1 Notations

Definition 1 (Notations for trees)**.** *We denote $\mathbb{T}$ the set of binary trees, with metavariable for trees ranging over s, t, u, v.... We denote by $\bullet$ the tree with just a leaf and by* $\overset{\bullet}{s \diagup \diagdown t}$ *the binary tree with sons t and s. The size $\#(t)$ is the number of nodes (and leafs) in t.*

For any $t \in \mathbb{T}$, the node (or leaf) of $x \in t$ are denoted by metavariable $x, y, z, ...$. Given a node $x \in t$, we denote t_x the subtree of x with root x. For $x, y \in t$, we denote $x \leq_t y$ if $y \in t_x$.

For $x \in t$, we denote $D_t(x) \in \mathbb{N}$ the depth of x in t (with the depth of the root being 0).

Given $s, t \in \mathbb{T}$ and $x \in t$ we denote $t\{s/x\}$ the tree t where the subtree t_x is erased and replaced by s.

Definition 2 (Duplication)**.** *given $x \in t$, we denote*

$$t^{2.x} \;:=\; t\left\{ \overset{\bullet}{t_x \diagup \diagdown t_x} \Big/ x \right\} \qquad t^{0.x} \;:=\; t\,\{\bullet/x\}$$

the tree where the subtree t_x hase been duplicated and erased, respectivelly.

Definition 3. *We define a probabilistic reduction of a tree as $t \to t^{2.x}$ with probability $\#(t)$ for any $x \in t$.*

Definition 4. *The expected value of $\#(t)$ after one reduction is denoted :*

$$\mathbb{E}_{\#}(t) \quad := \quad \frac{\sum_{x \in t} \#\left(t^{2.x}\right)}{\#(t)} \tag{1}$$

the expected growth of $\#(t)$ after one reduction is denoted :

$$\Delta_{\#}(t) \quad := \quad \mathbb{E}_{\#}(t) - \#(t) \;, \tag{2}$$

we also denote $\mathbb{E}^k_{\#}(t)$ and $\Delta^k_{\#}(t)$ the expexted and Growth values after k steps.

Conjecture 1 (Devroye)**.** *The serie given by the expected value of the size the n^{th} reduction of any tree $t \in \mathbb{T}$ is quasilinear :*

$$\mathbb{E}^n_{\#}(t) \;\sim\; n\log(n).$$

1.1 Counting large chains

Definition 5. *A (large) chain of t is a seqence $x_1 \leq_t \cdots \leq_t x_n$ of element of t. We use $\langle n\rangle(t)$ to denote the number of large chains of length n in t. We use $\langle n\rangle^*(t)$ to denote the number of large chains of length n in t that do not contain the root.*

$\mathbb{E}_{\langle n\rangle}(u)$ and $\Delta_{\langle n\rangle}(u)$ are defined as expected.

Proposition 1. *We know the following trivial equations :*

$$\langle 0\rangle^*(t) = \langle 0\rangle(t) = 1 \qquad\qquad \langle n\rangle(t) = \sum_{n' \leq n} \langle n'\rangle^*(t) \qquad\qquad \langle n'+1\rangle^*(t) = \langle n+1\rangle(t) - \langle n\rangle(t)\;.$$

Lemma 6.

$$\#(t) \quad = \quad [1](t)$$

Proof. Trivial ☐

Lemma 7. *For $n_1, n_2 \geq 0$:*

$$\sum_{x\in t}\binom{D_t(x)}{n_1}\langle n_2\rangle(t_x) \quad = \quad \langle n_1+n_2+1\rangle(t)$$

and in particular for $n_1 = 0$ and $n_2 = n$:

$$\sum_{x\in t}\langle n\rangle(t_x) \quad = \quad \langle n+1\rangle(t) \ .$$

Proof. The sum $\sum_{x\in t}\binom{D_t(x)}{n_1}[n_2](t_x)$ corresponds to counting the diferents output of the procedural :

- choose $x \in t$,
- choose n_1 elements $y_1 \leq_t \cdots \leq_t y_{n_1} \leq_t x$,
- choose n_2 elements $z_1 \leq_{t_x} \cdots \leq_{t_x} z_{n_2}$, *i.e.*, $x \leq_t z_1 \leq_t \cdots \leq_t z_{n_2}$

In the end, we are counting all the $y_1 \leq_t \cdots \leq_t y_{n_1} \leq_t x \leq_t z_1 \leq_t \cdots \leq_t z_{n_2}$, which gives $\langle n_1+n_2+1\rangle(t)$. □

Proposition 2.

$$\mathbb{A}_{\#}(t) \quad = \quad \mathbb{A}_{\langle 1\rangle}(t) \quad = \quad \frac{\langle 2\rangle(t)}{\langle 1\rangle(t)}+1$$

Proof. Recall that $\mathbb{A}_{\#}(t) := \frac{\sum_{x\in t}\#\left(t^{2.x}\right)}{\#(t)} - \#(t)$ thus we have first to calcul $\#\left(t^{2.x}\right)$. For this, we remark that the choosen $y \in t^{2.x}$ can be either in $t^{0.x}$ or in one of the two copies of t_x, thus :

$$\#\left(t^{2.x}\right) \quad = \quad \#\left(t^{0.x}\right) + 2.\#(t_x) \ .$$

But choosing an element of $t^{0.x}$ is the same as choosing an element of t which is not in t_x or which is x ; thus

$$\#\left(t^{0.x}\right) = \#(t) - \#(t_x) + 1 \ ,$$

so that

$$\#\left(t^{2.x}\right) \quad = \quad \#(t) + \#(t_x) + 1 \ .$$

In the end, we have :

$$\mathbb{A}_{\#}(t) \quad = \quad \frac{\sum_{x\in t}(\#(t)+\#(t_x)+1)}{\#(t)} - \#(t) \quad = \quad \frac{\sum_{x\in t}\#(t_x)}{\#(t)}+1$$

We can conclud by Lemmas 6 and 7 :

$$\mathbb{A}_{\#}(t) \quad = \quad \frac{\langle 2\rangle(t)}{\langle 1\rangle(t)}+1$$

□

This result can be generalised to any chain size :

Theorem 8. $\mathbb{A}_{[0]}(t) = 0$ *and for any $n \geq 1$:*

$$\mathbb{A}_{\langle n\rangle}(t) \quad = \frac{n\langle n+1\rangle(t) + (2n-1).\langle n\rangle(t) + (n-1).\langle n-1\rangle(t)}{\#(t)} \ .$$

Proof. Recall that $\mathbb{A}_{\langle n\rangle}(t) := \frac{\sum_{x\in t}\langle n\rangle(t^{2.x})}{\#(t)} - \langle n\rangle(t)$ thus we have first to calcul $\langle n\rangle(t^{2.x})$. For this, we remark that the choosen $y_1 \leq_{t^{2.x}} \cdots \leq_{t^{2.x}} y_n \in t^{2.x}$ are either all in $t^{0.x}$, or there is $n' < n$ such that $y_1 \leq_{t^{2.x}} \cdots \leq_{t^{2.x}} y_{n'} \leq_{t^{2.x}} x <_{t^{2.x}} y_{n'+1} \cdots \leq_{t^{2.x}} y_n$ with $y_{n'+1}, ..., y_n$ in one (and only one) of the copies of t_x, thus :

$$\langle n\rangle(t^{2.x}) \quad = \quad \langle n\rangle(t^{0.x}) + 2.\sum_{0\leq n'<n}\binom{D_t(x)+1}{n'}.\langle n-n'\rangle(t_x) \ .$$

But choosing a n-chain in $t^{0.x}$ is the same as choosing a n-chain in t with no elements in t_x exepts for x ; thus

$$\langle n\rangle(t^{0.x}) \quad = \quad \langle n\rangle(t) - \sum_{0\leq n'<n} \binom{D_t(x)+1}{n'} . \langle n-n'\rangle^*(t_x) \ ,$$

so that (notice that $n - n' \geq 0$ when $n' < n$)

$$\langle n\rangle(t^{2.x}) - \langle n\rangle(t) \quad = \quad \sum_{0\leq n'<n} \binom{D_t(x)+1}{n'} .(2.\langle n-n'\rangle(t_x) - \langle n-n'\rangle^*(t_x)) \quad = \quad \sum_{0\leq n'<n} \binom{D_t(x)+1}{n'} .(\langle n-n'\rangle(t_x) + \langle n-n'-1\rangle(t_x)) \quad ,$$

Using that $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$ we have :

$$\langle n\rangle(t^{2.x}) \quad = \quad \langle n\rangle(t) + \sum_{0\leq n'<n} (\binom{D_t(x)}{n'} + \binom{D_t(x)}{n'-1}).(2.\langle n-n'\rangle(t_x) - \langle n-n'\rangle^*(t_x)) \ ,$$

In the end, we have :

$$\mathbb{A}_{[n]}(t) \quad = \frac{\sum_{0\leq n'<n} \left(\sum_{x\in t} \binom{D_t(x)}{n'} \langle n-n'\rangle(t_x) + \sum_{x\in t} \binom{D_t(x)}{n'-1} \langle n-n'\rangle(t_x) + \sum_{x\in t} \binom{D_t(x)}{n'} \langle n-n'-1\rangle(t_x) + \sum_{x\in t} \binom{D_t(x)}{n'-1} \langle n-n'-1\rangle(t_x) \right)}{\#(t)} \ ,$$

By applying Lemma 7 whenever $n' \neq 0$, we get :

$$\mathbb{A}_{\langle n\rangle}(t) \quad = \frac{\sum_{0<n'<n} \Big(\langle n+1\rangle(t) + \langle n\rangle(t) + \langle n\rangle(t) + \langle n-1\rangle(t) \Big) + \Big(\langle n+1\rangle(t) + \langle n\rangle(t) \Big)}{\#(t)} \ ,$$

i.e.,

$$\mathbb{A}_{\langle n\rangle}(t) \quad = \frac{n\langle n+1\rangle(t) + (2n-1).\langle n\rangle(t) + (n-1).\langle n-1\rangle(t)}{\#(t)} \ .$$

□

1.1.1 Factorials ?

We use

$$A_n(t) := \sum_{1\leq i\leq n} \frac{i.!n}{n.!i} (-1)^{n-i} \langle i\rangle(t)$$

to denote the alternation of sequences.

Proposition 3. *for any* $n \geq 1$ *:*

$$\mathbb{A}_{A_n}(t) = \frac{\langle n+1\rangle(t) + \langle n\rangle(t)}{\#(t)} \ .$$

Proof. By induction on n :
For $n = 1$: $\mathbb{A}_{\langle 1\rangle}(t) = \frac{\langle 2\rangle(t)+\langle 1\rangle(t)}{\#(t)}$
For $n + 1$:

$$\begin{aligned}
\sum_{1\leq i\leq n+1} \frac{i.!n}{!i} (-1)^{n+1-i} \mathbb{A}_{\langle i\rangle}(t) &= \mathbb{A}_{\langle n+1\rangle}(t) - n \sum_{1\leq i\leq n} \frac{i.!n}{n.!i} (-1)^{n-i} \mathbb{A}_{\langle i\rangle}(t) \\
&= \mathbb{A}_{\langle n+1\rangle}(t) - n. \frac{\langle n+1\rangle(t) + \langle n\rangle(t)}{\#(t)} \\
&= \frac{(n+1)\langle n+2\rangle(t) + (2n+1).\langle n+1\rangle(t) + n.\langle n\rangle(t)}{\#(t)} - n. \frac{\langle n+1\rangle(t) + \langle n\rangle(t)}{\#(t)} \\
&= \frac{(n+1)\langle n+2\rangle(t) + (n+1).\langle n+1\rangle(t)}{\#(t)}
\end{aligned}$$

□

Proposition 4. *for any $n \geq 1$:*

$$A_1(\bullet) = 1 \qquad A_2(\bullet) = 0 \qquad A_n(\bullet) = !(n-1) \sum_{2 \leq i < n} \frac{(-1)^{n-i-1}}{!i} .$$

1.1.2 ...

Given a tree t, we can compute the characteristic $\langle \_ \rangle()$ for a virtual mean-tree $\mathbb{E}(t)$. We can itterate the process and compute the caracteristic of the virtual mean tree of the virtual meen tree and so on (remark that this is unfortunatelly not the virtual mean tree after k etape). We then define a sequence of functions g_k acting like

$$g_k(z) \simeq \sum_{n \geq 0} \langle n{+}1 \rangle ((\mathbb{E})^k(\bullet)) z^n$$

Let $u_{k,n}$ defined for $k \geq 0, n \geq 1$ by :

$$u_{0,n} := 1$$

$$u_{k+1,n} := u_{k,n} + \frac{n.u_{k,n+1} + (2n-1).u_{k,n} + (n-1).u_{k,n-1}}{u_{k,1}}$$

Let $v_{k,n}$ defined by $v_{k,n} := u_{k,n+1}$, i.e. :

$$v_{0,n} = 1$$

$$v_{k+1,n} = v_{k+1,n} + \frac{(n+1).v_{k,n+1} + (2n+1).v_{k,n} + n.v_{k,n-1}}{v_{k,0}}$$

Let g_k a function defined by :

$$g_k(z) = \sum_{n \geq 0} v_{k,n} z^n$$

So that :

$$g_0(z) = \frac{1}{1-z}$$

and

$$\begin{aligned}
g_{k+1}(z) &= g_k(z) + \sum_{n \geq 0} \left(\frac{(n+1).v_{k,n+1} + (2n+1).v_{k,n} + n.v_{k,n-1}}{v_{k,0}} \right) z^n \\
&= g_k(z) + \frac{\sum_{n \geq 0} \left((n+1).v_{k,n+1} + (2n+1).v_{k,n} + n.v_{k,n-1}\right) z^n}{v_{k,0}} \\
&= g_k(z) + \frac{\sum_{n \geq 0}(n+1).v_{k,n+1} z^n + \sum_{n \geq 0}(2n+1).v_{k,n} z^n + \sum_{n \geq 0} n.v_{k,n-1} z^n}{v_{k,0}} \\
&= g_k(z) + \frac{\sum_{n \geq 0} n.v_{k,n} z^{n-1} + \sum_{n \geq 0}(2n+1).v_{k,n} z^n + \sum_{n \geq 0}(n+1).v_{k,n} z^{n+1}}{v_{k,0}} \\
&= g_k(z) + \frac{\sum_{n \geq 0} v_{k,n}.n.(1+2z+z^2) z^{n-1} + \sum_{n \geq 0} v_{k,n}.(1+z)^n}{z} \ v_{k,0} \\
&= g_k(z) + \frac{(z+1)^2 g_k'(z) + (z+1) g_k(z)}{g_k(0)}
\end{aligned}$$

1.2 Profile evolution

Definition 9. *The nth slice of t, that we write t_n, is the number of node at depth n in t.*

The profile of t, writen ϕ_t is the polynomial $\phi_t(x) = \sum_n t_n.x^n$.

The profile of t on 1/2, writen $U(t)$ is the number $U(t) = \phi_t(\frac{1}{2}) = \sum_n \frac{t_n}{2^n}$.

$\mathbb{E}_{S_n}(t)$, $\mathbb{E}_\phi(t)$, $\mathbb{E}_U(t)$, $\mathbb{\Delta}_{S_n}(t)$, $\mathbb{\Delta}_\phi(t)$ and $\mathbb{\Delta}_U(t)$ are defined as expected.

Proposition 5.

$$\mathbb{\Delta}_\phi(t) \quad = \quad \frac{2x^2 - x}{\#(t)}\phi_t'(x) + \frac{2x}{\#(t)}\phi_t(x)$$

En particulier :

$$\mathbb{\Delta}_U(t) \quad = \quad \frac{\phi_t(1/2)}{\phi_t(1)}$$

Proof.

$$\mathbb{E}_{Sn}(t) = \frac{\sum_{i<n}((t_i - 1).t_n + 2t_{n-1}) + \sum_{i \geq n} t_n}{\#(t)}$$

$$\mathbb{\Delta}_{Sn}(t) = \frac{n.(2t_{n-1} - t_n)}{\#(t)}$$

$$\begin{aligned}
\mathbb{\Delta}_\phi(t) &= \sum_n \frac{n(2t_{n-1} - t_n)}{\#(t)} x^n \\
&= \frac{2}{\#(t)} \sum_n n t_{n-1} x^n - \frac{x}{\#(t)}\phi_t'(x) \\
&= \frac{2}{\#(t)} \sum_n (n-1) t_{n-1} x^n + \frac{2x}{\#(t)}\phi_t(x) - \frac{x}{\#(t)}\phi_t'(x) \\
&= \frac{2x^2}{\#(t)}\phi_t'(x) + \frac{2x}{\#(t)}\phi_t(x) - \frac{x}{\#(t)}\phi_i'(x)
\end{aligned}$$

□

Proposition 6.

$$\mathbb{\Delta}_U(t) \quad \leq \quad \frac{U(t)}{2^{\frac{U(t)}{2}}}$$

Proof. Because for any tree, $\phi_t(1/2) \leq 2.log_2(\phi_t(1))$ □

2 Old (for memory)

2.1 Devroye's conjecture

We consider a TRS $\mathbb{C}\mathtt{opy}$ with two symbols $\bullet : 0$ and $\boldsymbol{copy} : 2$ and with a unic rule :

$$\boldsymbol{copy}(x, y) \to \boldsymbol{copy}(\boldsymbol{copy}(x, y), \boldsymbol{copy}(x, y)).$$

The randomized strategy applied to this TRS corresponds to a variant of Remy's algorithm.Combinatoricians have studied this variation, which study leads to the following conjecture : Ref

Conjecture 2 (Devroye). *The serie given by the expected value of the size the n^{th} reduction of a term $M \in \mathbb{C}\mathtt{opy}$ is quasilinear :*

$$\mathbb{E}^n_{\#}(M) \sim n\log(n).$$

2.2 Approche par strates

Given a tree t, we denote :

— a node $x \in t$ is an occurence of $\boldsymbol{copy}$ in t,

— the depth of a node $x \in t$ is 0 if x is the root and $n + 1$ is x's father has depth n,

— the size $\#(t)$ of t is the number of nodes in t, which is also the number of t' such that $t \to t'$,

— the strata $\#_n t$ of t is the number of nodes of depth t, so that

$$\#(t) = \sum_{n\geq 0} \#_n t \tag{3}$$

— the mean depth $d(t)$ is given by

$$d(t) := \frac{\sum_{n\geq 0} n.\#_n t}{\#(t)}, \tag{4}$$

— the hight $H(t)$ is the maximal depth :

$$H(t) := \max\{n \mid \#_n t \neq 0\}, \tag{5}$$

— the expected value of $p(t)$, for $p \in \{\#, \#_n, d, H\}$, after one reduction is denoted :

$$\mathbb{E}_p(t) := \frac{\sum_{t'\leftarrow t} p(t)}{\#(t)} \tag{6}$$

— the expected growth of $p(t)$, for $p \in \{\#, \#_n, d, H\}$, after one reduction is denoted :

$$\Delta_p(t) := \mathbb{E}_p(t) - p(t), \tag{7}$$

— we also denote $\mathbb{E}^k_p(t)$ and $\Delta^k_p(t)$ the expexted and Growth values after k steps.

Lemma 10. *For all $n \geq 1$:*

$$\sum_{t\to t'} \#_n t' = \#t.\#_n t + n\Big(2\#_{n-1}t - \#_n t\Big) \tag{8}$$

so that :

$$\Delta_{\#_n}(t) = \frac{2n.\#_{n-1}t - n\#_n t}{\#t} \tag{9}$$

$$\Delta_{\#}(t) = 2 + d(t) \tag{10}$$

2.3 Approche par séquances

Pour u, t des arbres, on dénote :

— $t \in u$ si t est un suffix de u,

— $[n](u)$ est le nombre de séquence $x_1 > \cdots > x_n$ dans u,

— $\mathbb{E}_{[n]}(u)$ et $\mathbb{A}_{[n]}(u)$ sont définit de la même façon.

Remarquons d'abord que

$$\#u \;=\; [1](u) \qquad d(u) \;=\; \frac{[2](u)}{[1](u)} \qquad \sum_{t\in u}[n](t) \;=\; [n{+}1](u){+}[1](u) \qquad \sum_{t\in u}\binom{D_u t}{n_1}[n_2](t) \;=\; [n_1{+}n_2{+}1](u){+}[n_1{+}n$$

On a alors

$$\mathbb{A}_{[1]}(u) \quad = \quad \frac{[2](u)}{[1](u)} + 2$$

in general :

$$\mathbb{A}_{[n]}(u).[1](u) = n.\frac{[n+1](t) + 3.[n](t) + 2.[n-1](t)}{\#(t)}$$

2.4 Approche par sous-arbres

Pour u, t des arbres, on dénote :

— lorsque $t \in u$, u_t est le nombre de tels suffixes (sinon $u_t = 0$),

— $\mathbb{E}_{\ni t}(u) := \frac{\sum_{u' \leftarrow u} u'_t}{\#u}$,

— $\mathbb{A}_{\ni t}(u) := \mathbb{E}_{\ni t}(u) - u_t$,

Remarquons d'abord que

$$\#u \quad = \quad \sum_t u_t \quad = \quad 2.u_\bullet - 1$$

On a alors :

$$\mathbb{A}_{\ni v}(u) = \frac{\langle u, v\rangle - (\#v - 1).u_v + \sum_t t_v.u_t}{\#u}$$

où $< u, v >$ est une somme de $< u, v >= \sum_t k_t u_t$ où k_t est le nombre de réductions possible $t \to v$.

2.5 Counting strict chains

Definition 11. *A (strict) chain of t is a seqence $x_1 >_t \cdots >_t x_n$ of element of t. We use $[n](t)$ to denote the number of chains of length n in t. We use $[n]^*(t)$ to denote the number of chains of length n in t that do not contain the root.*

$\mathbb{E}_{[n]}(u)$ and $\mathbb{A}_{[n]}(u)$ are defined as expected.

Proposition 7. *The number of n-chains that do not contain the root is recursivelly given by :*

$$[0]^*(t) = 1 \qquad\qquad [n+1]^*(t) = [n+1](t) - [n]^*(t)\ .$$

Lemma 12.

$$\#(t) \quad = \quad [1](t)$$

Proof. Trivial □

Lemma 13. *For $n_1, n_2 \geq 0$:*

$$\sum_{x\in t}\binom{D_t(x)}{n_1}[n_2]^*(t_x) \quad = \quad [n_1 + n_2 + 1](t)$$

thus for $n_0 > 0$

$$\sum_{x\in t} \binom{D_t(x)}{n_1} [n_2](t_x) \quad = \quad [n_1+n_2+1](t) + [n_1+n_2](t)$$

and in particular for $n_1 = 0$ *and* $n_2 = n > 0$ *:*

$$\sum_{x\in t} [n](t_x) \quad = \quad [n+1](t) + [n](t)\ .$$

Notice that :

$$\sum_{x\in t} \binom{D_t(x)}{n_1} [0](t_x) \quad = \quad [n1+1](t)$$

Proof. The sum $\sum_{x\in t} \binom{D_t(x)}{n_1} [n_2](t_x)$ corresponds to counting the diferents output of the procedural :

- choose $x \in t$,
- choose n_1 elements $y_1 <_t \cdots <_t y_{n_1} <_t x$,
- choose n_2 elements $z_1 <_{t_x} \cdots <_{t_x} z_{n_2}$ that are not the root x, *i.e.*, $x <_t z_1 <_t \cdots <_t z_{n_2}$

In the end, we are counting all the $y_1 <_t \cdots <_t y_{n_1} <_t x <_t z_1 <_t \cdots <_t z_{n_2}$, which gives $[n_1+n_2+1](t)$. □

Proposition 8.

$$\mathbb{\Delta}_{\#}(t) \quad = \quad \mathbb{\Delta}_{[1]}(t) \quad = \quad \frac{[2](t)}{[1](t)} + 2$$

Proof. Recall that $\mathbb{\Delta}_{\#}(t) := \frac{\sum_{x\in t} \#\left(t^{2.x}\right)}{\#(t)} - \#(t)$ thus we have first to calcul $\#\left(t^{2.x}\right)$. For this, we remark that the choosen $y \in t^{2.x}$ can be either in $t^{0.x}$ or in one of the two copies of t_x, thus :

$$\#\left(t^{2.x}\right) \quad = \quad \#\left(t^{0.x}\right) + 2.\#(t_x)\ .$$

But choosing an element of $t^{0.x}$ is the same as choosing an element of t which is not in t_x or which is x ; thus

$$\#\left(t^{0.x}\right) = \#(t) - \#(t_x) + 1\ ,$$

so that

$$\#\left(t^{2.x}\right) \quad = \quad \#(t) + \#(t_x) + 1\ .$$

In the end, we have :

$$\mathbb{\Delta}_{\#}(t) \quad = \quad \frac{\sum_{x\in t} (\#(t) + \#(t_x) + 1)}{\#(t)} - \#(t) \quad = \quad \frac{\sum_{x\in t} \#(t_x)}{\#(t)} + 1$$

We can conclud by Lemmas 12 and 13 :

$$\mathbb{\Delta}_{\#}(t) \quad = \quad \frac{[2](t) + [1](t)}{[1](t)} + 1 \quad = \quad \frac{[2](t)}{[1](t)} + 2$$

□

This result can be generalised to any chain size :

Theorem 14. $\mathbb{\Delta}_{[0]}(t) = 0$ *and for any* $n \geq 1$ *:*

$$\mathbb{\Delta}_{[n]}(t) \quad = \frac{n.[n+1](t) + (3n-1).[n](t) + 2(n-1).[n-1](t)}{\#(t)}\ .$$

Proof. Recall that $\vartriangle_{[n]}(t) := \frac{\sum_{x\in t}[n](t^{2.x})}{\#(t)} - [n](t)$ thus we have first to calcul $[n](t^{2.x})$. For this, we remark that the choosen $y_1 <_{t^{2.x}} \cdots <_{t^{2.x}} y_n \in t^{2.x}$ are either all in $t^{0.x}$, or there is $n' < n$ such that $y_1 <_{t^{2.x}} \cdots <_{t^{2.x}} y_{n'} \leq_{t^{2.x}} x <_{t^{2.x}} y_{n'+1} \cdots y_n$ with $y_{n'+1}, ..., y_n$ in one (and only one) of the copies of t_x, thus :

$$[n](t^{2.x}) \quad = \quad [n](t^{0.x}) + 2. \sum_{0\leq n'<n} \binom{D_t(x)+1}{n'} .[n-n'](t_x) \ .$$

But choosing a n-chain in $t^{0.x}$ is the same as choosing a n-chain in t with no elements in t_x exepts for x ; thus

$$[n](t^{0.x}) \quad = \quad [n](t) - \sum_{0\leq n'<n} \binom{D_t(x)+1}{n'} .[n-n']^*(t_x) \ ,$$

so that (notice that $n - n' \geq 0$ when $n' < n$)

$$[n](t^{2.x}) \quad = \quad [n](t) + \sum_{0\leq n'<n} \binom{D_t(x)+1}{n'} .(2.[n-n'](t_x) - [n-n']^*(t_x)) \quad = \quad [n](t) + \sum_{0\leq n'<n} \binom{D_t(x)+1}{n'} .([n-n'](t_x) + [n-n'-1]^*(t_x)) \ ,$$

Using that $\binom{k+1}{n+1} = \binom{k}{n} + \binom{k}{n+1}$ we have :

$$[n](t^{2.x}) \quad = \quad [n](t) + \sum_{0\leq n'<n} (\binom{D_t(x)}{n'} + \binom{D_t(x)}{n'-1}).([n-n'](t_x) + [n-n'-1]^*(t_x)) \ ,$$

In the end, we have :

$$\vartriangle_{[n]}(t) \quad = \frac{\sum_{0\leq n'<n} \left(\sum_{x\in t} \binom{D_t(x)}{n'} [n-n'](t_x) + \sum_{x\in t} \binom{D_t(x)}{n'-1} [n-n'](t_x) + \sum_{x\in t} \binom{D_t(x)}{n'} [n-n'-1]^*(t_x) + \sum_{x\in t} \binom{D_t(x)}{n'-1} [n-n'-1]^*(t_x) \right)}{\#(t)} \ ,$$

By applying Lemma 13 whenever $n' \neq 0$, we get :

$$\vartriangle_{[n]}(t) \quad = \frac{\sum_{0<n'<n} \Big(([n+1](t) + [n](t)) + ([n](t) + [n-1](t)) + [n](t) + [n-1](t) \Big) + \Big(([n+1](t) + [n](t)) + [n](t) \Big)}{\#(t)} \ ,$$

i.e.,

$$\vartriangle_{[n]}(t) \quad = \frac{n[n+1](t) + (3n-1).[n](t) + 2(n-1).[n-1](t)}{\#(t)} \ .$$

□