathcal{A}_f(D)^{op})^2 \times D$.

Remark 1.6. In the literature (*e.g.* [**?**, **?**, **?**]), objects are preodered sets and the exponential object $D \Rightarrow D$ is defined by using finite subsets (or multisets) instead of the finite antichains. Our presentation is the quotient of the usual one by the equivalence relation induced by the preorder. The two presentations are equivalent (in terms of equivalence of category) but our choice simplifies the definition of hyperimmunity (Definition **??**).

Proposition 1.7. *The category* ScottL$_!$ *is isomorphic to the category prime algebraic complete lattice and Scott-continuous maps.*

Proof. Given a poset D, the initial segments $\mathcal{I}(D)$ form a prime algebraic complete lattice with $\{\downarrow \alpha \mid \alpha \in D\}$ as prime elements since $I = \bigcup_{\alpha\in I} \downarrow \alpha$. Conversely, the prime elements of a prime algebraic complete lattice form a poset. The two operations are inverse one to the other modulo ScottL$_!$-isomorphisms or, equivalently, Scott-continuous isomorphisms. □

1.2.2. *An algebraic presentation of K-models.*

Definition 1.8 ([**?**])**.** An *extensional K-model* is a pair (D, i_D) where:

- D is a poset.
- i_D is an order isomorphism between $D{\Rightarrow}D$ and D.

By abuse of notation we may denote the pair (D, i_D) simply by D when it is clear from the context we are referring to an extensional K-model.

Proposition 1.9. *Extensional K-models correspond exactly to extensional reflexive objects of* ScottL$_!$*,* i.e.*, an object D endowed with an isomorphism $abs_D : (D \Rightarrow D) \to D$ (and $app_D := abs_D^{-1}$).*

Proof. Given a K-model (D, i_D), the isomorphism between $D{\Rightarrow}D$ and D is given by:

$$\forall A \in \mathcal{I}(D{\Rightarrow}D), \qquad \mathrm{app}_D(A) = \{i_D(a,\alpha) \mid (a,\alpha) \in A\},$$
$$\forall B \in \mathcal{I}(D), \qquad \mathrm{abs}_D(B) = \{(a,\alpha) \mid i_D(a,\alpha) \in B\}.$$

Conversely, consider an extensional reflexive object (D, app_D, abs_D) of ScottL$_!$. Since abs_D is an isomorphism, it is linear (that is, it preserves all sups). For all $(a,\alpha) \in D{\Rightarrow}D$, we have

$$\downarrow(a,\alpha) = abs(app(\downarrow(a,\alpha))) = \bigcup_{\beta\in app(\downarrow(a,\alpha))} abs(\downarrow\beta).$$

Thus there is $\beta \in app(\downarrow(a,\alpha)$ such that $(a,\alpha) \in abs(\downarrow\beta)$, and since $abs(\downarrow\beta) \subseteq \downarrow(a,\alpha)$, this is an equality. Thus there is a unique β such that $app_D(a,\alpha) = \downarrow\beta$, this is $i_D(a,\alpha)$. □

In the following we will not distinguish between a K-model and its associated reflexive object, this is a model of the pure λ-calculus.

Definition 1.10. An *extensional partial K-model* is a pair (E, j_E) where E is an object of $\text{ScottL}_!$ and j_E is a partial function from $E \Rightarrow E$ to E that is an order isomorphism between $\text{Dom}(j_E)$ and E.

$$E \quad \overset{j_E}{\longleftrightarrow} \quad \text{Dom}(j_E) \quad \subseteq \quad (E \Rightarrow E)$$

Definition 1.11. The *completion of a partial K-model* (E, j_E) is the union

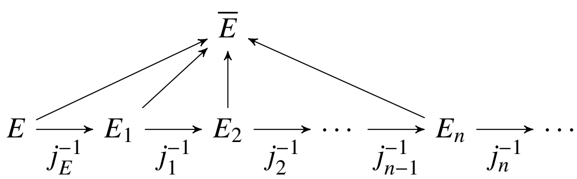
$$(\bar{E}, j_{\bar{E}}) = (\bigcup_{n\in\mathbb{N}} E_n, \bigcup_{n\in\mathbb{N}} j_{E_n})$$

of partial completions (E_n, j_{E_n}) that are extensional partial K-models defined by induction on n. $(E_0, j_{E_0}) = (E, j_E)$ and:

- $|E_{n+1}| = |E_n| \cup (|E_n \Rightarrow E_n| - Dom(j_{E_n}))$
- $j_{E_{n+1}}$ is defined only over $|E_n \Rightarrow E_n| \subseteq |E_{n+1} \Rightarrow E_{n+1}|$ by $j_{E_{n+1}} = j_{E_n} \cup id_{|E_n \Rightarrow E_n| - Dom(j_{E_n})}$
- $\leq_{E_{n+1}}$ is given by $j_{E_{n+1}}(a,\alpha) \leq_{E_{n+1}} (b,\beta)$ if $a \geq_{\mathcal{A}_f(E_n)} b$ and $\alpha \leq_{E_n} \beta$.

Remark that E_{n+1} corresponds to $E_n \Rightarrow E_n$ up to isomorphism, what leads to the equivalent definition:

Proposition 1.12. *The completion $(\bar{E}, j_{\bar{E}})$ of an extensional partial K-model (E, j_E) can be described as the categorical ω-colimit (in* ScottL*) of $(E'_n)_n$ along the injections $(j_n^{-1})_n$ where $(E'_0, j_0) = (E, j_E)$, $E'_{n+1} = E'_n \Rightarrow E'_n$ and j_{n+1}^{-1} is defined by $j_{n+1}^{-1}(a,\alpha) = (j_n(a), j_n(\alpha))$ if defined.*



Remark 1.13. The completion of an extensional partial K-model (E, j_E) is the smallest extensional K-model $\bar{E}$ containing E. In particular, any extensional K-model D is the extensional completion of itself: $D = \bar{D}$.

Example 1.14.

(1) *Scott's D_∞* [**?**] is the extensional completion of

$$|D| := \{*\}, \qquad \leq_D := id, \qquad j_D := \{(\emptyset, *) \mapsto *\}.$$

The completion is a triple $(|D_\infty|, \leq_{D_\infty}, j_{D_\infty})$ where $|D_\infty|$ is generated by:

$$\begin{array}{llcl} |D_\infty| & \alpha,\beta & ::= & * \mid a \to \alpha \\ |!D_\infty| & a,b & \in & \mathcal{A}_f(|D_\infty|) \end{array}$$

except that $\emptyset \to * \notin |D_\infty|$; j_{D_∞} is defined by $j_{D_\infty}(\emptyset, *) = *$ and $j_{D_\infty}(a,\alpha) = a \to \alpha$ for $(a,\alpha) \neq (\emptyset, *)$.

(2) *Park's P_∞* [**?**] is the extensional completion of

$$|P| := \{*\}, \qquad \leq_P := id, \qquad j_P := \{(\{*\}, *) \mapsto *\};$$

i.e., $|P_\infty|$ is defined by the previous grammar except that $(\{*\} \to *) \notin |P_\infty|$ while $\emptyset \to * \in |P_\infty|$.

(3) *Norm* or D_∞^* [**?**] is the extensional completion of

$$|E| := \{p, q\}, \qquad \leq_E := id \cup \{p < q\},$$
$$j_E := \{(\{p\}, q) \mapsto q, (\{q\}, p) \mapsto p\}.$$

$$[\![x_i]\!]_D^{\vec{x}} = \{(\vec{a},\alpha) \mid \alpha \leq \beta \in a_i\} \qquad [\![\lambda y.M]\!]_D^{\vec{x}} = \{(\vec{a}, b{\to}\alpha) \mid (\vec{a}b,\alpha) \in [\![M]\!]_D^{\vec{x}y}\}$$
$$[\![M\ N]\!]_D^{\vec{x}} = \{(\vec{a},\alpha) \mid \exists b, (\vec{a}, b{\to}\alpha) \in [\![M]\!]_D^{\vec{x}} \ \wedge \forall \beta{\in} b, (\vec{a},\beta) \in [\![N]\!]_D^{\vec{x}}\}$$

FIGURE 1. Direct interpretation of Λ in D

(4) *Well-stratified K-models* [?] are the extensional completions of some E respecting
$$\forall (a,\alpha){\in}\mathrm{Dom}(j_E), a{=}\emptyset.$$
(5) The *inductive* $\overline{\omega}$ is the extensional completion of
$$|E| := \mathbb{N}, \qquad \leq_E := id, \qquad j_E := \{(\{k \mid k < n\}, n){\mapsto} n \mid n \in \mathbb{N}\}.$$
(6) The *co-inductive* $\overline{\overline{\mathbb{Z}}}$ is the extensional completion of
$$|E| := \mathbb{Z}, \qquad \leq_E := id \qquad , j_E := \{(\{n\}, n+1){\mapsto} n{+}1 \mid n \in \mathbb{Z}\}.$$
(7) *Functionals* H^f (given $f : \mathbb{N} \to \mathbb{N}$) are the extensional completions of:
$$|E| := \{*\} \cup \{\alpha^n_j \mid n \geq 0,\ 1 \leq j \leq f(n)\}, \qquad \leq_E := id,$$
$$j_E := \Big\{(\emptyset, *) \mapsto *\Big\} \ \cup\ \Big\{(\emptyset, \alpha^n_{j+1}) \mapsto \alpha^n_j \mid 1 \leq j < f(n)\Big\} \ \cup\ \Big\{(\{\alpha^{n+1}_1\}, *) \mapsto \alpha^n_{f(n)} \mid n \in \mathbb{N}^*\Big\},$$
where $(\alpha^n_j)_{n,j}$ is a family of atoms different from $*$.

For the sake of simplicity, from now on we will work with a fixed extensional K-model D. Moreover, we will use the notation $a{\to}\alpha := i_D(a,\alpha)$. Notice that, due to the injectivity of i_D, any $\alpha \in D$ can be uniquely rewritten into $a{\to}\alpha'$, and more generally into $a_1{\to}\cdots{\to}a_n{\to}\alpha_n$ for any n.

Remark 1.15. Using this notations, the model H^f can be summarized by writing, for each n:
$$\alpha^n_1 \ = \ \underbrace{\emptyset{\to}\cdots{\to}\emptyset}_{f(n)}{\to}\{\alpha^{n+1}_1\}{\to}*$$

1.2.3. *Interpretation of the λ-calculus.*

The Cartesian closed structure of ScottL$_!$ endowed with the isomorphisms app_D and abs_D of the reflexive object induced by D (see Proposition **??**) defines a standard model of the λ-calculus.

A term M with at most n free variables $x_1, \ldots, x_n$ is interpreted as the graph of a morphism $[\![M]\!]_D^{x_1\ldots x_n}$ from D^n to D (when n is obvious, we can use $[\![.]\!]^{\bar{x}}$). By Equations (**??**) and (**??**) we have:
$$[\![M]\!]_D^{x_1\ldots x_n} \subseteq\ (D \Rightarrow \cdots \Rightarrow D \Rightarrow D)\ \simeq\ (\mathcal{A}_f(D)^{op})^n \times D.$$
In Figure **??**, we explicit the interpretation $[\![M]\!]_D^{x_1\ldots x_n}$ by structural induction on M.

Example 1.16.
$$[\![\lambda x.y]\!]_D^y = \{((a), b{\to}\alpha) \mid \alpha \leq_D \beta \in a\},$$
$$[\![\lambda x.x]\!]_D^y = \{((a), b{\to}\alpha) \mid \alpha \leq_D \beta \in b\},$$
$$[\![\boldsymbol{I}]\!]_D = \{a{\to}\alpha \mid \alpha \leq_D \beta \in a\},$$
$$[\![\underline{1}]\!]_D = \{a{\to}b{\to}\alpha \mid \exists c, c{\to}\alpha \leq_D \beta \in a,\ c \leq_{\mathcal{A}_f(D)} b\}.$$
In the last two cases, terms are interpreted in an empty environment. We, then, omit the empty sequence associated with the empty environment, *e.g.*, $a{\to}b{\to}\alpha$ stands for $((), a{\to}b{\to}\alpha)$.

$$\frac{\alpha \in a}{x : a \vdash x : \alpha} \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma, x : a \vdash M : \alpha} \qquad \frac{\Gamma \vdash M : \beta \qquad \alpha \leq \beta}{\Gamma \vdash M : \alpha}$$

$$\frac{\Gamma, x : a \vdash M : \alpha}{\Gamma \vdash \lambda x.M : a{\to}\alpha} \qquad \frac{\Gamma \vdash M : a{\to}\alpha \qquad \forall \beta \in a,\ \Gamma \vdash N : \beta}{\Gamma \vdash M\,N : \alpha}$$

FIGURE 2. Intersection type system computing the interpretation in D

We can verify that extensionality holds, indeed $[\![\underline{1}]\!]_D = [\![\boldsymbol{I}]\!]_D$, since $c{\to}\alpha \leq_D \beta \in a$ and $c \leq_{\mathcal{A}_f(D)} b$ exactly say that $b{\to}\alpha \leq_D \beta \in a$, and since any element of $\gamma \in D$ is equal to $d{\to}\delta$ for a suitable d and δ.

1.2.4. *Intersection types.*

It is folklore that the interpretation of the λ-calculus into a given K-model D is characterized by a specific *intersection type system*. In fact any element $\alpha \in D$ can be seen as an intersection type

$$\alpha_1 \wedge \cdots \wedge \alpha_n \to \beta \qquad \text{given by} \quad \alpha = \{\alpha_1, \ldots, \alpha_n\}{\to}\beta.$$

In Figure **??**, we give the intersection-type assignment corresponding to the K-model induced by D.

Proposition 1.17. *Let M be a term of Λ, the following statements are equivalent:*

- $(\vec{d}, \alpha) \in [\![M]\!]_D^{\vec{x}}$,
- *the type judgment $\vec{x} : \vec{d} \vdash M : \alpha$ is derivable by the rules of Figure* **??**.

Proof. By structural induction on the grammar of Λ. □

1.3. **The result.**

We state our main result, showing an equivalence between hyperimmunity (Def. **??**) and full abstraction for $\mathcal{H}^*$.

Definition 1.18 (*Hyperimmunity*)**.** A (possibly partial) extensional K-model D is said to be *hyperimmune* if for every sequence $(\alpha_n)_{n\geq 0} \in D^{\mathbb{N}}$, there is no recursive function $g : \mathbb{N}{\to}\mathbb{N}$ satisfying, the following condition for all $n{\geq}0$:

$$\alpha_n = a_{n,1}{\to}\cdots{\to}a_{n,g(n)}{\to}\alpha'_n \qquad \text{and} \qquad \alpha_{n+1} \in \bigcup_{k \leq g(n)} a_{n,k}. \tag{1.3}$$

Notice, in the above definition, that each antichain $a_{n,i}$ always exist and are uniquely determined by the isomorphism between D and $D \Rightarrow D$ that allow us to unfold any element α_i as an arrow (of any length).

The idea is the following. The sequence $(\alpha_n)_{n\geq 0}$ is morally describing a non well-founded chain of elements of D, through the isomorphism $D \simeq D \Rightarrow D$, allowing us to see any element α_i as an arrow (of any length):

$$\begin{array}{l} \alpha_0 = a_{0,1}\to\cdots a_{0,i_0}\cdots\to a_{0,g(0)}\to\alpha'_0 \\ \qquad\qquad\qquad \Cup \\ \qquad\qquad\qquad \alpha_1 = a_{1,1}\to\cdots\quad a_{1,i_1}\cdots\to a_{1,g(1)}\to\alpha'_1 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \Cup \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \alpha_2 = a_{2,1}\to\cdots a_{2,i_2}\cdots\to a_{2,g(2)}\to\alpha'_2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots \end{array}$$

The growth rate $(i_n)_n$ of the chain $(\alpha_n)_n$ depends on how many arrows must be displayed in α_i in order to see α_{i+1} as an element of the antecedent of one of them. Now, hyperimmunity means that if any such non-well founded chain $(\alpha_n)_n$ exists, then its growth rate $(i_n)_n$ cannot be bounded by any recursive function g.

Remark 1.19. It would not be sufficient to simply consider the function $n \mapsto i_n$ such that $\alpha_{n+1}{\in}a_{n,i_n}$ rather than the bounding function g. Indeed, $n \mapsto i_n$ may not be recursive even while g is.

Proposition 1.20. *For any extensional partial K-model E (Def.* **??***), the completion $\overline{E}$ (Def.* **??***) is hyperimmune iff E is hyperimmune.*

Proof. The left-to-right implication is trivial.
The right-to-left one is obtained by contradiction:
Assume to have a $(\alpha_n)_{n\geq 0} \in \bar{E}^{\mathbb{N}}$ and a recursive function $g : \mathbb{N} \to \mathbb{N}$ such that for all $n \geq 0$:

$$\alpha_n = a_{n,1}\to\cdots\to a_{n,g(n)}\to\alpha'_n \qquad\text{and}\qquad \alpha_{n+1} \in \bigcup_{i\leq g(n)} a_{n,i}$$

Recall that the sequence $(E_k)_{k\geq 0}$ of Definition **??** approximates the completion $\bar{E}$.
Then we have the following:

- There exists k such that $\alpha_0 \in E_k$, because $\alpha_0 \in \bar{E} = \bigcup_k E_k$.
- If $\alpha_n \in E_{j+1}$, then $\alpha_{n+1} \in E_j$, because there is $i \leq g(n)$ such that $\alpha_{n+1} \in a_{n,i} \subseteq E_j$.
- If $\alpha_n \in E_0 = E$, then $\alpha_{n+1} \in E$ by surjectivity of j_E.

Thus there is k such that $(\alpha_n)_{n\geq k} \in E^{\mathbb{N}}$, which would break hyperimmunity of E. □

Example 1.21.
- The well-stratified K-models of Example **??**(**??**) (and in particular D_∞ of Item (**??**)) are trivially hyperimmune: already in the partial K-model, there are not even α_1, α_2 and n such that $\alpha_1 = a_1\to\cdots\to a_n\to\alpha'_1$ and $\alpha_2 \in a_n$ (since $a_n = \emptyset$). The non-hyperimmunity of the partial K-model can be extended to the completion using Proposition **??**.
- The model $\overline{\omega}$ (Ex. **??**(**??**)) is hyperimmune. Indeed, any such $(\alpha_n)_n$ in the partial K-model would respect $\alpha_{n+1}{<_{\mathbb{N}}}\alpha_n$,hence $(\alpha_n)_n$ must be finite by well-foundation of $\mathbb{N}$.
- The models P_∞, D^*_∞ and $\overline{\mathbb{Z}}$ (Examples **??**(**??**), (**??**) and (**??**)) are not hyperimmune. Indeed for all of them $g = (n \mapsto 1)$ satisfies the condition of Equation (**??**), the respective non-well

founded chains $(\alpha_i)_i$ being $(*, *, \dots)$, $(p, q, p, q, \dots)$, and $(0, -1, -2, \dots)$:

$$\begin{array}{ccc} * = \{*\} \to * & p = \{q\} \to p & 0 = \{1\} \to 0 \\ \Cup & \Cup & \Cup \\ * = \{*\} \to * & q = \{p\} \to q & 1 = \{2\} \to 1 \\ \Cup & \Cup & \Cup \\ * = \{*\} \to * & p = \{q\} \to p & 2 = \{3\} \to 2 \\ \ddots & \ddots & \ddots \end{array}$$

- More interestingly, the model H^f (Ex. **??**(**??**)) is hyperimmune iff f is a *hyperimmune function* [**?**], *i.e.*, iff there is no recursive $g : \mathbb{N} \to \mathbb{N}$ such that $f \leq g$ (pointwise order); otherwise the corresponding sequence is $(\alpha_1^i)_i$.

$$\alpha_1^0 = \underbrace{\emptyset \to \cdots \to \emptyset}_{f(0)\ \text{times}} \to \{\alpha_1^1\} \to \emptyset \to \cdots \to \emptyset \to *$$

$$\Cup$$

$$\alpha_1^1 = \underbrace{\emptyset \to \cdots \to \emptyset}_{f(1)\ \text{times}} \to \{\alpha_1^2\} \to \emptyset \to \cdots \to \emptyset \to *$$

$$\Cup$$

$$\alpha_1^2 = \underbrace{\emptyset \to \cdots \to \emptyset}_{f(2)\ \text{times}} \to \{\alpha_1^3\} \to \emptyset \to \cdots \to \emptyset \to *$$

$$\Cup$$

$$\ddots$$

The following theorem constitutes the main result of the paper. It shows the equivalence between hyperimmunity and (inequational) full abstraction for $\mathcal{H}^*$ under a certain condition. This conditions, namely the test-sensibility, is a new property that will be defined in more details in Definition **??**.

Theorem 1.22. *For any extensional and test-sensible (Def.* **??***) K-model D, the following are equivalent:*

(1) *D is hyperimmune,*
(2) *D is inequationally fully abstract for $\mathcal{H}^*$,*
(3) *D is fully abstract for $\mathcal{H}^*$.*

Example 1.23. The model D_∞ (Ex.**??**(**??**)), the model $\overline{\omega}$ (Ex.**??**(**??**)) and the well-stratified K-models (Ex.**??**(**??**)) will be shown inequationally fully abstract, as well as the models H^f when f is hyperimmune. The models D_∞^*, $\overline{\mathbb{Z}}$ (Ex.**??**(**??**) and Ex.**??**(**??**)) will not be, as well as the model H^f for f not hyperimmune.

As for the traditional proof of full abstraction for the $\mathcal{H}^*$, the main idea of our proof is to use a middle step between our calculus and our models. However, this time the proxy will not be a kind of syntactical model (the Böhm trees), but a kind of semantical calculus, more exactly a set of calculi that we call λ-calculi with D-tests (Def. **??**). The traditional interest over Böhm trees lies in the fact that they are “syntactical models” directly inspired by the calculus (here the λ-calculus); thus, taking the opposite view, we will use "semantical calculi" that are directly inspired by the model (and that are dependent on the K-model D).

Given a K-model D, the λ-calculus with D-tests, denoted $\Lambda_{\tau(D)}$, is an extension of the untyped λ-calculus that can itself be interpreted in D (Def. **??**):

$$\begin{array}{ccc} \Lambda & \xrightarrow{\quad [\![.]\!] \quad} & D \\ & \searrow^{\subseteq} \quad \nearrow_{[\![.]\!]} & \\ & \Lambda_{\tau(D)} & \end{array}$$

The interest of $\Lambda_{\tau(D)}$ relies on the definition of sensibility for $\Lambda_{\tau(D)}$ (Def. **??**), which easily implies the full abstraction of D for $\Lambda_{\tau(D)}$ (Th. **??**), even if not for the λ-calculus. Therefore, it remains to understand when the observational equivalence is preserved from Λ to $\Lambda_{\tau(D)}$:

$$\Lambda \xrightarrow{\ \subseteq\ } \Lambda_{\tau(D)}$$

$$\begin{array}{ccc} M & \xmapsto{\ id\ } & M \\ \equiv_{\mathcal{H}^*} & & \equiv_{\tau(D)} \\ N & \xmapsto{\ id\ } & N \end{array}$$

The proof splits in the two directions: inequational full abstraction implies hyperimmunity (Sec. **??** and Th. **??**) and the non-full abstraction for $\mathcal{H}^*$ gives a counterexample to hyperimmunity (Sec. **??** and Th. **??**). However, the proofs will rely on syntactical properties of $\Lambda_{\tau(D)}$ such as confluence (Th. **??**) and standardization (Th. **??**).

2. λ-calculi with D-tests

2.1. **Syntax.**

The original idea of using *tests* to recover full abstraction (via a theorem of definability) is due to Bucciarelli *et al.* [**?**]. Here we define variants of Bucciarelli *et al.*'s calculus adapted to our framework.

Directly dependent on a given K-model D, the λ-calculus with D-tests $\Lambda_{\tau(D)}$ is, to some extent, an internal calculus for D. In fact, we will see that, for D to be fully abstract for $\Lambda_{\tau(D)}$, it is sufficient to be sensible (Th. **??**).

The idea is to introduce tests as a new kind in the syntax. Tests $Q \in \boldsymbol{T}_{\tau(D)}$ are sort of co-terms, in the sense that their interpretations are maps from the context to the dualizing object of the linear category ScottL ($\bot = \{*\}$):

$$[\![Q]\!]^{x_1 \ldots x_n} \in D^n \Rightarrow \bot$$

The type $\bot$ is the unit type, having only one value representing the convergence of the evaluation, seen as a success.[4]

The interaction between terms and tests is carried out by two groups of operations indexed by the elements $\alpha \in D$:

$$\tau_\alpha : \Lambda_{\tau(D)} \to \boldsymbol{T}_{\tau(D)} \quad \text{and} \quad \bar{\tau}_\alpha : \boldsymbol{T}_{\tau(D)} \to \Lambda_{\tau(D)}.$$

The first operation, τ_α, will verify that its argument $M \in \Lambda_{\tau(D)}$ has the point α in its interpretation. Intuitively, this is performed by recursively unfolding the Böhm tree of M and succeeding

[4] We will see in Remark **??** that in a polarized context, the behavior of test does not correspond to co-term (or stack), but to commands (or processes), *i.e.*, to interactions between usual terms and fictive co-terms extracted from the semantics.

(term)	$\Lambda_{\tau(D)}$	M, N	$::=$	$x \mid \lambda x.M \mid M\,N \mid \sum_{i\leq n} \bar{\tau}_{\alpha_i}(Q_i)$	$, \forall (\alpha_i)_i \in D^n, n \geq 0$
(test)	$\boldsymbol{T}_{\tau(D)}$	P, Q	$::=$	$\sum_{i\leq n} P_i \mid \prod_{i\leq n} P_i \mid \tau_\alpha(M)$	$, \forall \alpha \in D, n \geq 0$

FIGURE 3. Grammar of the calculus with D-tests

(*i.e.*, converging) when α is in the interpretation of the finite unfolded Böhm tree. If $\alpha \notin [\![M]\!]$, the test $\tau_\alpha(M)$ will either diverge or refute (raising a $\mathbf{0}$ considered as an error). Concretely, it is an infinite application that feeds its argument with empty $\bar{\tau}$ operators.

The second operator, $\bar{\tau}_\alpha$, simply constructs a term of interpretation $\downarrow\alpha$ if its argument succeeds and diverges otherwise. Concretely, it is an infinite abstraction that runs its test argument, but also tests each of its applicants using τ operators.

In addition to these operators, we use *sums* and *products* as ways to introduce may (for the addition) and must (for the multiplication) non-determinism; in the spirit of the λ+||-calculus [**?**]. Indeed, these two forms of non-determinism are necessary to explore the branching of Böhm trees.

The idea of these two operators is to use the parametricity of our terms toward their intersection types. As a result, $\bar{\tau}_\alpha(\epsilon)$ (further on denoted $\bar{\boldsymbol{\epsilon}}_\alpha$), that transfers the always succeeding test ϵ into a term of interpretation $\downarrow\alpha$, constitutes the canonical term of type α; its behavior is exactly the common behavior of every term of type α. Symmetrically, the test $\tau_\alpha(M)$ will verify whether M behaves like a term of type α.

Hereafter, D denotes a fixed extensional K-model.

Definition 2.1. The *λ-calculus with D-tests*, for short $\Lambda_{\tau(D)}$, is given by the grammar in Figure **??**. We denote the empty sum by $\mathbf{0}$, and the empty product by $\boldsymbol{\epsilon}$. Binary sums (resp. products) can be written with infix notation, *e.g.* $P{+}Q$ (resp $P{\cdot}Q$).

Moreover, we use the notation $\bar{\boldsymbol{\epsilon}}_\alpha := \bar{\tau}_\alpha(\boldsymbol{\epsilon})$ and $\bar{\boldsymbol{\epsilon}}_a := \sum_{\alpha\in a} \bar{\boldsymbol{\epsilon}}_\alpha$; which are terms.

Sums and products are considered as multisets, in particular we suppose associativity, commutativity and neutrality with, respectively, $\mathbf{0}$ and $\boldsymbol{\epsilon}$.

In the following, an *abstraction* can refer either to a λ-abstraction or to a sum of $\bar{\tau}$ operators. This notation is justified by the behavior of $\Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$ that mimics an infinite abstraction.

The operational semantics is given by three sets of rules in Figure **??**. The *main rules* of Figure **??** are the effective rewriting rules. The *distributive rules* of Figure **??** implement the distribution of the sum over the test-operators and the product. The small step semantics $\to$ is the free contextual closure (*i.e.*, by the rules of Figure **??**) of the rules of Figures **??** and **??**. The *contextual rules* of Figure **??** implement the *head reduction* $\to_h$ that is the specific contextual extension we are considering.

Example 2.2. The operational behavior of D-tests depends on D. Recall the K-models of Example **??**. In the case of Scott's D_∞ we have in $\Lambda_{\tau(D_\infty)}$:

$$\begin{aligned}\tau_*(\underline{(\lambda xy.x\,y)\,\bar{\boldsymbol{\epsilon}}_*}) &\overset{\beta}{\to}_h \underline{\tau_*(\lambda y}.\bar{\boldsymbol{\epsilon}}_*\,y) \overset{\tau}{\to}_h \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_*\,\bar{\boldsymbol{\epsilon}}_\emptyset})\\ &\overset{\bar{\tau}}{\to}_h \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_*}) = \underline{\tau_*(\bar{\tau}_*}(\boldsymbol{\epsilon})) \overset{\tau\bar{\tau}}{\to}_h \boldsymbol{\epsilon},\end{aligned}$$

$$\begin{aligned}\tau_*(\underline{(\lambda xy.y\,x)\,\bar{\boldsymbol{\epsilon}}_*}) &\overset{\beta}{\to}_h \underline{\tau_*(\lambda y}.y\,\bar{\boldsymbol{\epsilon}}_*) \overset{\tau}{\to}_h \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_\emptyset}\,\bar{\boldsymbol{\epsilon}}_*)\\ &= \tau_*(\underline{\mathbf{0}\,\bar{\boldsymbol{\epsilon}}_*}) \overset{\bar{\tau}}{\to}_h \underline{\tau_*(\mathbf{0})} \overset{\tau\bar{\tau}}{\to}_h \mathbf{0}.\end{aligned}$$

$$(\beta) \qquad (\lambda x.M)\,N \;\to\; M[N/x]$$

$$(\bar\tau) \qquad \forall \beta_i = a_i {\to} \alpha_i, \qquad (\Sigma_i \bar\tau_{\beta_i}(Q_i))\,N \;\to\; \Sigma_i \bar\tau_{\alpha_i}(Q_i \cdot \Pi_{\gamma\in a_i}\tau_\gamma(N))$$

$$(\tau) \qquad \forall \beta = a {\to} \alpha, \qquad \tau_\beta(\lambda x.M) \;\to\; \tau_\alpha(M[\bar{\epsilon}_a/x])$$

$$(\tau\bar\tau) \qquad \forall \alpha, \forall(\beta_i)_i, \qquad \tau_\alpha(\Sigma_i \bar\tau_{\beta_i}(Q_i)) \;\to\; \Sigma_{\{i|\alpha\le\beta_i\}} Q_i$$

(A) Main rules

$$(\cdot +) \qquad \Pi_{i\le n}\Sigma_{j\le k_i} Q_{i,j} \;\to\; \Sigma_{j_1\le k_1,\dots,j_n\le k_n}\Pi_{i\le n} Q_{i,j_i}$$

$$(\bar\tau +) \qquad \bar\tau_\alpha(\Sigma_i Q_i) \;\to\; \Sigma_i \bar\tau_\alpha(Q_i)$$

(B) Distribution of the sum

$$\frac{M \to_h M'}{\lambda x.M \to_h \lambda x.M'}\,(h\text{-}c\lambda) \qquad \frac{M \to_h M' \qquad M \text{ is an application}}{M\,N \to_h M'\,N}\,(h\text{-}c@)$$

$$\frac{M \to_h M' \qquad M \text{ is an application}}{\tau_\alpha(M) \to_h \tau_\alpha(M')}\,(h\text{-}c\tau) \qquad \frac{Q \to_h Q' \qquad Q \text{ is not a sum}}{\bar\tau_\alpha(Q) \to_h \bar\tau_\alpha(Q')}\,(h\text{-}c\bar\tau)$$

$$\frac{M \to_h M'}{M + N \to_h M' + N}\,(h\text{-}cs) \qquad \frac{Q \to_h Q'}{Q + P \to_h Q' + P}\,(h\text{-}c+) \qquad \frac{Q \to_h Q' \qquad Q \text{ is not a sum}}{Q{\cdot}P \to_h Q'{\cdot}P}\,(h\text{-}c\cdot)$$

(C) Contextual rules for the head reduction

$$\frac{M \to M'}{\lambda x.M \to \lambda x.M'}\,(c\lambda) \qquad \frac{M \to M'}{M\,N \to M'\,N}\,(c@L) \qquad \frac{N \to N'}{M\,N \to M\,N'}\,(c@R)$$

$$\frac{M \to M'}{\tau_\alpha(M) \to \tau_\alpha(M')}\,(c\tau) \qquad \frac{Q \to Q'}{\bar\tau_\alpha(Q) \to \bar\tau_\alpha(Q')}\,(c\bar\tau)$$

$$\frac{M \to M'}{M + N \to M' + N}\,(cs) \qquad \frac{Q \to Q'}{Q + P \to Q' + P}\,(c+) \qquad \frac{Q \to Q'}{Q{\cdot}P \to Q'{\cdot}P}\,(c\cdot)$$

(D) Contextual rules for the full reduction

FIGURE 4. Operational semantics of the calculus with D-tests

In the case of Park P_∞:

$$\underline{\tau_*(\lambda x.xx)} \overset{\tau}{\to}_h \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_*\ \bar{\boldsymbol{\epsilon}}_*}) \overset{\bar\tau}{\to}_h \underline{\tau_*(\bar\tau_*}(\underline{\tau_*(\bar{\boldsymbol{\epsilon}}_*}))) \overset{\tau\bar\tau}{\to}_h \overset{\tau\bar\tau}{\to}_h \boldsymbol{\epsilon}.$$

In the case of Norm:

$$\underline{\tau_p(\lambda x.x)} \overset{\tau}{\to}_h \underline{\tau_p(\bar{\boldsymbol{\epsilon}}_q)} \overset{\tau\bar\tau}{\to}_h \boldsymbol{\epsilon}, \qquad \underline{\tau_q(\lambda x.x)} \overset{\tau}{\to}_h \underline{\tau_q(\bar{\boldsymbol{\epsilon}}_p)} \overset{\tau\bar\tau}{\to}_h \mathbf{0}.$$

Example 2.3. In any K-model D, given $\alpha = a_1 \to \cdots \to a_{n+1} \to \beta \in D$, and if we denote $\alpha' = a_2 \to \cdots \to a_{n+1} \to \beta$ we have:

$$\begin{aligned} \bar{\boldsymbol{\epsilon}}_\alpha\ M_1 \cdots M_{n+1} &\overset{\bar\tau}{\to}_h \bar\tau_{\alpha'}(\Pi_{\gamma \in a_1} \tau_\gamma(M_1))\ M_2 \cdots M_{n+1} \\ &\overset{\bar\tau}{\to}{}^n_h \bar\tau_\beta(\Pi_{i \le n+1} \Pi_{\gamma \in a_i} \tau_\gamma(M_i)) \end{aligned}$$

Remark 2.4. In a polarized (or classical) framework with explicit co-terms (or stacks) as the framework presented in [**?**], tests would correspond to commands (or processes), or, more exactly, to conjunctions and disjunctions of commands. Indeed, a test $\tau_\alpha(M)$ is nothing else than the command $\langle M \mid \pi_\alpha \rangle$ where π_α would be the canonical co-term of interpretation $\uparrow\alpha$, the same way that $\bar{\boldsymbol{\epsilon}}_\alpha$ is the canonical term of interpretation $\downarrow\alpha$. Similarly, the term $\bar\tau(Q)$ can be seen as the canonical term $\bar{\boldsymbol{\epsilon}}_\alpha$ endowed with a parallel composition referring to the set of commands Q. To resume, we have:

$$\tau_\alpha(M) \simeq \langle M \mid \uparrow\alpha \rangle \qquad\qquad \langle \bar\tau_\alpha(Q) \mid \pi \rangle \simeq \langle \downarrow\alpha \mid \pi \rangle \cdot Q$$

Remark 2.5. In the conference version [**?**], the rule $(\tau\bar\tau)$ is decomposed into three rules (the distribution of the sum over τ, denoted $(\tau+)$ and two versions of $(\tau\bar\tau)$ depending on whether $\alpha \le \beta$). This decomposition was easier to understand as more atomic, but ultimately it always reproduces our actual rule $(\tau\bar\tau)$ and does not permit to use Theorem **??**.

Proposition 2.6. *A test is in* head-normal form *iff it has the following shape:*

$$\Sigma_{i \le k} \Pi_j \tau_{\alpha_{i,j}}(x_{i,j}\ M^1_{i,j} \cdots\ M^n_{i,j}),$$

with $k \ge 1$ *and* $M^k_{i,j}$ *any term.*
A term is in head-normal form *if it has one of the following shapes:*

$$\lambda x_1 \ldots x_n.y\ M_1 \cdots\ M_m, \qquad or \qquad \lambda x_1 \ldots x_n.\Sigma_{i \le k} \bar\tau_{\alpha_i}(Q_i),$$

where $m, n \ge 0$, $k \ge 1$, $(\alpha_i)_i \in D^k$, M_i *is any term, and every* Q_i *any test in head-normal form without sums.*

Proof. By structural induction on the grammar of $\Lambda_{\tau(D)}$. In particular, notice that any test of the shape $\tau_\alpha(\lambda x.M)$ is not a head-normal form because i_D is surjective and thus $\alpha = a \to \beta$ for some a, β and we can apply Rule (τ). □

Definition 2.7. A term (resp. test) is *head-converging* if it head reduces to a *may-head-normal form* (denoted mhnf) that is either a head-normal form or a term (resp. test) of the form

$$\lambda x_1 \ldots x_n.(\bar\tau_\alpha(Q) + N) \qquad\qquad \text{resp. } Q_1 + Q_2$$

with $\bar\tau_\alpha(Q)$ (resp. Q_1) in head-normal form and N any term (resp. Q_2 any test). This corresponds to a *may-convergence* for the sum. Coherently with the head convergence in λ-calculus, the convergence will be denoted by $\Downarrow^{\mathrm{h}}$ and the divergence by $\Uparrow^{\mathrm{h}}$.

Example 2.8. For any $n \in \mathbb{N}$, the term $\underline{n}\ (\lambda x.\bar\tau_\alpha(\tau_\alpha(x) + \tau_\beta(x)))\ \bar{\boldsymbol{\epsilon}}_\alpha$ may-head-converges.

(term-context)	$\Lambda^{(\!|\cdot|\!)}_{\tau(D)}$	$C ::= x \mid (\!|.|\!) \mid C\,C' \mid \lambda x.C \mid \sum_{i\leq n}\bar{\tau}_{\alpha_i}(K_i)$	$,\forall(\alpha_i)_i\in D^n, n\geq 0$
(test-context)	$\boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}$	$K ::= \sum_{i\leq n} K_i \mid \prod_{i\leq n} K_i \mid \tau_\alpha(C)$	$,\forall\alpha\in D, n\geq 0$

FIGURE 5. Grammar of the contexts in a calculus with D-tests

$$[\![x_i]\!]^{\vec{x}}_D = \{(\vec{a},\alpha) \mid \alpha\leq\beta\in a_i\} \qquad [\![\lambda y.M]\!]^{\vec{x}}_D = \{(\vec{a},(b\to\alpha)) \mid (\vec{a}b,\alpha)\in[\![M]\!]^{\vec{x}y}_D\}$$
$$[\![M\,N]\!]^{\vec{x}}_D = \{(\vec{a},\alpha) \mid \exists b, (\vec{a},(b\to\alpha))\in[\![M]\!]^{\vec{x}}_D \ \wedge \forall\beta{\in}b, (\vec{a},\beta)\in[\![N]\!]^{\vec{x}}_D\}$$

(A) Interpretation of Λ

$$[\![\Sigma_{i\leq k}\bar{\tau}_{\alpha_i}(Q_i)]\!]^{\vec{x}}_D = \bigcup_{i\leq k}\{(\vec{a},\beta) \mid \vec{a}\in[\![Q_i]\!]^{\vec{x}}_D \ \wedge\ \beta\leq_D \alpha_i\} \qquad [\![\mathbf{0}]\!]^{\vec{x}}_D = \emptyset$$
$$[\![\tau_\alpha(M)]\!]^{\vec{x}}_D = \{\vec{a} \mid (\vec{a},\alpha)\in[\![M]\!]^{\vec{x}}_D\}$$
$$[\![\Pi_{i\leq k}Q_i]\!]^{\vec{x}}_D = \bigcap_{i\leq k}[\![Q_i]\!]^{\vec{x}}_D \qquad [\![\boldsymbol{\epsilon}]\!]^{\vec{x}}_D = \mathcal{A}_f(D)^{\vec{x}} \qquad [\![\Sigma_{i\leq k}Q_i]\!]^{\vec{x}}_D = \bigcup_{i\leq k}[\![Q_i]\!]^{\vec{x}}_D \qquad [\![\mathbf{0}]\!]^{\vec{x}}_D = \emptyset$$

(B) Interpretation of tests extensions

FIGURE 6. Direct interpretation in D

Let us notice that this calculus enjoys the properties of confluence and standardization (Th. **??** and Th. **??**). We also have another syntactical theorem stating invariance wrt the head-convergence in at most n steps, denoted $\Downarrow^{\mathrm{h}}_n$ (Theorem **??**). This means that performing a non-head reduction can only reduce the length of convergence.

Definition 2.9. Grammars of *term-contexts* $\Lambda^{(\!|\cdot|\!)}_{\tau(D)}$ and *test-contexts* $\boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}$ are given in Figure **??**.

Definition 2.10. The *observational preorder* $\sqsubseteq_{\tau(D)}$ of $\Lambda_{\tau(D)}$ is defined by:

$$M \sqsubseteq_{\tau(D)} N \ \text{ iff } \ (\forall K{\in}\boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)},\ K(\!|M|\!)\Downarrow^{\mathrm{h}} \text{ implies } K(\!|N|\!)\Downarrow^{\mathrm{h}}).$$

We denote by $\equiv_{\tau(D)}$ the *observational equivalence*, *i.e.*, the equivalence induced by $\sqsubseteq_{\tau(D)}$.

Remark 2.11. The observational preorder could have been defined using term-contexts rather than test-contexts, but this appears to be equivalent and test-contexts are easier to manipulate (because normal forms for tests are simpler).

Proof. For any test Q and for any α, $Q\Downarrow^{\mathrm{h}}$ iff $\bar{\tau}_\alpha(Q)\Downarrow^{\mathrm{h}}$. Conversely, for all M, there is $n\in\mathbb{N}$ and $\alpha\in D$ such $M\Downarrow^{\mathrm{h}}$ iff $\tau_\alpha(M\underbrace{x_0\cdots x_0}_{n})\Downarrow^{\mathrm{h}}$ (remark that if N diverges, then $\tau_\alpha(N\underbrace{x_0\cdots x_0}_{n \text{ times}})\Uparrow^{\mathrm{h}}$). □

2.2. Semantics.

The standard interpretation of Λ into D (Fig. **??** and recalled here in Figure **??**) can be extended to $\Lambda_{\tau(D)}$ (Fig. **??**).

Definition 2.12. A term M with n free variables is *interpreted* as a morphism (Scott-continuous function) from D^n to D and a test Q with n free variables as a morphism from D^n to the dualizing object $\{*\}$ (singleton poset):

$$[\![M]\!]^{x_1,\dots,x_n}_D \ \subseteq\ (D\Rightarrow\cdots\Rightarrow D\Rightarrow D)\ \simeq\ (\mathcal{A}_f(D)^{op})^n\times D$$
$$[\![Q]\!]^{x_1,\dots,x_n}_D \ \subseteq\ (D\Rightarrow\cdots\Rightarrow D\Rightarrow\{*\})\ \simeq\ (\mathcal{A}_f(D)^{op})^n$$

This interpretation is given in Figure **??** by structural induction.

$$\frac{\alpha \in a}{x : a \vdash x : \alpha} \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma, x : a \vdash M : \alpha} \qquad \frac{\Gamma \vdash M : \beta \qquad \alpha \leq \beta}{\Gamma \vdash M : \alpha}$$

$$\frac{\Gamma, x : a \vdash M : \alpha}{\Gamma \vdash \lambda x.M : a{\to}\alpha} \qquad \frac{\Gamma \vdash M : a{\to}\alpha \qquad \forall \beta \in a,\ \Gamma \vdash N : \beta}{\Gamma \vdash M\,N : \alpha}$$

$$\frac{\exists i \leq n,\ \Gamma \vdash Q_i}{\Gamma \vdash \Sigma_{i\leq n}\bar{\tau}_{\alpha_i}(Q_i) : \alpha_i} \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma \vdash \tau_\alpha(M)} \qquad \frac{\exists i \leq n,\ \Gamma \vdash Q_i}{\Gamma \vdash \Sigma_{i\leq n} Q_i} \qquad \frac{\forall i \leq n,\ \Gamma \vdash Q_i}{\Gamma \vdash \Pi_{i\leq n} Q_i}$$

FIGURE 7. Intersection type system associated with tests extensions

Proposition 2.13. *For any extensional K-model D, D is a model of the λ-calculus with D-tests,* i.e.*, the interpretation is invariant under reduction.*

Proof. The invariance under β-reduction is obtained, as usual, by the Cartesian closedness of $\text{ScottL}_!$. The other rules are easy to check directly. □

Proposition 2.14. *For any extensional K-model D, the interpretation is invariant by context,* i.e.*, $[\![M]\!]^{\vec{x}} = [\![N]\!]^{\vec{x}}$ implies that for any test/term-context C, $[\![C(\!|M|\!)]\!]^{\vec{x}} = [\![C(\!|N|\!)]\!]^{\vec{x}}$.*

Proof. By easy induction on C. □

The idea of intersection types can be generalized to $\Lambda_{\tau(D)}$. We introduce in Figure **??** a type assignment system associating with any term $M \in \Lambda_{\tau(D)}$ an element of D under an environment $(x_i{:}a_i)_i$ with $a_i \in \mathcal{A}_f(D)$. The following theorem gives the equivalence between the interpretation of a term and the set of judgments derivable from the type system.

Theorem 2.15 (Intersection types)**.** *Let M be a term of $\Lambda_{\tau(D)}$, (resp. Q be a test of $\boldsymbol{T}_{\tau(D)}$), the following statements are equivalent:*

- $(\vec{a}, \alpha) \in [\![M]\!]_D^{\vec{x}}$ *(resp. $\vec{a} \in [\![Q]\!]_D^{\vec{x}}$),*
- *the type judgment $\vec{x} : \vec{a} \vdash M : \alpha$ (resp. $\vec{x} : \vec{a} \vdash Q$) is derivable by the rules of Figure* **??**.

Proof. By structural induction on the grammar of $\Lambda_{\tau(D)}$. □

Remark 2.16. In particular, an easy induction gives that if $\vdash M[N/x] : \alpha$ then there is a such that $N : a \vdash M : \alpha$.

2.2.1. *Full abstraction and sensibility for tests.*
The main theorem (Th. **??**) uses the assumption of sensibility of D for $\Lambda_{\tau(D)}$. The sensibility is simply asking for the diverging terms $M \in \Lambda_{\tau(D)}$ to have empty interpretation as specified in Definition **??**. Its interest is in implying directly the inequational full abstraction of D for $\Lambda_{\tau(D)}$ (*i.e.* for its observational preorder) as we will see in Theorem **??**. The proof of Theorem **??** needs a technical counterpart that is basically the *definability* of $\Lambda_{\tau(D)}$ stated in Theorem **??**. This definability theorem is not usual and appears to be stronger and more useful for future developments.

First we recall the definition of sensibility:

Definition 2.17. An extensional K-model D is *sensible for* $\Lambda_{\tau(D)}$ whenever diverging terms (resp. tests) correspond exactly to the terms (resp. tests) having empty interpretation, *i.e.*, for all $M \in \Lambda_{\tau(D)}$ and $Q \in \boldsymbol{T}_{\tau(D)}$:

$$M{\Uparrow}^{\mathrm{h}} \quad \Leftrightarrow \quad [\![M]\!]_D^{\vec{x}} = \emptyset \qquad\qquad Q{\Uparrow}^{\mathrm{h}} \quad \Leftrightarrow \quad [\![Q]\!]_D^{\vec{x}} = \emptyset$$

Lemma 2.18. If D is sensible for $\Lambda_{\tau(D)}$ then:

$$(\vec{a}b,\alpha) \in [\![M]\!]^{\vec{y}x} \quad\Leftrightarrow\quad (\vec{a},\alpha) \in [\![M[\bar{\boldsymbol{\epsilon}}_b/x]]\!]^{\vec{y}},$$
$$(\vec{a},\alpha) \in [\![M]\!]^{\vec{y}} \quad\Leftrightarrow\quad \vec{a} \in [\![\tau_\alpha(M)]\!]^{\vec{y}}.$$

Proof. This lemma and its test counterpart is proved by a straightforward induction on M (and Q of the test version). □

Theorem 2.19 (Definability)**.** *If D is sensible for $\Lambda_{\tau(D)}$ then:*

$$(\vec{a},\alpha) \in [\![M]\!]^{\vec{x}} \quad\Leftrightarrow\quad \tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])\Downarrow^{\mathrm{h}}.$$

Proof. If $(\vec{a},\alpha) \in [\![M]\!]^{\vec{x}}$ then $[\![\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])]\!]$ is not empty by Lemma **??**, thus it converges by sensibility. Conversely, if $\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])\Downarrow^{\mathrm{h}}$ then its interpretation is non empty, which means that in particular $*\in[\![\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])]\!]$ (where $*$ denotes the only inhabitant of $\bot$) and thus, by Lemma **??**, $(\vec{a},\alpha) \in [\![M]\!]^{\vec{x}}$. □

Theorem 2.20 (full abstraction)**.** *For any extensional K-model D, if D is sensible for $\Lambda_{\tau(D)}$, then D is inequationally fully abstract for the observational preorder of $\Lambda_{\tau(D)}$:*

$$[\![M]\!] \subseteq [\![N]\!] \quad\Leftrightarrow\quad \forall C \in \boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}, C(\!|M|\!)\Downarrow^{\mathrm{h}} \Rightarrow C(\!|N|\!)\Downarrow^{\mathrm{h}}.$$

Proof. Let $[\![M]\!] \subseteq [\![N]\!]$ and $C(\!|M|\!)\Downarrow^{\mathrm{h}}$. Then by sensibility we have that $[\![C(\!|M|\!)]\!]$ is non-empty. Moreover, by Proposition **??** we have that $[\![C(\!|M|\!)]\!] \subseteq [\![C(\!|N|\!)]\!]$. Thus $[\![C(\!|N|\!)]\!]$ is non-empty and by sensibility, $C(\!|N|\!)\Downarrow^{\mathrm{h}}$.
Conversely, suppose that for all context $C \in \boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}, C(\!|M|\!)\Downarrow^{\mathrm{h}} \Rightarrow C(\!|N|\!)\Downarrow^{\mathrm{h}}$ and let $(\vec{a},\alpha) \in [\![M]\!]^{\vec{x}}$:
Then by Theorem **??**, $\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])\Downarrow^{\mathrm{h}}$ where n is the length of $\vec{a}$. Thus, after stating the context $C = \tau_\alpha((\lambda x_{.}..x_n.(\!|.|\!))\,\bar{\boldsymbol{\epsilon}}_{a_1}\cdots\bar{\boldsymbol{\epsilon}}_{a_n})$, we have $C(\!|M|\!) \to_h^n \tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])\Downarrow^{\mathrm{h}}$ which implies that $C(\!|N|\!)\Downarrow^{\mathrm{h}}$. However, there is no choice[5] for the n first head reductions of $C(\!|N|\!)$, those are forced to be $C(\!|N|\!) \to_h^n \tau_\alpha(N[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])$ so that this term is head-converging. Then by applying the reverse implication of Theorem **??** we conclude $(\vec{a},\alpha) \in [\![N]\!]^{\vec{x}}$. □

2.3. **Technical theorems.**

2.3.1. *Confluence.*

This section is dedicated to the proof of Theorem **??** stating the confluence of the reduction $\to$ in $\Lambda_{\tau(D)}$. The proof uses the diamond property of the full parallel reduction, following the proof of [**?**] for the λ-calculus.

We define first the *parallel reduction* $\Rightarrow$ in Figure **??**, allowing the parallel reduction of independent redexes.

Lemma 2.21. If $M \Rightarrow N$ then $M \to^* N$ and if $M \to^* N$ then $M \Rightarrow^* N$.
In particular we have $\Rightarrow^* = \to^*$.

Proof. Firstly remark that $\Rightarrow$ is reflexive. Indeed, when we proceed by induction the only difficult case is $\epsilon \Rightarrow \epsilon$ that is obtained by Rule $(P\text{-}\cdot\text{+})$ for $n = 0$.
Rules with similar names are then simulating each other except for

[5] We have to verify that this are the only possible reductions because in general the head-reduction is not determonistic in $\Lambda_{\tau(D)}$.

$$\frac{M \Rightarrow M' \qquad N \Rightarrow N'}{(\lambda x.M)\, N \Rightarrow M'[N'/x]}\ (P\text{-}\beta) \qquad \frac{M \Rightarrow M' \qquad \forall i,\ Q_i \Rightarrow \Sigma_j Q'_{ij}}{\Sigma_i \bar\tau_{a_i\to\alpha_i}(Q_i)\ M \Rightarrow \Sigma_{ij}\bar\tau_{\alpha_i}(Q'_{ij}\cdot \Pi_{\gamma\in a_i}\tau_\gamma(M'))}\ (P\text{-}\bar\tau)$$

$$\frac{M \Rightarrow M'}{\tau_{a\to\alpha}(\lambda x.M) \Rightarrow \tau_\alpha(M'[\bar\epsilon_a/x])}\ (P\text{-}\tau) \qquad \frac{\forall_i, Q_i \Rightarrow Q'_i}{\tau_\alpha(\Sigma_i\bar\tau_{\beta_i}(Q_i)) \Rightarrow \Sigma_{\{i|\alpha\leq\beta_i\}} Q'_i}\ (P\text{-}\tau\bar\tau)$$

(A) Main rules

$$\frac{\forall i,\ Q_i \Rightarrow Q'_i}{\bar\tau_\alpha(\Sigma_i Q_i) \Rightarrow \Sigma_i\bar\tau_\alpha(Q'_i)}\ (P\text{-}\bar\tau+) \qquad \frac{\forall ij,\ Q_{ij} \Rightarrow Q'_{ij}}{\Pi_{i\leq n}\Sigma_{j\leq k_i} Q_{ij} \Rightarrow \Sigma_{j_1\leq k_1,\dots,k_n\leq k_n}\Pi_{i\leq n} Q'_{ij_i}}\ (P\text{-}\cdot+)$$

(B) Distribution of the sum

$$\frac{}{x \Rightarrow x}\ (P\text{-}id) \qquad \frac{M \Rightarrow M'}{\lambda x.M \Rightarrow \lambda x.M'}\ (P\text{-}c\lambda) \qquad \frac{M \Rightarrow M' \qquad N \Rightarrow N'}{M\,N \Rightarrow M'\,N'}\ (P\text{-}c@)$$

$$\frac{M \Rightarrow M'}{\tau_\alpha(M) \Rightarrow \tau_\alpha(M')}\ (P\text{-}c\tau) \qquad \frac{\forall i,\ M_i \Rightarrow M'_i}{\Sigma_i M_i \Rightarrow \Sigma_i M'_i}\ (P\text{-}cs)$$

(c) Contextual rules

FIGURE 8. Operational Semantics of parallel reduction

- $(c@L)$ and $(c@R)$ that are simulated by $(P\text{-}c@)$.
- $(P\text{-}id)$ that is simulated by $\to^\epsilon$ (the reduction in 0 step).
- $(c+)$ that is a particular case of $(P\text{-}\cdot+)$ with $n = 1$ and $k_1 = 2$.
- $(c\cdot)$ that is a particular case of $(P\text{-}\cdot+)$ with $n = 2$ and $k_1 = k_2 = 1$.
- $(c\bar\tau)$ that is a particular case of $(P\text{-}\bar\tau+)$ where the sum has one element.

□

For a term M (resp. a test Q) we define the *maximal parallel reduct* M^+ (resp. Q^+) by induction on M and Q in Figure **??**. Recall that by abstractions, we not only mean λ-abstractions, but also terms of the form $\Sigma_i\bar\tau_{\alpha_i}(Q_i)$.

Lemma 2.22. For any M (resp. Q), M^+ (resp. Q^+) is well defined.

Proof. By induction, since it is always the case that exactly one rule is applied. □

Lemma 2.23. If $M \Rightarrow N$ (resp. $Q \Rightarrow P$) then $N \Rightarrow M^+$ (resp. $P \Rightarrow Q^+$).

Proof. By induction on M:

- If $M = x$:
  Then $N = x \Rightarrow x = M^+$.
- If $M = \lambda x.M'$:
  Then $N = \lambda x.N'$ for some N' such that $M' \Rightarrow N'$.
  By IH, $N' \Rightarrow M'^+$ and thus $N \Rightarrow \lambda x.M'^+ = M^+$.
- If $M = M_1\ M_2$:
  - If M_1 is not an abstraction:
    Then $N = N_1\ N_2$ with $M_1 \Rightarrow N_1$ and $M_2 \Rightarrow N_2$.
    By IH, $N_1 \Rightarrow M_1^+$ and $N_2 \Rightarrow M_2^+$, thus $N \Rightarrow M_1^+\ M_2^+ = M^+$.

$$\frac{}{((\lambda x.M)\ N)^+ := M^+[N^+/x]}\ (T\text{-}\beta) \qquad \frac{\forall i,\quad Q_i^+ = \Sigma_j Q'_{ij} \qquad \forall j,\ Q'_{i,j} \text{ are not sums}}{((\Sigma_i \bar\tau_{a_i\to\alpha_i}(Q_i))\ M)^+ := \Sigma_{ij}\bar\tau_{\alpha_i}(Q'_{ij}\cdot \Pi_{\gamma\in a_i}\tau_\gamma(M^+))}\ (T\text{-}\bar\tau)$$

$$\frac{}{\tau_{a\to\alpha}(\lambda x.M)^+ := \tau_\alpha(M^+[\bar\epsilon_a/x])}\ (T\text{-}\tau) \qquad \frac{\forall i\in I,\ \alpha\leq_D \beta_i \qquad \forall i\in J,\ \alpha\not\leq_D\beta_i}{\tau_\alpha(\Sigma_{i\in I\cup J}\bar\tau_{\beta_i}(Q_i))^+ := \Sigma_{i\in I}Q_i^+}\ (T\text{-}\tau\bar\tau)$$

(A) Main rules

$$\frac{\forall i,\ Q_i \text{ are not sums}}{\bar\tau_\alpha(\Sigma_i Q_i)^+ := \Sigma_i\bar\tau_\alpha(Q_i^+)}\ (T\text{-}\bar\tau+) \qquad \frac{n\neq 1 \text{ or } k_1\neq 1 \qquad \text{the } Q_{ij} \text{ are not sums}}{(\Pi_{i\leq n}\Sigma_{j\leq k_i}Q_{ij})^+ := \Sigma_{j_1\leq k_1,\dots,k_n\leq k_n}\Pi_{i\leq n}Q^+_{ij_i}}\ (T\text{-}\cdot+)$$

(B) Distribution of the sum

$$\frac{}{x^+ := x}\ (T\text{-}id) \qquad \frac{}{(\lambda x.M)^+ \Rightarrow \lambda x.M^+}\ (T\text{-}c\lambda) \qquad \frac{M \text{ is not an abstraction}}{(M\ N)^+ := M^+\ N^+}\ (T\text{-}c@)$$

$$\frac{M \text{ is not an abstraction}}{\tau_\alpha(M) := \tau_\alpha(M^+)}\ (T\text{-}c\tau) \qquad \frac{k\neq 1}{(\Sigma_{i\leq k}M_i)^+ := \Sigma_{i\leq k}M_i^+}\ (T\text{-}cs)$$

(c) Contextual rules

Figure 9. Full parallel reduction

- If $M_1 = \lambda x.M_0$:
  - Either $N = (\lambda x.N_0)\ N_2$ with $M_i \Rightarrow N_i$ (for $i\in\{0,2\}$).
    By IH, $N_i \Rightarrow M_i^+$ and $N \Rightarrow M_0^+[M_2^+/x] = M^+$.
  - Or $N = N_1[N_2/x]$ with $M_i \Rightarrow N_i$ (for $i\in\{0,2\}$).
    By IH, $N_i \Rightarrow M_i^+$ and $N \Rightarrow M_0^+[M_2^+/x] = M^+$.
- If $M_1 = \Sigma_{i\in I}\bar\tau_{a_i\to\alpha_i}(Q_i)$:
  - Either $N = (\Sigma_{i,j}\bar\tau_{a\to\alpha_i}(P_{i,j}))\ N_2$ with $M_2 \Rightarrow N_2$ and $Q_i = \Sigma_i P'_{i,j}$ and $P'_{i,j} \Rightarrow P_{i,j}$.
    By IH, $N_2 \Rightarrow M_2^+$ and, moreover,
    $P_{i,j} \Rightarrow Q^+_{i,j} = \Sigma_k Q'_{i,j,k}$ where $Q'_{i,j,k}$ that are not sums.
    Thus $N \Rightarrow \Sigma_{i,j,k}\bar\tau_{\alpha_i}(Q'_{i,j,k}\cdot\Pi_{\gamma\in a_i}\tau_\gamma(M_2^+)) = M^+$.
  - Or $N = \Sigma_{i,j}\bar\tau_{\alpha_i}(P_{i,j}\cdot\Pi_{\gamma\in a_i}\tau_\gamma(N_2))$ with $M_2 \Rightarrow N_2$ and $Q_i \Rightarrow \Sigma_j P_{i,j}$.
    By IH, $N_2 \Rightarrow M_2^+$ and, moreover,
    $\Sigma_j P_{i,j} \Rightarrow Q_i^+ = \Sigma_{j,k}Q'_{i,j,k}$ where $Q'_{i,j,k}$ that are not sums and $P_{i,j} \Rightarrow \Sigma_k Q'_{i,j,k}$.
    Thus $N \Rightarrow \Sigma_{i,j,k}\bar\tau_{\alpha_i}(Q'_{i,j,k}\cdot\Pi_{\gamma\in a_i}\tau_\gamma(M_2^+)) = M^+$.

- If $Q = \tau_\alpha(M)$:
  - If M is not an abstraction:
    Then $P = \tau_\alpha(N)$ for some N such that $M \Rightarrow N$.
    By IH, $N \Rightarrow M^+$ and thus $P \Rightarrow \lambda x.M^+ = Q^+$.
  - If $\alpha = a\to\alpha$ and $M = \lambda x.M'$:
    - Either $P = \tau_{a\to\alpha}(\lambda x.N)$ with $M \Rightarrow N$.
      By IH, $N \Rightarrow M'^+$ and $P \Rightarrow \tau_\alpha(M'^+[\bar\epsilon_a/x]) = Q^+$.
    - Or $P = \tau_\alpha(N[\bar\epsilon_a/x])$ with $M' \Rightarrow N$.
      By IH, $N \Rightarrow M'^+$ and $P \Rightarrow \tau_\alpha(M'^+[\bar\epsilon_a/x]) = Q^+$.

– If $M = \Sigma_i \bar\tau_{\beta_i}(Q_i)$:
  * Either $N = \tau_\alpha(\Sigma_{i,j}\bar\tau_{\beta_i}(P'_{i,j}))$ with $Q_i = \Sigma_j P_{i,j}$ and $P_{i,j} \Rightarrow P'_{i,j}$.
    By IH, $P'_{i,j} \Rightarrow P^+_{i,j}$. Thus, $N \Rightarrow \Sigma_{\{i|\alpha\leq\beta_i\}}\Sigma_j P^+_{i,j} = \Sigma_{\{i|\alpha\leq\beta_i\}} Q^+_i = Q^+$.
  * Or $N = \Sigma_{\{i|\alpha\leq\beta_i\}} Q'_i$ with $Q_i \Rightarrow Q'_i$.
    By IH, $Q'_i \Rightarrow Q^+_i$. Thus, $N \Rightarrow \Sigma_{i|\alpha\leq\beta_i} Q^+_i = Q^+$.

- If $M = \Sigma_i M_i$:
  Then $N = \Sigma_i N_i$ with $M_i \Rightarrow N_i$.
  By IH, $N_i \Rightarrow M^+_i$ and $N \Rightarrow \Sigma_i M^+_i = M^+$.
- If $M = \bar\tau_\alpha(\Sigma_i Q_i)$ where none of the Q_i are sums:
  Then we can only apply rules (P-$\bar\tau$+) and (P-$\cdot$+). Thus there are J and a surjective function $\phi : I \to J$ such that $N = \Sigma_{j\in J}\bar\tau_\alpha(\Sigma_{i\in\phi^{-1}(j)} P_i)$ and $Q_i \Rightarrow P_i$.
  By IH, $P_i \Rightarrow Q^+_i$ and $N \Rightarrow \Sigma_{i\in I}\bar\tau_\alpha(Q^+_i) = M^+$.
- If $Q = \Pi_{i\leq n}\Sigma_{j\leq k_i} Q_{ij}$ where none of the Q_{ij} are sums and where either $n \neq 1$ or one of the $k_i \neq 1$:
  Then there are, for all $i \leq n$, J_i and $\phi_i : [\![1, k_i]\!] \to J_i$ such that $P = \Sigma_{(t_i)_i\in(J_i)_i}\Pi_{i\leq n}\Sigma_{j|\phi(j)=t_i} P_{ij}$ with $Q_{ij} \Rightarrow P_{ij}$.
  By IH, $P_{ij} \Rightarrow Q^+_{ij}$ and $P \Rightarrow \Sigma_{j_1\leq k_1\ldots j_n\leq k_n}\Pi_{i\leq n} Q^+_{ij_i} = Q^+$.

□

Theorem 2.24 (Confluence)**.** *The calculus $\Lambda_{\tau(D)}$ with the reduction $\to$ is confluent:*

$$\begin{array}{ccc} M & \to^* & M_2 \\ \downarrow_* & \rightsquigarrow & \downarrow_* \\ M_1 & \to^* & M' \end{array}$$

Proof. By Lemma **??**, $\Rightarrow$ is strongly confluent. This means that, for any $M_1 \Leftarrow M \Rightarrow M_2$, we have $M_1 \Rightarrow M^+ \Leftarrow M_2$. By chasing diagrams, we obtain the confluence of $\Rightarrow$ and we conclude by Lemma **??** stating that $\Rightarrow^* = \to^*$. □

2.3.2. *Standardization theorem.*
This section is dedicated to the proof of Theorem **??** stating a version of the standardization theorem for $\Lambda_{\tau(D)}$. The proof is directly inspired by Kashima's proof [**?**].

Definition 2.25. The *standard reduction*, denoted by $\Rightarrow_{st}$ is defined in Figure **??**.

Proposition 2.26. *We have the following inclusions:*

- $\Rightarrow_{st} \subseteq \to^*$,
- $id \subseteq \Rightarrow_{st}$, i.e., $\Rightarrow_{st}$ *is reflexive,*
- $\to^*_h \subseteq \Rightarrow_{st}$,
- $\Rightarrow_{st} \subseteq \to^*_h \to^*_{\not h}$ *where $\to^*_{\not h}$ is the reflexive transitive closure of $\to_{\not h} = \to - \to_h$.*

Proof.
- The inclusion $\Rightarrow_{st} \subseteq \to^*$ is obtain by easy induction (using each time the transitivity on $\to^*_h \subseteq \to^*$ and on the corresponding contextual rule of Figure **??** applied on the inductive hypothesis).
- The inclusion $id \subseteq \Rightarrow_{st}$ derives from an easy induction using $id \subseteq \to^*_h$.
- The inclusion $\to^*_h \subseteq \Rightarrow_{st}$ is obtained from a case analysis and the inclusion $id \subseteq \Rightarrow_{st}$.
- Let $M, N \in \Lambda_{\tau(D)}$ (resp. $P, Q \in \boldsymbol{T}_{\tau(D)}$) be such that $M \Rightarrow_{st} N$ (resp. $P \Rightarrow_{st} Q$). We will show that $M \to^*_h \to^*_{\not h} N$ (resp. $P \to^*_h \to^*_{\not h} Q$) by induction on N (resp. Q):

$$\frac{M \to_h^* x}{M \Rightarrow_{st} x}\ (S\text{-}x) \qquad \frac{M \to_h^* \lambda x.M_0 \qquad M_0 \Rightarrow_{st} N_0}{M \Rightarrow_{st} \lambda x.N_0}\ (S\text{-}\lambda)$$

$$\frac{M \to_h^* M_1\ M_2 \qquad M_1 \Rightarrow_{st} N_1 \qquad M_2 \Rightarrow_{st} N_2}{M \Rightarrow_{st} N_1\ N_2}\ (S\text{-}@)$$

$$\frac{P \to_h^* \tau_\alpha(M) \qquad M \Rightarrow_{st} N}{P \Rightarrow_{st} \tau_\alpha(N)}\ (S\text{-}\tau) \qquad \frac{M \to_h^* \Sigma_i \bar{\tau}_{\alpha_i}(P_i) \qquad \forall i,\ P_i \Rightarrow_{st} Q_i}{M \Rightarrow_{st} \Sigma_i \bar{\tau}_{\alpha_i}(Q_i)}\ (S\text{-}\bar{\tau})$$

$$\frac{P \to_h^* \Sigma_i P_i \qquad \forall i,\ P_i \Rightarrow_{st} Q_i}{P \Rightarrow_{st} \Sigma_i Q_i}\ (S\text{-}+) \qquad \frac{P \to_h^* \Pi_i P_i \qquad \forall i,\ P_i \Rightarrow_{st} Q_i}{P \Rightarrow_{st} \Pi_i Q_i}\ (S\text{-}\cdot)$$

FIGURE 10. Definition of the standard reduction

- If $N = x$ with $M \to_h^* x$: trivial.
- If $N = \lambda x.N_0$, then $M \to_h^* \lambda x.M_0$ and $M_0 \Rightarrow_{st} N_0$. By IH $M_0 \to_h^* \to_{\not h}^* N_0$ so that Rule (h-$c\lambda$) gives $M \to_h^* \lambda x.M_0 \to_h^* \to_{\not h}^* \lambda x.N_0$.
- If $N = N_1\ N_2$, then $M \to_h^* M_1\ M_2$, $M_1 \Rightarrow_{st} N_1$ and $M_2 \Rightarrow_{st} N_2$. By induction hypothesis $M_1 \to_h^* M_1' \to_{\not h}^* N_1$ for some $M_1' \in \Lambda_{\tau(D)}$.
  - If M_1' is not an abstraction, then there is no abstraction in the sequence $M_1 \to_h \cdots \to_h M_1'$ and by Rule (h-$c@$), $M \to_h^* M_1\ M_2 \to_h^* M_1'\ M_2 \to_{\not h}^* N_1\ M_2$.
  - Otherwise, there is a first abstraction M_1'' such that $M_1 \to_h^* M_1'' \to^* M_1'$ with no abstraction in the sequence $M_1 \to_h \cdots \to_h M_1''$.
    In this case, by Rule (h-$c@$),
    $M \to_h^* M_1\ M_2 \to_h^* M_1''\ M_2 \to_{\not h}^* M_1'\ M_2 \to_{\not h}^* N_1\ M_2 \to_{\not h}^* N_1\ N_2$.
- If $Q = \tau_\alpha(N)$, then the argument is similar:
  There is M such that $P \to_h^* \tau_\alpha(M)$ and $M \Rightarrow_{st} N$. By IH, there is M' such that $M \to_h^* M' \to_{\not h}^* N$. Either M' is not an abstraction and since there is no abstraction in the sequence $M \to_h \cdots \to_h M'$, we have, by Rule (h-$c\tau$), that $P \to_h \tau_\alpha(M) \to_h^* \tau_\alpha(M') \to_{\not h}^* \tau_\alpha(N)$.
  Otherwise there is a first abstraction M'' in the sequence $M \to_h \cdot \to_h M'' \to_h \cdots \to_h M'$, and we have, by Rule ($h$-$c\tau$), that $P \to_h \tau_\alpha(M) \to_h^* \tau_\alpha(M'') \to_{\not h}^* \tau_\alpha(N)$.
- If $N = \Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$, there are $(P_i)_i$ such that $M \to_h^* \Sigma_i \bar{\tau}_{\alpha_i}(P_i)$ and $P_i \Rightarrow_{st} Q_i$ for all i. By IH, for all i, $P_i \to_h^* P_i' \to_{\not h}^* Q_i$ for some $P_i' \in \Lambda_{\tau(D)}$. For all i, if P_i' is not a sum (with $n \neq 1$ arguments) we set $P_i'' = P_i'$, otherwise there is a first sum P_i'' such that $P_i \to_h^* P_i'' \to_h^* P_i'$.
  Then, using Rule (h-$c\bar{\tau}$) we have, for all i, $\bar{\tau}_{\alpha_i}(P_i) \to_h^* \bar{\tau}_{\alpha_i}(P_i'') \to_{\not h}^* \bar{\tau}_{\alpha_i}(Q_i)$.
  Thus, using Rule (h-cs), we have $M \to_h^* \Sigma_i \bar{\tau}_{\alpha_i}(P_i) \to_h^* \Sigma_i \bar{\tau}_{\alpha_i}(P_i'') \to_{\not h}^* \Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$.
- If $Q = \Pi_i(Q_i)$ then the argument is similar:
  There are $(P_i)_i$ such that $P \to_h^* \Pi_i P_i$ and $P_i \Rightarrow_{st} Q_i$ for all i. By IH, for all i, $P_i \to_h^* P_i' \to_{\not h}^* Q_i$ for some $P_i' \in \Lambda_{\tau(D)}$. For all i, if P_i' is not a sum (with $n \neq 1$ arguments) we set $P_i'' = P_i'$, otherwise there is a first sum P_i'' such that $P_i \to_h^* P_i'' \to_h^* P_i'$.
  Then, using Rule (h-$c\cdot$), we have $P \to_h^* \Pi_i P_i \to_h^* \Sigma_i P_i'' \to_{\not h}^* \Sigma_i Q_i$.
- If $Q = \Sigma_i(Q_i)$, there are $(P_i)_i$ such that $P \to_h^* \Sigma_i P_i$ and $P_i \Rightarrow_{st} Q_i$ for all i. By IH, for all i, $P_i \to_h^* P_i' \to_{\not h}^* Q_i$ and, by Rule (h-$\cdot$), $\Sigma_i P_i \to_h^* \Sigma_i P_i' \to_{\not h}^* \Sigma_i Q_i$.

□

Lemma 2.27. Ultimately, sums will necessarily commutes with $\bar{\tau}$, with products and with τ:

(1) If $P \to_h^* \Sigma_{j\le k} Q_j$, then there for all $j \le k$ is $P_j \to_h^* Q_j$ such that
$$\bar\tau_\alpha(P) \to_h^* \Sigma_{j\le k}\bar\tau_\alpha(P_j).$$
(2) Similarly, if $P \to_h^* \Sigma_{j\le k} Q_j$, then for all $j \le k$, there is $P_j \to_h^* Q_j$ such that
$$Q{\cdot}P \to_h^* \Sigma_j(Q{\cdot}P_j).$$
(3) Similarly, if $M \to_h^* \Sigma_{j\le k}\bar\tau_{\beta_j}(Q_j)$, then for all $j \le k$, there is $P_j \to_h^* Q_j$ such that
$$\tau_\alpha(M) \to_h^* \Sigma_{\{j|\beta_j\ge\alpha\}} P_j.$$

Proof. The proof follows the exact same pattern for each cases.

(1) Let $P \to_h^n \Sigma_{j\le k} Q_j$. The proof is by induction on the lexicographically ordered (n, P).
- If $n = 0$ then this is Rule $(\bar\tau+)$.
- If $P = \Sigma_{i\le k'} P'_i$ with $k' \ne 1$, there is a surjective $\phi : [1, k] \to [1, k']$ such that $P'_i \to_h^{n_i} \Sigma_{j\in\phi^{-1}(i)} Q_j$ with $n = \Sigma_i n_i$. By IH on each P'_i, there are $(P_j)_{j\in\phi^{-1}(i)}$ such that, for all $i \le k'$, $\bar\tau_\alpha(P'_i) \to_h^* \Sigma_{j\in\phi^{-1}(i)}\bar\tau_\alpha(P_j)$ with $P_j \to_h^* Q_j$. Thus $\bar\tau_\alpha(P) \xrightarrow{\bar\tau+}_h \Sigma_{i\le k'}\bar\tau_\alpha(P'_i) \to_h^* \Sigma_{i\le k'}\Sigma_{j\in\phi^{-1}(i)}\bar\tau_\alpha(P_j)$.
- Otherwise, we can decompose the reduction by $P \to_h P' \to_h^{n-1} \Sigma_{j\le k} Q_j$. Since P is not a sum we can apply the rule $H\text{-}c\bar\tau$ so that $\bar\tau_\alpha(P) \to_h \bar\tau_\alpha(P')$ and we conclude since by IH, $\bar\tau_\alpha(P') \to_h^* \Sigma_{j\le k}\bar\tau_\alpha(P_j)$.

(2) Let $P \to_h^n \Sigma_{j\le k} Q_j$. The proof is by induction on the lexicographically ordered (n, P).
- If $n = 0$ then this is Rule $(\cdot+)$.
- If $P = \Sigma_{i\le k'} P'_i$ with $k' \ne 1$, there is a surjective $\phi : [1, k] \to [1, k']$ such that $P'_i \to_h^{n_i} \Sigma_{j\in\phi^{-1}(i)} Q_j$ with $n = \Sigma_i n_i$. By IH on each P'_i, there are $(P_j)_{j\in\phi^{-1}(i)}$ such that, for all i, $(Q{\cdot}P_i) \to_h^* \Sigma_{j\in\phi^{-1}(i)}(Q{\cdot}P_j)$ with $P_j \to_h^* Q_j$. Thus $Q{\cdot}P \xrightarrow{\cdot+}_h \Sigma_{i\le k'}(Q{\cdot}P_i) \to_h^* \Sigma_{i\le k'}\Sigma_{j\in\phi^{-1}(i)}(Q{\cdot}P_j)$.
- Otherwise, we can decompose the reduction by $P \to_h P' \to_h^{n-1} \Sigma_{j\le k} Q_j$. Since P is not a sum we can apply the rule $H\text{-}c{\cdot}$ so that $Q{\cdot}P \to_h Q{\cdot}P'$ and we conclude since by IH, $Q{\cdot}P' \to_h^* \Sigma_{j\le k} Q{\cdot}P_j$.

(3) Let $M \to_h^n \Sigma_{j\le k}\bar\tau_\alpha(Q_j)$. The proof is by induction on the lexicographically ordered (n, M):
- If $n = 0$ then this is Rule $(\tau\bar\tau)$.
- If $M = \Sigma_{i\le k'}\bar\tau_{\gamma_i}(P'_i)$ with $k' \ne 1$, there is a surjective $\phi : [1, k] \to [1, k']$ such that $\bar\tau_{\gamma_i}(P'_i) \to_h^{n_i} \Sigma_{j\in\phi^{-1}(i)}\bar\tau_{\beta_j} Q_j$ with $n = \Sigma_i n_i$. By IH on each $\bar\tau_{\gamma_i}(P'_i)$, there are $(P_j)_{j\le\phi^{-1}(i)}$ such that, for all i, $\tau_\alpha(\bar\tau_{\gamma_i}(P'_i)) \to_h^* \Sigma_{\{j\in\phi^{-1}(i)|\alpha\le\beta_j\}} P_j$ with $P_j \to_h^* Q_j$. Since the only head reduction that can be applied on each $\tau_\alpha(\bar\tau_{\gamma_i}(P'_i))$ is $(h\text{-}\tau\bar\tau)$, we have that $\tau_\alpha(M) \to_h \Sigma_{\{1|\alpha\le\gamma_i\}} P_i \to_h^* \Sigma_j Q_j$.
- The case $M = \lambda x.M'$ is impossible since $M \to^* \Sigma_j\bar\tau_{\beta_j}(Q_j)$ and no rule can erase a λ in first position.
- Otherwise, we can decompose the reduction by $M \to_h M' \to_h^{n-1} \Sigma_{j\le k}\bar\tau_{\beta_j}(Q_j)$. Since M is not an abstraction we can apply the rule $(h\text{-}\tau)$ so that $\tau_\alpha(M) \to_h \tau_\alpha(M')$ and we conclude since by IH, $\tau_\alpha(M') \to_h^* \Sigma_{\{j|\beta_j\ge\alpha\}} P_j$.

□

Lemma 2.28. For all $M, N, N' \in \Lambda_{\tau(D)}$ such that $M \Rightarrow_{st} N \to N'$, there is M' such that $M \Rightarrow_{st} N'$. Similarly, for all $P, Q, Q' \in \boldsymbol{T}_{\tau(D)}$ such that $P \Rightarrow_{st} Q \to Q'$, there is P' such that $P \Rightarrow_{st} Q'$.

Proof. We proceed by structural induction on N:
- The case $N = x$ is impossible since x is a normal form.

- If $N = \lambda x.N_0$ then $N_0 \to N_0'$ with $N' = \lambda x.N_0'$. By definition of $\Rightarrow_{st}$, $M \to_h^* \lambda x.M_0$ and $M_0 \Rightarrow_{st} N_0$. By IH, $M_0 \Rightarrow_{st} N_0'$, thus $M \Rightarrow_{st} \lambda x.N_0'$.
- The case $N = \mathbf{0}$ is impossible since $\mathbf{0}$ is a normal form.
- If $N = \bar{\tau}_\alpha(Q)$ then the only rule that can change the form of the expression is $(\bar{\tau}+)$ applied in head position:
  - Either $N = \bar{\tau}_\alpha(\Sigma_j Q_j) \xrightarrow{\bar{\tau}+}_h N' = \Sigma_j \bar{\tau}_\alpha(Q_j)$. By definition of $\Rightarrow_{st}$, $M \to_h^* \bar{\tau}_\alpha(P)$ and $P \to_h^* \Sigma_j P_j$ with $P_j \Rightarrow_{st} Q_j$. Thus, by Lemma **??**, there is $(P_j')_j$ such that $M \to_h^* \Sigma_j \bar{\tau}_\alpha(P_j')$ with $P_j' \to^* P_j \Rightarrow_{st} Q_j$, so that $M \Rightarrow_{st} N'$.
  - Otherwise, $Q \to Q'$ and $N' = \bar{\tau}_\alpha(Q')$. In this case, since $M \to_h^* \bar{\tau}_\alpha(P)$ and $P \Rightarrow_{st} Q \to Q'$, we can apply the IH so that $P \Rightarrow_{st} Q'$ and $M \Rightarrow_{st} \bar{\tau}_\alpha(Q')$.
- Let $N = \Sigma_{i\leq n} N_i$ with $n > 0$. Then, modulo commutativity of the sum, we can assume that $N_n \to N_n'$, so that $N' = \Sigma_{i<n} N_i + N_n'$. By definition of $\Rightarrow_{st}$, $M \to_h^* \Sigma_{i\leq n} M_i$ with $M_i \Rightarrow_{st} N_i$. By induction hypothesis, $M_n \Rightarrow_{st} N_n'$ and we can set $M \Rightarrow_{st} N'$.
- Let $N = N_1\ N_2$, then $M \to_h^* M_1\ M_2$ with $M_1 \Rightarrow_{st} N_1$ and $M_2 \Rightarrow_{st} N_2$. There are different cases:
  - Either $N_1 \to_h N_1'$ and $N' = N_1'\ N_2$. In this case, the IH on $M_1 \Rightarrow_{st} N_1 \to_h N_1'$ gives $M_1 \Rightarrow_{st} N_1'$, so that $M \Rightarrow_{st} N'$.
  - Or $N_2 \to_h N_2'$ and $N' = N_1\ N_2'$. In this case, the IH on $M_2 \Rightarrow_{st} N_2 \to_h N_2'$ gives $M_2 \Rightarrow_{st} N_2'$, so that $M \Rightarrow_{st} N'$.
  - Or $N_1 = \lambda x.N_0$ and $N' = N_0[N_2/x]$. By definition of $\Rightarrow_{st}$, $M_1 \to_h^* \lambda x.M_0$ with $M_0 \Rightarrow_{st} N_0$. By easy induction on $\Rightarrow_{st}$, one can see that $M_0[M_2/x] \Rightarrow_{st} N_0[N_2/x]$. We can conclude since $\to^*\Rightarrow_{st}\subseteq\Rightarrow_{st}$.
  - Or $N_1 = \Sigma_{i\leq n} \bar{\tau}_{a_i\to\alpha_i}(Q_i)$ and $N' = \Sigma_{i\leq n} \bar{\tau}_{\alpha_i}(Q_i \cdot \Pi_{\gamma\in a_i} \tau_\gamma(N_2))$. By definition of $\Rightarrow_{st}$, $M_1 \to_h^* \Sigma_{i\leq n} \bar{\tau}_{a_i\to\alpha_i}(P_i)$ and $P_i \Rightarrow_{st} Q_i$ for all i. By definition of $\Rightarrow_{st}$, one can see that $\Sigma_{i\leq n} \bar{\tau}_{\alpha_i}(P_i \cdot \Pi_{\gamma\in a_i} \tau_\gamma(M_2)) \Rightarrow_{st} \Sigma_{i\leq n} \bar{\tau}_{\alpha_i}(\Pi_{\gamma\in a_i} \tau_\gamma(N_2))$ so that $M \Rightarrow_{st} N$.
- If $Q = \tau_{a\to\alpha}(N)$, then $P \to_h^* \tau_{a\to\alpha}(M)$ with $M \Rightarrow_{st} N$ and there are different cases:
  - Either $N \to N'$ and $Q' = \tau_{a\to\alpha}(N')$. In this case, the IH on $M \Rightarrow_{st} N \to N'$ gives $M \Rightarrow_{st} N'$, so that $P \Rightarrow_{st} Q'$.
  - Or $N = \lambda x.N_0$ and $Q' = \tau_\alpha(N_0[\bar{\boldsymbol{\epsilon}}_a/x])$. By definition of $\Rightarrow_{st}$, $M \to_h^* \lambda x.M_0$ with $M_0 \Rightarrow_{st} N_0$. By easy induction on $\Rightarrow_{st}$, one can see that $M_0[\bar{\boldsymbol{\epsilon}}_a/x] \Rightarrow_{st} N_0[\bar{\boldsymbol{\epsilon}}_a/x]$. We can conclude since $\to^*\Rightarrow_{st}\subseteq\Rightarrow_{st}$.
  - Or $N = \Sigma_{i\leq n} \bar{\tau}_{\beta_i}(Q_i)$ and $N' = \Sigma_{i\leq n|\beta_i\geq\alpha} Q_i$. By definition of $\Rightarrow_{st}$, $M \to_h^* \Sigma_{i\leq n} \bar{\tau}_{\beta_i}(P_i)$ and $P_i \Rightarrow_{st} Q_i$ for all i. By Lemma **??**, there is $(P_i')_i$ such that $\tau_\alpha(M) \to_h^* \Sigma_{i\leq n|\beta_i\geq\alpha} P_i'$ and $P_i' \Rightarrow_{st} Q_i$ so that $P \Rightarrow_{st} Q'$.
- If $Q = \Sigma_{i\leq n} Q_i$ then (up to commutativity of the sum) $Q_n \to Q_n'$ and $Q' = \Sigma_{i<n} Q_i + Q_n'$. By definition of $\Rightarrow_{st}$, $P \to_h^* \Sigma_{i\leq n} P_i$ with $P_i \Rightarrow_{st} Q_i$ for all i. By IH on $P_n \Rightarrow_{st} Q_n \to Q_n'$, $P_n \Rightarrow_{st} Q_n'$ so that $P \Rightarrow_{st} Q'$.
- If $Q = \Pi_{i\leq n} Q_i$ then the only rule that changes the form of the expression is $(\cdot+)$ applied in head position. There are two cases:
  - Either $Q = \Pi_i \Sigma_{j\leq k_i} Q_{ij} \xrightarrow{\cdot+}_h Q' = \Sigma_{(j_i)_i} \Pi_i Q_{ij_i}$. By definition of $\Rightarrow_{st}$ (used 2 times), $P \to_h^* \Pi_i P_i$ and $P_i \to_h^* \Sigma_{j\leq k_i} P_{ij}$ with $P_{ij} \Rightarrow_{st} Q_{ij}$ for all i, j. Thus, by Lemma **??**, there is $(P_{ij}')_{ij}$ such that $P \to_h^* \Sigma_{(j_i)_i} \Pi_i P_{ij}'$ with $P_{ij}' \to^* P_{ij} \Rightarrow_{st} Q_{ij}$, so that $M \Rightarrow_{st} N'$.
  - Otherwise (and up to commutativity of the sum), $Q_n \to Q_n'$ and $Q' = \Pi_{i<n} Q_i \cdot Q_i'$. By definition of $\Rightarrow_{st}$, $P \to_h^* \Pi_i P_i$ and $P_i \Rightarrow_{st} Q_i$. We can apply the IH on $P_n \Rightarrow_{st} Q_n \to_h Q_n'$, so that $P \Rightarrow_{st} Q'$.

□

Theorem 2.29 (Standardization)**.** *For any reduction $M \to^* N$ (resp. $P \to^* Q$), there is a standard reduction $M \Rightarrow_{st} N$ (resp. $P \Rightarrow_{st} Q$). In particular, any term M (resp. test Q) head converges iff it reduces to a may head-normal form:*

$$M\Downarrow^{\mathrm{h}} \quad\Leftrightarrow\quad \exists N \in mhnf, M \to^* N' \qquad\qquad P\Downarrow^{\mathrm{h}} \quad\Leftrightarrow\quad \exists Q \in mhnf, P \to^* Q'.$$

Proof. By applying successively Lemma **??**. The equivalence between $\Downarrow^{\mathrm{h}}$ and having a may-head-normal form is an immediate consequence once noticed that whenever $M \to_{\not h} M'$ then $M \in$ mhnf iff $M' \in mhnf$. □

2.4. **Invariance for the convergence.**

We will see in this section that the head convergence in at most n steps is invariant wrt the reduction. This means that performing a non-head reduction can only reduce the length of convergence.

Theorem 2.30 (Invariance for the convergence)**.** *For any terms $M \to N$ (resp. test $P \to Q$) and any $n \in \mathbb{N}$:*

$$M\Downarrow^{\mathrm{h}}_n \quad\Rightarrow\quad N\Downarrow^{\mathrm{h}}_n \qquad\qquad P\Downarrow^{\mathrm{h}}_n \quad\Rightarrow\quad Q\Downarrow^{\mathrm{h}}_n$$

Proof. By recursive invocations of Lemma **??**, for any k we can close the diagrams:

$$\begin{array}{ccc} M & \to_h & M_1 \\ \downarrow_k & \searrow & \downarrow_* \\ M_2 & \to_h^? & M' \end{array} \qquad\qquad \begin{array}{ccc} Q & \to_h & Q_1 \\ \downarrow_k & \searrow & \downarrow_* \\ Q_2 & \to_h^? & Q' \end{array}$$

where $\to_h^? := (\to_h \cup id)$ is either a head reduction or an equality.
Recursively invoking this diagrams, for any n we can now close the diagrams:

$$\begin{array}{ccc} M & \to_h^n & M_1 \\ \downarrow_* & \searrow & \downarrow_* \\ M_2 & \to_h^{\leq n} & M' \end{array} \qquad\qquad \begin{array}{ccc} Q & \to_h & Q_1 \\ \downarrow_* & \searrow & \downarrow_* \\ Q_2 & \to_h^{\leq n} & Q' \end{array}$$

where $\to_h^{\leq n} = \bigcup_{i\leq n} \to_h^i$ represents at most n iterations of $\to$.
In particular, if $M \to_h^* M'$ with $M' \in mhnf$ (*i.e.* M converges), since $M \to N$, there is N_0 such that $N \to_h^{\leq n} N_0$ and $N \to^* N_0$, from the last we deduce that $N_0 \in mhnf$ and conclude. The same goes for tests. □

In order to prove this theorem we need a stronger notion of confluence for the cases where one of the reduction is a head reduction.

Lemma 2.31. Any pick, $M \to_h M_1$ and $M \to M_2$ (resp. $Q \to_h Q_1$ and $Q \to Q_2$), between a head reduction and any reduction verifies the diamond:

$$\begin{array}{ccc} M & \to_h & M_1 \\ \downarrow & \searrow & \downarrow_* \\ M_2 & \to_h^? & M' \end{array} \qquad\qquad \begin{array}{ccc} Q & \to_h & Q_1 \\ \downarrow & \searrow & \downarrow_* \\ Q_2 & \to_h^? & Q' \end{array}$$

where $\to_h^? := (\to_h \cup id)$ is either a head reduction or an equality.

Proof. By induction on M and Q:

- The cases $M = x$ and $M = \mathbf{0}$ are impossible since $M \to_h M_1$.

- If $M = \lambda x.N$: then $M_1 = \lambda x.N_1$ and $M_2 = \lambda x.N_2$ so that $N_1 {}_h\!\leftarrow N \to N_2$, thus, by induction, there is N' such that $N_1 \to^* N' \,{}^?_h\!\leftarrow N_2$, finally we can choose $M' = \lambda x.N'$.
- If $M = \Sigma_{i\leq n+2} N^i$: then, modulo commutativity of the sum, $M_1 = N_1^{n+2} + \Sigma_{i\leq n+1} N^i$ with $N^{n+2} \to_h N_1^{n+2}$.
  - Either (modulo commutativity of the sum), $M_2 = N_2^{n+2} + \Sigma_{i\leq n+1} N^i$ with $N^{n+2} \to N_2^{n+2}$ and by induction there is $N_1^{n+2} \to^* N_0^{n+2} \,{}^?_h\!\leftarrow N_2^{n+2}$ such that $M' = N_0^{n+2} + \Sigma_{i\leq n+1} N^i$.
  - Or (modulo commutativity of the sum), $M_2 = N^{n+2} + N_2^{n+1} + \Sigma_{i\leq n+1} N^i$ with $N^{n+1} \to N_1^{n+1}$, so that $M' = N_1^{n+2} + N_2^{n+1} + \Sigma_{i\leq n+1} N^i$.
- If $M = \bar\tau_\alpha(Q)$ with Q that is not a sum: then $M_1 = \bar\tau_{\alpha_i}(Q_1)$ and $M_2 = \bar\tau_\alpha(Q_2)$ with $Q_1 {}_h\!\leftarrow Q \to Q_2$, thus, by induction, there is Q' such that $Q_1 \to^* Q' \,{}^?_h\!\leftarrow Q_2$, finally we can fix $M' = \bar\tau_\alpha(Q')$.
- If $M = \bar\tau_\alpha(\Sigma_{i\leq n+1} Q^i)$ and $M_1 = \Sigma_{i\leq n+1} \bar\tau_\alpha(Q^i)$:
  - Either $M_2 = \bar\tau_\alpha(Q_2^{n+1} \Sigma_{i\leq n} Q^i)$ and $M' = \bar\tau_\alpha(Q^{n+1}) \Sigma_{i\leq n} \bar\tau_\alpha(Q^i)$.
  - Or $Q^i = \Sigma_j P^{i,j}$ and $M_2 = \Sigma_j \bar\tau_\alpha(P^{i,j})$, then $M' = \Sigma_{i,j} \bar\tau_\alpha(P^{i,j})$.
- If $M = N\ L$:
  - If N is not an abstraction: then $M_1 = N_1\ L$ with $N \to_h N_1$. Moreover
    * Either $M_2 = N_2\ L$ with $N \to N_2$ and N_2 that is not an abstraction. By induction there is N' such that $N_1 \to^* N' \,{}^?_h\!\leftarrow N_2$, and $M' = N'\ L$.
    * Or $M_2 = (\lambda x.N_2)\ L$ with $N \to N_2$ and N_2 that is an abstraction: since N is not an abstraction, this can only be the result of a (β) or a $\bar\tau$ reduction in outermost position in N. In both cases, necessary $M_1 = M_2$.
    * Or $M_2 = N\ L_2$ with $L \to L_2$: then $M' = N_1\ L_2$.
  - If $N = \lambda x.N'$: then $M_1 = N'[L/x]$ and
    * Either $M' = M_2 = M_1$.
    * Or $M_2 = (\lambda x.N_2')\ L$ with $N' \to N_2$, thus $M' = N_2'[L/x]$.
  - If $N = \Sigma_i \bar\tau_{\alpha_i}(Q_i)$: idem.
- If $Q = \tau_\alpha(M)$:
  - If M is not an abstraction: then $Q_1 = \tau_\alpha(M_1)$ and $Q_2 = \tau_\alpha(M_2)$ with $M_1 {}_h\!\leftarrow M \to M_2$ and by induction hypothesis, there is M' so that $M_1 \to^* M' \,{}^?_h\!\leftarrow M_2$.
    * Either M_2 is not an abstraction and $Q' = \tau_\alpha(M')$.
    * Or $M \to M_2$ is an abstraction created by a (β) or a $(\bar\tau)$ outermost reduction. In both cases, necessary $M_1 = M_2$.
  - If $M = \lambda x.N$: then $Q_1 = \tau_{\alpha'}(N[\bar{\boldsymbol\epsilon}_a/x])$ and
    * Either $Q_2 = Q_1 = Q'$.
    * Or $Q_2 = \tau_\alpha \lambda x.N_2$ with $N \to N_2$, thus $Q' = \tau_{\alpha'}(N_2[\bar{\boldsymbol\epsilon}_a/x])$.
  - If $M = \Sigma_{i\leq n+1} \bar\tau_{\beta_i}(P^i)$: then $Q_1 = \Sigma_{\{i\leq n+1 | \alpha\leq\beta_i\}} P^i$ and
    * Either $Q_2 = Q_1 = Q'$.
    * Or $Q_2 = \tau_\alpha\Big(\Sigma_{i\leq n} \bar\tau_{\alpha_i}(P^i) + \Sigma_j \bar\tau_{\beta_n}(R^j)\Big)$ with $\bar\tau_{\alpha_{n+1}}(P^{n+1}) \to \Sigma_j \bar\tau_{\beta_j}(R^j)$, thus $Q' = \Sigma_{\{j|\alpha\leq\beta_n\}} R^j + \Sigma_{\{i\leq n|\alpha\leq\beta_i\}} P^i$.
- If $Q = P{+}R$: then, modulo commutativity of the sum, $Q_1 = P_1{+}R$ with $P \to_h P_1$.
  - Either $Q_2 = P_2{+}R$ with $P \to P_2$ and the induction hypothesis gives P' so that $M' = P'{+}R$.
  - Or $Q_2 = P{+}R_2$ and $M' = P_1{+}R_2$.
- If $Q = P{\cdot}R$: same as for $Q = P{+}R$ except if a rule $(\cdot{+})$ is used in outermost position. In this case, either only one of the reduction is a $(\cdot{+})$ and the two reductions are independents, or both of them are $(\cdot{+})$, which is similar to the case $M = \bar\tau_\alpha(\Sigma_{i\leq n+1} Q^i)$.

□

3. Proof

3.1. **Hyperimmunity implies full abstraction.**

In this subsection we show that if D is sensible for $\Lambda_{\tau(D)}$ and is hyperimmune, D is inequationally fully abstract for Λ, that is Theorem **??**. We use the full abstraction of D for $\Lambda_{\tau(D)}$ of Theorem **??** (or rather its technical counterpart: Theorem **??**) in order to express the problem in a purely syntactical form:

$$\begin{aligned}
[\![M]\!] \neq [\![N]\!] &\iff \exists \alpha \in P, \quad \alpha \in [\![M]\!] - [\![N]\!] && \text{or conv.}\\
&\overset{(1)}{\iff} \exists \alpha \in P, \quad \tau_\alpha(M)\Downarrow \text{ and } \tau_\alpha(N)\Uparrow && \text{or conv.}\\
&\overset{(2)}{\implies} \exists C \in \Lambda^{(\!|\cdot|\!)}, C(\!|M|\!)\Downarrow \text{ and } C(\!|N|\!)\Uparrow && \text{or conv.}\\
&\iff M \not\equiv_{\mathcal{H}^*} N
\end{aligned}$$

Here (1) is given by Theorem **??** so that we only have to prove (2) which is done in the proof of Theorem **??** by induction on the finite reduction $\tau_\alpha(M)\Downarrow$. However, the proof require a specific treatment of the case where $M = \boldsymbol{I}$ (we have some $\eta\infty$-ex pensions issues) this is the purpose of the key-lemma (Lemma **??**). This key-lemma is assuming that (2) is false for $M = \boldsymbol{I}$ (and any N) then co-inductively constructs a counterexample $(\alpha_n)_n$ the hyperimmunity by unfolding $\tau_\alpha(N)\Uparrow$.

Before that, we need the technical Lemma **??** in order to refute the operational equivalence between two λ-terms in easy cases.

Lemma 3.1 ([**?**])**.** Let $M = \lambda x_1...x_n.y\ M_1 \cdots M_k \in \Lambda$ and let $N = \lambda x_1...x_{n'}.y'\ N_1 \cdots N_{k'} \in \Lambda$ be λ-terms such that $M \sqsubseteq_{\mathcal{H}^*} N$. Then:

(1) $y = y'$,
(2) $n - k = n' - k'$,
(3) if $i \leq k$ and $i \leq k'$ then $M_i \sqsubseteq_{\mathcal{H}^*} N_i$,
(4) if $i > k$ and $i \leq k'$ then $x_{i-k} \sqsubseteq_{\mathcal{H}^*} N_i$,
(5) if $i \leq k$ and $i > k'$ then $M_{i-k} \sqsubseteq_{\mathcal{H}^*} x_i$.

Proof. From each $i \leq 5$, assuming statements (1)...(i-1) and refuting statement (i), we can exhibit a context $C \in \Lambda^{(\!|\cdot|\!)}$ such that $C[\![M]\!]\Downarrow^{\mathrm{h}}$ and $C[\![N]\!]\Uparrow^{\mathrm{h}}$. □

3.1.1. *The key-lemma.*

From now on, we consider an extensional K-model D that is hyperimmune and sensible for $\Lambda_{\tau(D)}$. The following lemma is a key lemma that introduces the hyperimmunity in the picture. It basically states that if $\tau_\alpha(N[\bar{\boldsymbol{\epsilon}}_\alpha/x_0])\Uparrow^{\mathrm{h}}$ then $N \not\sqsupseteq_{\mathcal{H}^*} x_0$.

Lemma 3.2. Let $\alpha \in D$ and $a_0, \dots, a_k \in \mathcal{A}_f(D)$ be such that $\alpha \in a_0$.
Let $N \in \Lambda$ and $x_0, \dots, x_k$ be such that $\tau_\alpha(N[\bar{\boldsymbol{\epsilon}}_{a_0}/x_0, \dots, \bar{\boldsymbol{\epsilon}}_{a_k}/x_k])\Uparrow^{\mathrm{h}}$. Then $N \not\sqsupseteq_{\mathcal{H}^*} x_0$.

Proof. We define the recursive function $g_{N'}$ for any $N' \in \Lambda$ such that $N' \sqsupseteq_{\mathcal{H}^*} x_0$, it is done by recursively defining $g_{N'}(k)$ for $k \in \mathbb{N}$:
Since $N' \sqsupseteq_{\mathcal{H}^*} x_0$, N' is converging, and by Lemma **??** $N' \to_h^* \lambda y_1...y_n.x_0\ N_1 \cdots N_n$ with $N_m \sqsupseteq_{\mathcal{H}^*} y_m$ for all $m \leq n$. We then define $g_{N'}(0) = n$ and $g_{N'}(k+1) = max_{i \leq n} g_{N_i}(k)$.
We will show that assuming $N \sqsupseteq_{\mathcal{H}^*} x_0$ contradicts the hyperimmunity of D by showing that:

There exists $(\alpha_n)_{n\geq 0}$ with $\alpha_0 = \alpha$ and for all n, $\alpha_n = a_1^n \to \cdots a_{g_N(n)}^n \to \alpha'_n$ and $\alpha_{n+1} \in \bigcup_{i \leq g_N(n)} a_i^n$.

We are constructing $(\alpha_n)_n$ by co-induction.
Since $N \sqsupseteq_{\mathcal{H}^*} x_0$, it is converging, and by Lemma **??**, $N \to^* \lambda y_1...y_n.x_0\ N_1 \cdots N_n$ with $N_m \sqsupseteq_{\mathcal{H}^*} y_m$ for all $m \leq n$.
We will assume that $\alpha = b_1 \to \cdots \to b_n \to \alpha'$ and $a_0 = \{\alpha, \beta_1, \ldots, \beta_t\}$ with $\beta_i = c_1^i \to \cdots \to c_n^i \to \beta'_i$ (always possible since "$\to$" is a bijection).
Then (notice the use of a calculation done in Example **??**)

$$\begin{aligned}
\tau_\alpha(N[s]) &\to^* \tau_\alpha(\lambda y_1...y_n.\bar{\epsilon}_{a_0}\ N_1[s] \cdots N_n[s]) \\
&\overset{\tau}{\to}{}_h^* \tau_{\alpha'}(\bar{\epsilon}_{a_0}\ N_1[s,s'] \cdots N_n[s,s']) \\
&\overset{Ex??}{\longrightarrow}{}^* \tau_{\alpha'}(\Sigma_{d_1 \to \cdots d_n \to \delta \in a_0} \bar{\tau}_\delta(\Pi_{m\leq n} \Pi_{\gamma \in d_m} \tau_\gamma(N_m[s,s']))) \\
&\overset{\tau\bar{\tau}}{\to}{}_h \Pi_{m \leq n} \Pi_{\gamma \in b_m} \tau_\gamma(N_m[s,s']) \quad + \quad \Sigma_{\{i \leq t \mid \alpha' \leq \beta'_i\}} \Pi_{m \leq n} \Pi_{\gamma \in c_m^i} \tau_\gamma(N_m[s,s'])
\end{aligned}$$

with $[s] = [\bar{\epsilon}_{a_0}/x_0, \ldots, \bar{\epsilon}_{a_k}/x_k]$ and $[s'] = [\bar{\epsilon}_{b_1}/y_1, \ldots, \bar{\epsilon}_{b_n}/y_n]$.
Since $\tau_\alpha(N[s])$ diverges, by standardization theorem (Th. **??**), the test $\Pi_{m\leq n}\Pi_{\gamma \in b_m} \tau_\gamma(N_m[s,s'])$ diverges. In particular there is $m \leq n$ and $\gamma \in b_m$ such that $\tau_\gamma(N_m[s,s'])$ diverges.
Since $N_m \sqsupseteq_{\mathcal{H}^*} y_m$ and $\tau_\gamma(N_m[s,s'])\Uparrow^{\mathrm{h}}$, the co-induction gives $(\gamma_k)_k$ such that $\gamma_0 = \gamma$ and for all k, $\gamma_k = c_1^k \to \cdots c_{g_{N_m}(k)}^k \to \gamma'_k$ and $\gamma_{k+1} \in \bigcup_{i \leq g_{N_m}(k)} a_i^k$. In this case we can define $(\alpha_k)_k$ as follows:

$$\alpha_0 = \alpha \qquad\qquad \forall k,\ \alpha_{k+1} = \gamma_k$$

This is sufficient since:

$$m \leq n = g_N(0) \qquad\qquad g_{N_m}(k) \leq sup_{j \leq n} g_{N_j}(k) = g_N(k+1)$$

□

3.1.2. *Inequational completeness.*

Theorem 3.3 (Inequational full completeness)**.** *For all $M, N \in \Lambda$,*

$$M \sqsubseteq_{\mathcal{H}^*} N \qquad \Rightarrow \qquad [\![M]\!]^{\vec{x}} \subseteq [\![N]\!]^{\vec{x}}.$$

Proof. We will prove the equivalent (by Theorem **??**) statement:

Let $\alpha \in D$ and $a_0, \ldots, a_k \in \mathcal{A}_f(D)$.
Let $\{x_0, \ldots, x_k\} \supseteq \mathrm{FV}(M)$ be a set of variables, and let $[s] = [\bar{\epsilon}_{a_0}/x_0 \cdots \bar{\epsilon}_{a_k}/x_k]$.
If[6] $\tau_\alpha(M[s])\Downarrow_n^{\mathrm{h}}$ and $\tau_\alpha(N[s])\Uparrow^{\mathrm{h}}$ then $M \not\sqsubseteq_{\mathcal{H}^*} N$.

The statement is proved by induction on the length n of the reduction $\tau_\alpha(M[s])\Downarrow_n^{\mathrm{h}}$:

- The case $n = 0$:
  Then $\tau_\alpha(M[s])$ is in normal form without free variables, which is impossible.

[6]Recall that $M\Downarrow_n^{\mathrm{h}}$ means that M may-head converges in at most n steps

- The case $n \geq 1$:
Since $\tau_\alpha(M[s])\Downarrow^{\mathrm{h}}_n$, by applying the sensibility for $\Lambda_{\tau(D)}$, the interpretation of $\tau_\alpha(M[s])\Downarrow^{\mathrm{h}}_n$ is non empty. By Remark **??**, the interpretation of M is also non empty. Thus, reapplying the sensibility, M is converging to a head-normal form $M \to_h^* \lambda y_1...y_n.z\ M_1 \cdots M_m$. We can then make some assumptions:
  - We can assume that $N \to_h^* \lambda y_1...y_{n'}.z'\ N_1 \cdots N_{m'}$:
  In fact, if N does not converge then trivially $M \not\sqsubseteq_{\mathcal{H}^*} N$.
  - We can assume that $n' \geq n$:
  In fact, if $n'{<}n$ then we can always define $N' = \lambda y_1...y_{n'} y_{n'+1}...y_n.z'\ N_1 \cdots N_{m'}\ y_{n'+1} \cdots y_n$ (with $y_{n'+1}...y_n \notin \mathrm{FV}(z'\ N_1 \cdots N_{m'})$), and we would have $N' \equiv_{\mathcal{H}^*} N$ and $\tau_\alpha(N'[s])\Uparrow^{\mathrm{h}}$.
  - We can assume that $n{=}0$:
  In fact, let $a_0{\to}\cdots a_n{\to}\alpha'{=}\alpha$, $[s']{=}[\bar{\boldsymbol{\epsilon}}_{a_0}/y_1, \ldots, \bar{\boldsymbol{\epsilon}}_{a_n}/y_n]$, $N' = \lambda y_{n+1}...y_{n'}.z'\ N_1 \cdots N_{m'}$ and $M' = z\ M_1 \cdots M_m$. Since $\tau_\alpha(M[s]) \to^* \tau_{\alpha'}(M'[s,s'])$ (resp. $\tau_\alpha(N[s]) \to^* \tau_{\alpha'}(N'[s,s'])$), by confluence and standardization theorems (Th. **??** and Th.**??**), the convergences of $\tau_\alpha(M[s])$ (resp. $\tau_\alpha(N[s])$) and $\tau_{\alpha'}(M'[s,s'])$ (resp. $\tau_{\alpha'}(N'[s,s'])$) are equivalent. Applying Theorem **??**, we thus have $\tau_{\alpha'}(M'[s,s'])\Downarrow^{\mathrm{h}}_n$ and $\tau_{\alpha'}(N'[s,s'])\Uparrow^{\mathrm{h}}$.
  Moreover $M' \sqsubseteq_{\mathcal{H}^*} N' \Leftrightarrow M \sqsubseteq_{\mathcal{H}^*} N$ so that the property on M' and N' is equivalent to the same property on M and N.
  - We can assume that $z' = z = x_0$:
  Since $\{x_0 \ldots x_k\} \supseteq \mathrm{FV}(M)$, there is $j \leq k$ such that $z = x_j$, for simplicity we assume that $j = 0$. Then we can remark that by Item (**??**) of Lemma **??**, either $M \not\sqsubseteq_{\mathcal{H}^*} N$ or $z' = z = x_0$, we will thus continue with the second case.

Altogether we have:

$$M \to_h^* x_0\ M_1 \cdots M_m \qquad\qquad N \to_h^* \lambda y_1...y_{n'}.x_0\ N_1 \cdots N_{m'}$$

The case $M = x_0$ corresponds exactly to the hypothesis of Lemma **??** that concludes by $M = x_0 \not\sqsubseteq_{\mathcal{H}^*} N$. We are now assuming that $m \geq 1$.

By Lemma **??**, either $M \not\sqsubseteq_{\mathcal{H}^*} N$ or the following holds:
  - $m = m' - n'$, and in particular $m \leq m'$
  - for $i \leq m$, $M_i \sqsubseteq_{\mathcal{H}^*} N_i$
  - for $m < i \leq m'$, $y_{i-m} \sqsubseteq_{\mathcal{H}^*} N_i$.

We will assume that $m = m' - n'$ and then refute $M_i \sqsubseteq_{\mathcal{H}^*} N_i$ or $y_i \sqsubseteq_{\mathcal{H}^*} N_{m+i}$ for some $i \leq n'$; we then conclude that $M \not\sqsubseteq_{\mathcal{H}^*} N$.

In the following we unfold
  - $\alpha = b_1{\to}\cdots{\to}b_{n'}{\to}\alpha'$,
  - $a_0 = \{\beta_0 \ldots \beta_r\}$,
  - for all $t \leq r, \beta_t = c^t_1{\to}\cdots c^t_m{\to}\beta'_t$,
  - and for all $t \leq r, \beta'_t = c^t_{m+1}{\to}\cdots c^t_{m'}{\to}\beta''_t$.

Moreover we set $[s'] = [\bar{\boldsymbol{\epsilon}}_{b_1}/y_1 \ldots \bar{\boldsymbol{\epsilon}}_{b_{n'}}/y_{n'}]$.

Then we have:

$$\tau_\alpha(M[s]) \quad \to^* \quad \tau_\alpha(\bar{\boldsymbol{\epsilon}}_{a_0}\ M_1[s] \cdots M_m[s]) \tag{3.1}$$

$$\overset{\bar{\tau}}{\to}{}^m_h \overset{\tau\bar{\tau}}{\to}_h \quad \Sigma_{\{t \leq r \mid \alpha \leq \beta'_t\}} \Pi_{i \leq m} \Pi_{\gamma \in c^t_i} \tau_\gamma(M_i[s]). \tag{3.2}$$

By Theorem **??**, $\tau_\alpha(\bar{\boldsymbol{\epsilon}}_{a_0}\ M_1[s] \cdots M_m[s])\Downarrow^{\mathrm{h}}_n$. Moreover, since the head reduction (**??**) is prefix of any head reduction sequence starting from $\tau_\alpha(\bar{\boldsymbol{\epsilon}}_{a_0}\ M_1[s] \cdots M_m[s])$, the test $\Sigma_{\{t\leq r|\alpha\leq\beta'_t\}}\Pi_{i\leq m}\Pi_{\gamma\in c_i^t}\tau_\gamma(M_i[s])$ head converges in $(n-m-1)$ steps so that there exists $t_0\leq r$ such that $\alpha\leq\beta'_{t_0}$ and for all $i\leq m$ and all $\gamma\in c_i^{t_0}$, we have $M_i[s]{\Downarrow^{\mathrm{h}}}_{n-1}$.

Similarly we have:

$$\begin{array}{rll} \tau_\alpha(N[s]) & \to^* & \tau_\alpha(\lambda y_1...y_{n'}.\bar{\epsilon}_{a_0}\ N_1[s]\cdots N_{m'}[s]) \\ & \xrightarrow{\tau}{}^{n'} & \tau_{\alpha'}(\bar{\epsilon}_{a_0}\ N_1[s,s']\cdots N_{m'}[s,s'])) \\ & \xrightarrow{\bar\tau}{}^{m'} & \tau_{\alpha'}(\Sigma_{t\leq r}\bar\tau_{\beta''_t}(\Pi_{i\leq m'}\Pi_{\gamma\in c_i^t}\tau_\gamma(N_i[s,s']))) \\ & \xrightarrow{\tau\bar\tau} & \Sigma_{t\leq r|\alpha'\leq\beta''_t}\Pi_{i\leq m'}\Pi_{\gamma\in c_i^t}\tau_\gamma(N_i[s,s']). \end{array}$$

Thus, by standardization (Th. **??**), $\Sigma_{t\leq r|\alpha'\leq\beta''_t}\Pi_{i\leq m'}\Pi_{\gamma\in c_i^t}\tau_\gamma(N_i[s,s'])$ diverges. Thus there are two cases:

- Either $\alpha'\not\leq\beta''_{t_0}$: which is impossible since $\alpha\leq\beta'_{t_0}$.
- Or there are $i\leq m'$ and $\gamma\in c_i^{t_0}$ such that $\tau_\gamma(N_i[s,s'])$ diverges.
  - $*$ Either $i\leq m$:
    Then since $\tau_\gamma(M_i[s,s'])=\tau_\gamma(M_i[s]){\Downarrow^{\mathrm{h}}}_{n-1}$, the induction hypothesis yields that $M_i\not\sqsubseteq_{\mathcal{H}^*}N_i$.
  - $*$ Or $m<i$:
    Since $\alpha\leq\beta'_{t_0}$ we have $b_{i-m}\geq c_i^{t_0}$ and $\gamma\leq\gamma'\in b_{i-m}$. Moreover, using Theorem **??** and $\gamma\leq\gamma'$, we have that $\tau_{\gamma'}(N_i[s,s'])$ diverges. Thus we can apply Lemma **??** that results in $y_{i-m}\not\sqsubseteq_{\mathcal{H}^*}N_i$.

□

Theorem 3.4 (Hyperimmunity implies full abstraction)**.** *Any extensional K-model D that is hyperimmune and sensible for $\Lambda_{\tau(D)}$ is inequationally fully abstract for the pure λ-calculus.*

Proof. Inequational adequacy: inherited from the inequational sensibility of D for $\Lambda_{\tau(D)}$. Indeed, for any $M,N\in\Lambda$ and $C\in\Lambda^{(\!|\cdot|\!)}$, if $[\![M]\!]_D^{\vec{x}}\subseteq[\![N]\!]_D^{\vec{x}}$ and if $C(\!|M|\!)\Downarrow^{\mathrm{h}}$, then by sensibility $[\![C(\!|N|\!)]\!]_D^{\vec{x}}\supseteq[\![C(\!|M|\!)]\!]_D^{\vec{x}}\neq\emptyset$ and (still by sensibility) $[\![C(\!|N|\!)]\!]_D^{\vec{x}}$ converges.

Inequational completeness: for all $M,N\in\Lambda$ such that $[\![M]\!]^{\vec{x}}\not\subseteq[\![N]\!]^{\vec{x}}$, there is $(\vec{a},\alpha)\in[\![M]\!]^{\vec{x}}-[\![N]\!]^{\vec{x}}$, thus by Theorem **??**, $M\not\sqsubseteq_{\mathcal{H}^*}N$. □

3.2. Full abstraction implies hyperimmunity.

3.2.1. *The counterexample.*

In this section, we are assuming that D is sensible for $\Lambda_{\tau(D)}$ but is not hyperimmune. Then we will construct a counterexample $(\boldsymbol{J}_g\ \underline{0})$ for the full abstraction such that $(\boldsymbol{J}_g\ \underline{0})\equiv_{\mathcal{H}^*}\boldsymbol{I}$ and $[\![\boldsymbol{J}_g\ \underline{0}]\!]\neq[\![\boldsymbol{I}]\!]$ resulting in Theorem **??**.

By Definition **??**, if D is hyperimmune, then there exist a recursive $g:(\mathbb{N}\to\mathbb{N})$ and a family $(\alpha_n)_{n\geq 0}\in D^{\mathbb{N}}$ such that $\alpha_n=a_{n,1}\to\cdots\to a_{n,g(n)}\to\alpha'_n$ with $\alpha_{n+1}\in\bigcup_{k\leq g(n)}a_{n,k}$.

We will use the function g for defining a term $\boldsymbol{J}_g$ (Eq. **??**) such that $(\boldsymbol{J}_g\ \underline{0})$ is observationally equal to the identity in Λ (Lemma **??**) but can be distinguished in $\Lambda_{\tau(D)}$ (Cor. **??**). From this latter statement and the full abstraction for $\Lambda_{\tau(D)}$ (Th. **??**), we will obtain that $[\![\boldsymbol{J}_g\ \underline{0}]\!]_D\neq[\![\boldsymbol{I}]\!]_D$, and thus we conclude with Theorem **??**.

Let $(\boldsymbol{G}_n)_{n\in\mathbb{N}}$ be the sequence of closed λ-terms defined by:

$$\boldsymbol{G}_n:=\lambda uex_1...x_{g(n)}.e\ (u\ x_1)\ \cdots\ (u\ x_{g(n)}) \tag{3.3}$$

The recursivity of g implies that of the sequence $\boldsymbol{G}_n$. We can thus use the Proposition **??** that build $\boldsymbol{G} \in \Lambda$ such that:

$$\boldsymbol{G}\ \underline{n} \to^* \boldsymbol{G}_n. \tag{3.4}$$

Recall that $\boldsymbol{S}$ denotes the Church successor function and $\boldsymbol{\Theta}$ the Turing fixedpoint combinator. We define:

$$\boldsymbol{J}_g := \boldsymbol{\Theta}\ (\lambda uv.\boldsymbol{G}\ v\ (u\ (\boldsymbol{S}\ v))). \tag{3.5}$$

Then:

$$\boldsymbol{J}_g\ \underline{n} \to^* \boldsymbol{G}_n\ (\boldsymbol{J}_g\ \underline{n{+}1}). \tag{3.6}$$

Lemma 3.5. For all $n \in \mathbb{N}$, all $\alpha \in D$ and all $b = \{\beta_1, ..., \beta_k\} \subseteq D$, let:

- $\alpha = a_1 \to \cdots \to a_{g(n)} \to \alpha'$,
- for all $j \leq k, \beta_j = b_{j,1} \to \cdots \to b_{j,g(n)} \to \beta'_j$,

we have:

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\ \to^* \to_h\ \Sigma_{\{j\leq k \mid \alpha' \leq \beta'_j\}} \Pi_{i\leq g(n)} \Pi_{\gamma\in b_{j,i}} \tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i}).$$

Proof. We can reduce:

$$\begin{aligned}
\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b) &\overset{Eq(\mathbf{??})}{\longrightarrow}{}^* \quad \tau_\alpha(\boldsymbol{G}\ \underline{n}\ (\boldsymbol{J}_g\ \underline{n{+}1})\ \bar{\boldsymbol{\epsilon}}_b) \\
&\overset{Eq(\mathbf{??})}{\longrightarrow}{}^* \quad \tau_\alpha(\boldsymbol{G}_n\ (\boldsymbol{J}_g\ \underline{n{+}1})\ \ \bar{\boldsymbol{\epsilon}}_b) \\
&\overset{Eq(\mathbf{??})}{\longrightarrow}{}^* \quad \tau_\alpha\Big((\lambda ue\vec{x}^{g(n)}.e\ (u\ x_1)\ \cdots\ (u\ x_{g(n)}))\ (\boldsymbol{J}_g\ \underline{n{+}1})\ \ \bar{\boldsymbol{\epsilon}}_b\Big) \\
&\overset{\beta}{\to}{}_h^2 \quad \tau_\alpha\Big(\lambda\vec{x}^{g(n)}.\bar{\boldsymbol{\epsilon}}_b\ (\boldsymbol{J}_g\ \underline{n{+}1}\ x_1)\ \cdots\ (\boldsymbol{J}_g\ \underline{n{+}1}\ x_{g(n)})\Big) \\
&\overset{\tau}{\to}{}_h^{g(n)} \quad \tau_{\alpha'}\Big(\bar{\boldsymbol{\epsilon}}_b\ (\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_1})\ \cdots\ (\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_{g(n)}})\Big) \\
&\overset{\bar{\tau}}{\to}{}_h^{g(n)} \quad \tau_{\alpha'}(\Sigma_{j\leq k} \bar{\tau}_{\beta'_j}(\Pi_{i\leq g(n)} \Pi_{\gamma\in b_{j,i}} \tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_i))) \\
&\overset{\tau\bar{\tau}}{\to}{}_h \quad \Sigma_{\{j\leq k \mid \alpha' \leq \beta'_j\}} \Pi_{i\leq g(n)} \Pi_{\gamma\in b_{j,i}} \tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i})
\end{aligned}$$

□

Lemma 3.6. For all n, we have $\boldsymbol{J}_g\ \underline{n} \equiv_{\mathcal{H}^*} \boldsymbol{I}$.

Proof. Let D_∞ be defined as in Example **??**, it is fully abstract for $\mathcal{H}^*$.[7] It results that it is sufficient to verify that $[\![\boldsymbol{J}_g\ \underline{n}]\!]_{D_\infty} = [\![\boldsymbol{I}]\!]_{D_\infty}$, or equivalently (Th. **??**) to verify that :

$$\forall \alpha \in D_\infty, \tau_\alpha(\boldsymbol{J}_g\ \underline{n}) \Downarrow^{\mathrm{h}} \Leftrightarrow \tau_\alpha(\boldsymbol{I}) \Downarrow^{\mathrm{h}}.$$

Trivially $\tau_{a_0\to\alpha}(\boldsymbol{I})$ converges iff there is β such that $\alpha \leq \beta \in a_0$. Conversely we can prove by induction on a_0 that $\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_{a_0})$ converges iff there is β such that $\alpha \leq \beta \in a_0$ and conclude by extensionality.

If we denote $\alpha = a_1 \to \cdots \to a_{g(n)} \to \alpha'$, Lemma **??** gives that:

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_{a_0})\quad \to^* \to_h \quad \Sigma_{\{b_1\to\cdots b_{g(n)}\to\beta' \in a_0 \mid \alpha' \leq \beta'\}} \Pi_{i\leq g(n)} \Pi_{\gamma\in b_i} \tau_\gamma(\boldsymbol{J}_g\ \underline{n{+}1}\ \bar{\boldsymbol{\epsilon}}_{a_i}).$$

By induction hypothesis and standardisation, this test converges iff there is $\beta = b_1 \to \cdots b_{g(n)} \to \beta' \in a_0$ such that $\alpha' \leq \beta'$ and for all $i \leq g(n)$ and all $\gamma \in b_i$, $\gamma \leq \delta \in a_i$, *i.e.*, for all i, $b_i \leq a_i$. Equivalently,

[7] Notice that the full abstraction of D_∞ for $\mathcal{H}^*$, that has been proved for decade [**?**, **?**], can be recovered as we have seen in Example **??** that D_∞ is hyperimmune.

this test converges iff $\alpha \leq \beta \in a_0$. Thus, using the standardization (Th. **??**), $\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_{a_0})$ converges iff $\alpha \leq \beta \in a_0$. □

Lemma 3.7. For all $n \in \mathbb{N}$, all $\alpha \in D$ and all $b \in \mathcal{A}_f(D)$, if $\beta \not\geq \alpha$ for all $\beta \in b$, then:

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\Uparrow^{\mathrm{h}}$$

Proof. Let $\{\beta_1, ..., \beta_l\} = b$ and, for all $j \leq l$, let $b_{j,1} \to \cdots \to b_{j,l} \to \beta'_j = \beta_j$.
We are proving by induction on k that there is no convergence in k steps:[8]
We assume that $\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\Downarrow^{\mathrm{h}}_{k+1}$.
From Lemma **??**, we have:

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\ \ \to^*\to_h\ \ \Sigma_{\{j\leq l \mid \beta'_j\leq\alpha'\}}\Pi_{i\leq g(n)}\Pi_{\gamma\in b_{j,i}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n+1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$$

By Theorem **??**, and since the last head reduction was necessary, the resulting term converges in k steps. Thus there exists $j \leq l$ such that $\beta'_j \geq \alpha'$ and for all $i \leq g(n)$ and each $\gamma \in b_{j,i}$, $\tau_\gamma(\boldsymbol{J}_g\ \underline{n+1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$ converges in k steps.
Let $j \leq l$ be such that $\beta'_j \geq \alpha'$. Since $\beta_j \not\geq \alpha$, there is i such that $b_{j,i} \not\leq a_i$, *i.e.*, there is $\gamma \in b_{j,i}$ such that for all $\delta \in a_i$, $\gamma \not\leq \delta$ and by induction we get a contradiction to $\tau_\gamma(\boldsymbol{J}_g\ \underline{n+1}\ \bar{\boldsymbol{\epsilon}}_{a_i})\Downarrow^{\mathrm{h}}_k$. □

We recall that $(\alpha_n)_n$ is given by the counterexample of the hyperimmunity, and that for all n, $\alpha_n = a_{n,1} \to \cdots \to a_{n,g(n)} \to \alpha'_n$ and $\alpha_{n+1} \in \bigcup_{k\leq g(n)} a_{n,k}$.

Lemma 3.8. For any $n \in \mathbb{N}$ and any anti-chain $b = \{\alpha_n, \beta_1, ..., \beta_k\}$, then:

$$\tau_{\alpha_n}((\boldsymbol{J}_g\ \underline{n})\ \bar{\boldsymbol{\epsilon}}_b)\Uparrow^{\mathrm{h}}.$$

In particular, $\tau_{\alpha_0}(\boldsymbol{J}_g\ \underline{0}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0})\Uparrow^{\mathrm{h}}$.

Proof. We unfold $\beta_j = b_{j,1} \to \cdots \to b_{j,g(n)} \to \beta'_j$.
We are proving by induction on k that there is no convergence in k steps:[9]
We assume that $\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\Downarrow^{\mathrm{h}}_{k+1}$.
From Lemma **??**, we have:

$$\tau_\alpha(\boldsymbol{J}_g\ \underline{n}\ \bar{\boldsymbol{\epsilon}}_b)\quad \to^*\to_h\quad \Pi_{i\leq g(n)}\Pi_{\gamma\in a_{ni}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n+1}\ \bar{\boldsymbol{\epsilon}}_{a_i})\quad +\quad \Sigma_{\{j\leq l\mid \alpha'_n\leq\beta'_j\}}\Pi_{i\leq g(n)}\Pi_{\gamma\in b_{ji}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n+1}\ \bar{\boldsymbol{\epsilon}}_{a_i}).$$

By Theorem **??**, and since the last head reduction was necessary, the resulting term converges in k steps. Thus one of the addends should converges in k steps, however:

- The fist member $\Pi_{i\leq g(n)}\Pi_{\gamma\in a_{ni}}\tau_\gamma(\boldsymbol{J}_g\ \underline{n+1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$ does not since there is $i \leq g(n)$ such that $\alpha_{n+1} \in a_{ni}$ and by induction, $\tau_{\alpha_{n+1}}(\boldsymbol{J}_g\ \underline{n+1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$ cannot converges in k steps.
- The second member of the sum diverges by Lemma **??**.
  For any $j \leq l$ such that $\beta'_j \geq \alpha'_n$ we know that $\beta_j \not\geq \alpha_n$ since $\{\alpha_n, \beta_1, ..., \beta_l\}$ is an anti-chain. Thus there is always $i \leq g(n)$ such that $b_{j,i} \not\leq a_{n,i}$, *i.e.*, there is $\gamma \in b_{j,i}$ such that for all $\delta \in a_{n,i}$, $\gamma \not\leq \delta$. We can conclude by Lemma **??** that $\tau_\gamma(\boldsymbol{J}_g\ \underline{n+1}\ \bar{\boldsymbol{\epsilon}}_{a_i})$ diverges.

□

Theorem 3.9 (Full abstraction implies Hyperimmunity)**.** *If D is not hyperimmune, but sensible for $\Lambda_{\tau(D)}$, then it is not fully abstract for the λ-calculus.*

Proof. Since $\tau_{\alpha_0}(\boldsymbol{I}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0}) \overset{\beta}{\to}_h \overset{\tau\bar{\tau}}{\to}_h \epsilon$, we have that $[\![\tau_{\alpha_0}(\boldsymbol{I}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0})]\!]\neq\emptyset$, while by Lemma **??** we have that $[\![\tau_{\alpha_0}(\boldsymbol{J}_g\ \underline{0}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0})]\!]=\emptyset$, and thus $[\![\boldsymbol{J}_g\ \underline{0}]\!] \neq [\![\boldsymbol{I}]\!]$. Hence, by Lemma **??**, D is not fully abstract. □

[8] We could have used a co-induction, but justifying the productivity is not easy (it uses Theorem **??**).
[9] See footnote **??**

Appendix A. Appendix

A.1. **Lemma ??.**

Lemma ?? *If D is sensible for $\Lambda_{\tau(D)}$ then:*

$$(\vec{a}b,\alpha)\in [\![M]\!]^{\vec{y}x} \quad\Leftrightarrow\quad (\vec{a},\alpha)\in [\![M[\bar{\epsilon}_b/x]]\!]^{\vec{y}},$$
$$(\vec{a},\alpha)\in [\![M]\!]^{\vec{y}} \quad\Leftrightarrow\quad \vec{a}\in [\![\tau_\alpha(M)]\!]^{\vec{y}}.$$

Proof. For this proof we use the intersection type system of Figure **??**. Such a change of viewpoint replaces the statement by:

$$\Gamma,x:a\vdash M:\alpha \quad\Leftrightarrow\quad \Gamma\vdash M[\bar{\epsilon}_a/x]:\alpha$$
$$\Gamma\vdash M:\alpha \quad\Leftrightarrow\quad \Gamma\vdash \tau_\alpha(M)$$

- $\Gamma,x:a\vdash M:\alpha \;\Rightarrow\; \Gamma\vdash M[\bar{\epsilon}_a/x]:\alpha$ and $\Gamma,x:a\vdash Q \;\Rightarrow\; \Gamma\vdash Q[\bar{\epsilon}_a/x]$:
  By structural induction on M and Q:
  - If $M=x$: then $\alpha\leq\beta\in a$ and by definition $\Gamma\vdash\bar{\epsilon}_a:\alpha$.
  - If $M=y\neq x$: trivial.
  - If $M=\lambda y.N$: then $\alpha=b{\to}\beta$ and $\Gamma,y:b,x:a\vdash N:\beta$ thus by IH, $\Gamma,y:b\vdash N[\bar{\epsilon}_a/x]:\beta$ and thus $\Gamma\vdash M[\bar{\epsilon}_a/x]:\alpha$.
  - If $M=N_1\ N_2$: then there exists b such that $\Gamma,x:a\vdash N_1:b{\to}\alpha$ and for all $\beta\in b$, $\Gamma,x:a\vdash N_2:\beta$. Thus by IH, $\Gamma\vdash N_1[\bar{\epsilon}_a/x]:b{\to}\alpha$ and for all $\beta\in b$, $\Gamma\vdash N_2[\bar{\epsilon}_a/x]:\beta$ and thus $\Gamma\vdash M[\bar{\epsilon}_a/x]:\alpha$.
  - If $M=\Sigma_i\bar{\tau}_{\alpha_i}(Q_i)$: then there exists i such that $\alpha=\alpha_i$ and $\Gamma,x:a\vdash Q_i$. Thus by IH, $\Gamma\vdash Q_i[\bar{\epsilon}_a/x]$ and thus $\Gamma\vdash M[\bar{\epsilon}_a/x]:\alpha$.
  - If $Q=\Sigma_iQ_i$: then there exists i such that $\Gamma,x:a\vdash Q_i$. Thus by IH, $\Gamma\vdash Q_i[\bar{\epsilon}_a/x]$ and thus $\Gamma\vdash Q[\bar{\epsilon}_a/x]$.
  - If $Q=\Pi_iQ_i$: then for all i, $\Gamma,x:a\vdash Q_i$. Thus by IH, for all i, $\Gamma\vdash Q_i[\bar{\epsilon}_a/x]$ and thus $\Gamma\vdash Q[\bar{\epsilon}_a/x]$.
  - If $Q=\tau_\beta(M)$: then $\Gamma,x:a\vdash M:\beta$. Thus by IH, $\Gamma\vdash M[\bar{\epsilon}_a/x]:\beta$ and thus $\Gamma\vdash Q[\bar{\epsilon}_a/x]$.
- $\Gamma,x:a\vdash M:\alpha \;\Leftrightarrow\; \Gamma\vdash M[\bar{\epsilon}_a/x]:\alpha$:
  and $\Gamma,x:a\vdash Q \;\Leftrightarrow\; \Gamma\vdash Q[\bar{\epsilon}_a/x]$:
  By structural induction on M and Q:
  - If $M=x$ then $\Gamma\vdash\bar{\epsilon}_a:\alpha$ and by definition $\Gamma,x:a\vdash x:\alpha$, *i.e*, $\Gamma,x:a\vdash M:\alpha$
  - If $M=y\neq x$: trivial.
  - If $M=\lambda y.N$: then $\alpha=i_D(b{\to}\beta)$ and $\Gamma,y:b\vdash N[\bar{\epsilon}_a/x]:\beta$ thus by IH, $\Gamma,y:b,x:a\vdash N:\beta$ and thus $\Gamma,x:a\vdash M:\alpha$.
  - If $M=N_1\ N_2$: then there exists b such that $\Gamma\vdash N_1[\bar{\epsilon}_a/x]:b{\to}\alpha$ and for all $\beta\in b$, $\Gamma\vdash N_2[\bar{\epsilon}_a/x]:\beta$. Thus by IH, $\Gamma,x:a\vdash N_1:b{\to}\alpha$ and for all $\beta\in b$, $\Gamma,x:a\vdash N_2:\beta$ and thus $\Gamma,x:a\vdash M:\alpha$.
  - If $M=\Sigma_i\bar{\tau}_{\alpha_i}(Q_i)$: then there exists i such that $\alpha=\alpha_i$ and $\Gamma\vdash Q_i[\bar{\epsilon}_a/x]$. Thus by IH, $\Gamma,x:a\vdash Q_i$ and thus $\Gamma,x:a\vdash M:\alpha$.
  - If $Q=\Sigma_iQ_i$: then there exists i such that $\Gamma\vdash Q_i[\bar{\epsilon}_a/x]$. Thus by IH, $\Gamma,x:a\vdash Q_i$ and thus $\Gamma,x:a\vdash Q$.
  - If $Q=\Pi_iQ_i$: then for all i, $\Gamma\vdash Q_i[\bar{\epsilon}_a/x]$. Thus by IH, for all i, $\Gamma,x:a\vdash Q_i$ and thus $\Gamma,x:a\vdash Q$.

  - If $Q = \tau_\beta(M)$: then $\Gamma \vdash M[\bar{\epsilon}_a/x] : \beta$. Thus by IH, $\Gamma, x : a \vdash M : \beta$ and thus $\Gamma, x : a \vdash Q$.
- $\Gamma \vdash \tau_\alpha(M) \quad\Leftrightarrow\quad \Gamma \vdash M : \alpha$: by definition of the inference rule for τ_α

□

A.1.1. *Lemma* **??**.

Lemma ?? *Let $M = \lambda x_1...x_n.y\ M_1 \cdots M_k \in \Lambda$ and let $N = \lambda x_1...x_{n'}.y'\ N_1 \cdots N_{k'} \in \Lambda$ be such that $M \sqsubseteq_{\mathcal{H}^*} N$. Then:*

1. *$y = y'$,*
2. *$n - k = n' - k'$,*
3. *if $i \le k$ and $i \le k'$ then $M_i \sqsubseteq_{\mathcal{H}^*} N_i$,*
4. *if $i > k$ and $i \le k'$ then $x_{i-k} \sqsubseteq_{\mathcal{H}^*} N_i$,*
5. *if $i \le k$ and $i > k'$ then $M_{i-k} \sqsubseteq_{\mathcal{H}^*} x_i$.*

Proof. In the following, $M = \lambda x_1...x_n.y\ M_1 \cdots M_k$ and $N = \lambda x_1...x_{n'}.y'\ N_1 \cdots N_{k'}$.

If $y \neq y'$, then $M \not\sqsubseteq_{\mathcal{H}^*} N$, indeed:

- If y' is free in M and N then by setting $C(\!|.|\!) = (\lambda y'.(\!|.|\!))\ \Omega$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.
- If $y' = x_j$ for $j \le n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1\ \cdots\ x_{j-1}\ \Omega$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

Now we suppose that $M = \lambda x_1...x_n.y\ M_1 \cdots M_k$ and $N = \lambda x_1...x_{n'}.y\ N_1 \cdots N_{k'}$.

If $n - k \neq n' - k'$, then $M \not\sqsubseteq_{\mathcal{H}^*} N$:

- If y is free in M and N, then by setting $C(\!|.|\!) = (\lambda y.(\!|.|\!)\ x_1 \cdots x_{n'+k})\ (\lambda z_1...z_{k'+k}u.u)\ \mathbf{\Omega}$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$:
- If $y = x_j$ for $j \le n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1 \cdots x_{j-1}\ (\lambda z_1...z_{k'+k}u.u)\ x_{i+1} \cdots x_{n'+k}\ \mathbf{\Omega}$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

Now we suppose that $n - k = n' - k'$.

If there is i such that $i \le k$, $i \le k'$ and $M_i \not\sqsubseteq_{\mathcal{H}^*} N_i$ then there is $C'(\!|.|\!)$ such that $C'(\!|M_i|\!)\Downarrow^{\mathrm{h}}$ and $C'(\!|N_i|\!)\Uparrow^{\mathrm{h}}$:

- If y is free in M and N, then by setting $C(\!|.|\!) = (\lambda y.(\!|.|\!))\ (\lambda z_1...z_{k+k'}.C'(\!|z_i|\!))$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.
- If $y = x_j$ for $j \le n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1 \cdots x_{j-1}\ (\lambda z_1...z_{k+k'}.C(\!|z_i|\!))$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

If there is i such that $k < i \le k'$ and $x_{i-k} \not\sqsubseteq_{\mathcal{H}^*} N_i$ then there is $C'(\!|.|\!)$ such that $C'(\!|x_{i-k}|\!)\Downarrow^{\mathrm{h}}$ and $C'(\!|N_i|\!)\Uparrow^{\mathrm{h}}$:

- If y is free in M and N, then by setting $C(\!|.|\!) = (\lambda y.(\!|.|\!)\ x_1 \cdots\ x_{n+k})\ (\lambda z_1...z_{k+k'}.C'(\!|z_i|\!))$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.
- If $y = x_j$ for $j \le n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1 \cdots x_{j-1}\ (\lambda z_1...z_{k+k'}.C(\!|z_i|\!))\ x_{j+1} \cdots\ x_{n+k}$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

If there is i such that $k' < i \le k$ and $M_i \not\sqsubseteq_{\mathcal{H}^*} x_{i-k'}$ then there is $C'(\!|.|\!)$ such that $C'(\!|M_i|\!)\Downarrow^{\mathrm{h}}$ and $C'(\!|x_{i-k'}|\!)\Uparrow^{\mathrm{h}}$:

- If y is free in M and N, then by setting $C(\!|.|\!) = (\lambda y.(\!|.|\!)\ x_1 \cdots\ x_{n+k})\ (\lambda z_1...z_{k+k'}.C'(\!|z_i|\!))$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.
- If $y = x_j$ for $j \le n'$, then by setting $C(\!|.|\!) = (\!|.|\!)\ x_1 \cdots x_{j-1}\ (\lambda z_1...z_{k+k'}.C(\!|z_i|\!))\ x_{j+1} \cdots\ x_{n+k}$ we have $C(\!|M|\!)\Downarrow^{\mathrm{h}}$ and $C(\!|N|\!)\Uparrow^{\mathrm{h}}$.

□