The resource lambda calculus is short-sighted in its relational model

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Abstract. Relational semantics is one of the simplest and categorically most natural semantics of Linear Logic. The co-Kleisli category MRel associated with its exponential comonad contains a fully abstract model of the untyped λ-calculus. That particular object of MRel is also a model of the resource λ-calculus, deriving from Ehrhard and Regnier's differential extension of Linear Logic and related to Boudol's λ-calculus with multiplicities. Bucciarelli et al. conjectured that model to be fully-abstract also for the resource λ-calculus. We give a counter-example to the conjecture. As a by-product we achieve a context lemma for the resource λ-calculus.

Keywords: Full abstraction, resource λ-calculus, linear logic, nondeterminism.

1 Introduction

Rel. The category Rel of set and relations is known to model Linear Logic, andits construction is all free. Indeed, Rel can be seen as the free infinite biproduct completion of the boolean ring seen as a category with one objects and two morphisms (true and false), the conjonction being the identity [**?**]. the exponential modality ! of linear logic is given by the multisets comonad that precisely is the free commutative comonad in Rel [**?**]. Moreover, the biproduct that seems to be a degeneration morally still preserves all proofs, *i.e.* the interpretation of cut free proofs is injective up to isomorphism[1] [**?**].

This multiset comonoid $!A$ of a set A is the set of finites multisets of element in A. Intuitively a finite multiset in $a \in !A$ is a resource that behave as $\invamp_{\alpha\in a}\alpha$, *i.e.* like a resource that have to be used by a program exactly once per element in a (with multiplicities). This behavior enabling an interesting resource management, it was natural to develop a syntactical counterpart.

Resource λ-calculus. A restricted version was previously introduced by Boudol in 1993 [**?**]. Boudol's resource λ-calculus extends the call-by-value λ-calculus with a special resource sensitive application (able to manage finite resources) that involves multisets of affine arguments each one used at most once. Independent from our considerations on Rel, this was seen as a natural way to

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[1] up to technical details, but the unrestricted injectivity is strongly conjectured

export resource sensitiveness into the functional setting. However, restricted by an evaluation strategy, it was not fully explored. Later on, Ehrhard and Regnier, working on the implement of behaviors discovered in Rel, comes to a similar calculus, the differential λ-calculus [?], which enjoys many syntactical and semantical properties (confluence, Taylor expansion). In Ehrhard and Regnier's differential λ-calculus the resource-sensitiveness is obtained by adding to the λ-calculus a derivative operation $\frac{\partial M}{\partial x}(N)$ (will be implemented in our notations as the term $M\langle N/x\rangle$, see section 2). This operator syntactically corresponds to a substitution of exactly one occurrence of x by N in M (introducing non determinism on the choice of the substituted occurrence); confluence is recovered, then, by performing all the possible choices at once. This linear substitution takes place when β-reducing specific applications where an argument is marked as linear, in order to be used exactly once. We will adopt the syntax of [?] that re-implements improvements from differential λ-calculus into Boudol's calculus, and we will call it resource λ-calculus or $\partial\lambda$-calculus.

MRel. For Rel as for most categorical models of Linear Logic, the interpretation of the exponential modality induces a comonad from which we can construct the Kleisli category that contains a model of the λ-calculus. In the case of Rel, this new category, MRel, corresponds to the category whose objects are sets and whose morphisms from A to B are the relations from $\mathbb{N}\langle A\rangle$ (the set of finite multisets over A) to B. It is then a model of both λ and $\partial\lambda$-calculi. This construction being very natural, MRel is one of the most-studied models of the $\partial\lambda$-calculus.

MRel and $\partial\lambda$-calculus. The depth of the connection between the reflexive elements of MRel and the $\partial\lambda$-calculus is precisely the purpose of our work. More precisely, we investigate the question of the full abstraction of $\mathcal{M}_\infty$, a reflexive object for the $\partial\lambda$-calculus [?]. We also endowed $\partial\lambda$-calculus with the particular choice of reduction that is the may-outer-reduction; this is not the only choice, but this correspond to the intuition that conduct from Rel to Ehrhard-Regnier's differential calculus. Until now we knew that $\mathcal{M}_\infty$ was adequate for the $\partial\lambda$-calculus [?], *i.e.* that two terms carrying the same interpretations in $\mathcal{M}_\infty$ behave the same way in all contexts. But we did not know anything about the converse, the full abstraction.

Full abstraction. The full abstraction of $\mathcal{M}_\infty$ has been thoroughly studied. For lack of direct results, the full abstraction has been studied for restrictions and extensions of the $\partial\lambda$-calculus: For the untyped λ-calculus (which deterministic and linear-free fragment of the $\partial\lambda$-calculus); For the orthogonal bang-free restriction where the applications only accepts bags of linear arguments; For the extension with tests of [?], an extension with must non-determinism and with operators inspired by 0-ary *par* and *tensor product* that could be added freely in DiLL-proof nets.

These studies were encouraging since they systematically showed MRel to be fully abstract for these calculi ([?] for untyped λ-calculus , [?] for the bag-free restriction and [?] for resource λ-calculus with tests). Therefore Bucciarelli *et.al.* [?] conjectured a full abstraction for the $\partial\lambda$-calculus.

The counter-example. The purpose of this article is to expose a highly unexpected counter-example to this conjecture. We will see how an untyped fixpoint and a may sum can combine to produce a term $\boldsymbol{A}$ (Equation 7) behaving like an infinite sum $\Sigma_{i\geq 1}\boldsymbol{B}_i$ where every $\boldsymbol{B}_i$ begins with $(i{+}1)$ λ-abstractions, put its $(i{+}1)^{th}$ argument in head position but otherwithe behave as the identity in applicative context with exactly i arguments; that how $\boldsymbol{A}$ can be thought to have an arbitrary number of λ-abstraction. Such a term can thus look for an argument further than the length of any bounded applicative context. There lies the immediate interest of achieving a context lemma (what have not yet been done for this calculus) in order to prove that the observational equivalence is so short-sighted. This will refute the inequational full abstraction since the relational semantics can sublimate this short-sightedness. More concretely we will see that $\boldsymbol{A}$ is observationally above the identity but not denotationally. It is not difficult, then, to refute the equational full abstraction.

We proceed in this order. Section 2 present the $\partial\lambda$-calculus and its properties. Section 3 describe MRel and its reflexive object $\mathcal{M}_\infty$, and see how it is related to $\partial\lambda$-calculus. Section 4 gives our results with the context lemma followed by the counter-example (Theorem 8). We will also discuss the generality of this counter-example in the conclusion and explain how it is representative of an unhealthy interaction between untyped fixpoints and *may-non-determinism* that can be reproduced in other calculi like the *may-non-deterministic* extention of λ-calculus.

2 Syntax

2.1 $\partial\lambda$-calculus

In this section we give some background on the $\partial\lambda$-calculus, a lambda calculus with resources. Here is the grammar of its syntax:

$$
\begin{array}{llll}
\text{(terms)} & \Lambda : & L, M, N ::= & x \quad | \quad \lambda x.M \quad | \quad M\ P \\
\text{(bags)} & \Lambda^b : & P, Q ::= & 1 \quad | \quad [M] \quad | \quad [M^!] \quad | \quad P{\cdot}Q \\
\text{(sums)} & \mathbb{A}, \mathbb{A}^b : & \mathbb{L}, \mathbb{M} \in \mathbb{N}\langle\Lambda\rangle & \mathbb{P}, \mathbb{Q} \in \mathbb{N}\langle\Lambda^b\rangle
\end{array}
$$

Fig. 1: Grammar of the $\partial\lambda$-calculus

The $\partial\lambda$-calculus extends the standard λ-calculus in two directions. First, it is a non deterministic λ-calculus. The argument of an application is a superposition of inputs, called *bag of resources* and denoted by a multiset in multiplicative notation (namely $P{\cdot}Q$ is the disjoint union of P and Q). Symetricaly, the result of a reduction step is a superposition of outputs denoted by a multiset in additive notation (namely $\mathbb{L}{+}\mathbb{M}$ is the disjoint union of $\mathbb{L}$ and $\mathbb{M}$).We also have empty multisets, expressing an absence of available inputs (denoted by 1) or of results (denoted by 0).

Second, the $\partial\lambda$-calculus distinguishes between *linear* and *reusable* resources. The formers will never suffer any duplication or erasing regardless of the reduction strategy. A reusable resource will be denoted by a banged term $M^!$ in a bag, e.g. $[N^!, L, L]$ is a bag of two linear occurrences of the resource L and a reusable occurrence of the resource N. We use the notation $N^{(!)}$ whenever we do not explicit whether M occurs linearly or not in a bag.

Finally, keeping all possible results of a reduction step into a finite multiset $\Sigma_i M_i$ of outcomes allows to have a confluent rewriting system in such a non-deterministic setting [?].

Small Latin letters $x, y, z, ...$ will range over an infinite set of λ-calculus variables. Capital Latin letters L, M, N (resp. P, Q, R) are meta-variables for terms (resp bags). Initial capital Latin letters E, F will denote indifferently terms and bags and will be called *expressions*. Finally, the meta-variables $\mathbb{L}, \mathbb{M}, \mathbb{N}$ (resp $\mathbb{P}, \mathbb{Q}, \mathbb{R}$) vary over sums (*i.e.* multisets in additive notation) of terms (resp. bags). Bags and sums are multisets, so we are assuming associativity and commutativity of the disjoint union and neutrality of the empty multiset.

Notice that the sum operator is always at the top level of the syntax trees. This is a design choice taken from [?] allowing for a lighter syntax. However, it is sometimes convenient to write sums inside an expression as a short notation for the expression obtained by distributing the sums following the conventions:

$$\lambda x.(\Sigma_i M_i) := \Sigma_i(\lambda_x.M_i) \qquad (\Sigma_i M_i)\ (\Sigma_j P_j) := \Sigma_{i,j}(M_i\ P_j)$$
$$[(\Sigma_i M_i)^!]\cdot P \ := \ [M_1^!, \ldots, M_n^!]\cdot P \qquad [\Sigma_i M_i]\cdot P \ := \ \Sigma_i [M_i]\cdot P$$

Notice, every construct is (multi)-linear but the bang $()^!$, where we apply the linear logic equivalence $[(M+N)^!] = [M^!]\cdot[N^!]$ which is reminiscent of the standard exponential rule $e^{a+b} = e^a \cdot e^b$.

Since we have two kind of resources, we need two different substitutions: the usual one, denoted $\{.\}$, and the linear one, denoted $\langle . \rangle$. Supposing that $x \neq y$, $x \neq z$ and $z \notin \mathrm{FV}(N)$:

$$x\langle N/x\rangle := N \qquad y\langle N/x\rangle := 0 \qquad (\lambda z.M)\langle N/x\rangle := \lambda z.(M\langle N/x\rangle)$$
$$(M\ P)\langle N/x\rangle := (M\langle N/x\rangle\ P) + (M\ P\langle N/x\rangle)$$
$$[M^!]\langle N/x\rangle := [M\langle N/x\rangle, M^!] \qquad [M]\langle N/x\rangle := [M\langle N/x\rangle]$$
$$(P\cdot Q)\langle N/x\rangle := (P\langle N/x\rangle)\cdot Q + P\cdot(Q\langle N/x\rangle) \qquad 1\langle N/x\rangle := 0$$

Notice that in the above definition we are heavily using the notational convention of the distributing sums. For example, $[x^!, y]\langle N/x\rangle := [x\langle N/x\rangle, x^!, y] + [x^!, y\langle N/x\rangle] := [N, x^!, y] + [x^!, 0] = [N, x^!, y] + 0 = [N, x^!, y]$. Substitutions enjoy the following commutation properties:

Lemma 1 ([?]). *For an expression E and terms M, N, if $x \notin \mathrm{FV}(N)$ and $y \notin \mathrm{FV}(M)$ (potentially $x{=}y$) then:*

$$E\langle M/x\rangle\langle N/y\rangle = E\langle N/y\rangle\langle M/x\rangle \qquad E\{(M{+}x)/x\}\langle N/y\rangle = E\langle N/y\rangle\{(M{+}x)/x\}$$
$$E\{(M{+}x)/x\}\{(N{+}y)/y\} = E\{(N{+}y)/y\}\{(M{+}x)/x\}\}$$

Hence the notion of substitution of variables by bags, denoted $\langle\!\langle s\rangle\!\rangle$ (where s is a list of substitutions P/x), may be defined as follows: if $x \notin \mathrm{FV}(N) \cup \mathrm{FV}(P)$:

$$M\langle\!\langle 1/x\rangle\!\rangle \ := \ M\{0/x\} \qquad M\langle\!\langle [N^!]\cdot P/x\rangle\!\rangle \ := \ M\{(x{+}N)/x\}\langle\!\langle P/x\rangle\!\rangle$$
$$M\langle\!\langle [N]\cdot P/x\rangle\!\rangle \ := \ M\langle N/x\rangle\langle\!\langle P/x\rangle\!\rangle \qquad M\langle\!\langle s_1; s_2\rangle\!\rangle \ := \ M\langle\!\langle s_1\rangle\!\rangle\langle\!\langle s_2\rangle\!\rangle$$

2.2 Beta and *outer reduction*

Reduction is defined essentially as the contextual closure of the β-rule.

$$\frac{}{(\lambda x.M)\ P \to M\langle\!\langle P/x\rangle\!\rangle}\ \beta \qquad \frac{M \to \mathbb{M}}{M\ P \to \mathbb{M}\ P}\ \text{left} \qquad \frac{M \to \mathbb{M}}{\lambda x.M \to \lambda x.\mathbb{M}}\ \text{abs}$$

$$\frac{N \to \mathbb{N}}{M\ [N]{\cdot}P \to M\ [\mathbb{N}]{\cdot}P}\ \text{lin} \qquad \frac{N \to \mathbb{N}}{M\ [N^!]{\cdot}P \to M\ [\mathbb{N}^!]{\cdot}P}\ !$$

$$\frac{M \to \mathbb{M}' \qquad \mathbb{N} \to \mathbb{N}'}{M{+}\mathbb{N} \to \mathbb{M}'{+}\mathbb{N}'}\ s_1 \qquad \frac{M \to \mathbb{M}'}{M{+}\mathbb{N} \to \mathbb{M}'{+}\mathbb{N}}\ s_2$$

Fig. 2: Reduction rules

Rules s_1 and s_2 allow to reduce one or more terms of a sum in a single step (this is used in Theorem 1).
In the following example and all along this article we denote:

$$\boldsymbol{\omega} := \lambda x.x[x^!] \qquad \boldsymbol{I} := \lambda x.x \qquad \boldsymbol{\Delta} := \lambda gu.u\ [(g\ [g^!]\ [u^!])^!] \qquad \boldsymbol{\Theta} := \boldsymbol{\Delta}[\boldsymbol{\Delta}^!]$$

Example 1.

$$\boldsymbol{I}\ [u^!, v^!] \to u{+}v \qquad (\lambda x.y\ [(x\ [y])^!])\ [u, v^!] \to y\ [u\ [y], (v\ [y])^!] \tag{1}$$

$$(\lambda x.x\ [x, x^!])\ [u, v^!] \to (u\ [v, v^!]){+}(v\ [u, v^!]){+}(v\ [v, u, v^!]) \tag{2}$$

$$u\ [\boldsymbol{I}\ 1] \to 0 \qquad u\ [(\boldsymbol{I}\ 1)^!] \to u\ 1 \tag{3}$$

$$\boldsymbol{\omega}\ [\boldsymbol{\omega}^!] \to \boldsymbol{\omega}\ [\boldsymbol{\omega}^!] \qquad \boldsymbol{\omega}\ [\boldsymbol{\omega}] \to 0 \tag{4}$$

$$\boldsymbol{\Theta}\ [v^!] \to\!\!\to (v\ [(\boldsymbol{\Theta}\ [v^!])^!]) \tag{5}$$

$$\boldsymbol{\Theta}\ [u, v^!] \to\!\!\to (u\ [(\boldsymbol{\Theta}\ [v^!])^!]) + (v\ [(\boldsymbol{\Theta}[u, v^!]), (\boldsymbol{\Theta}[v^!])^!]) \tag{6}$$

As customary, a notion of convergence will be used for relating the operational and denotational semantics of the $\partial\lambda$-calculus.

In this paper, we consider the *may-outer convergence* of [**?**]. The attribute *may* refers to an angelic notion of non-determinism, hence $M{+}N$ will converges whenever at least one of the two converges. Indeed, the demonic (must) convergence is also of great interest, however it is harder to deal with (see [**?**]), in fact the demonic non-determinism bad interacts with the Taylor expansion, which is a crucial tool in our analysis (section 2.3). Moreover, the attribute *outer* refers to the fact that we reduce only redexes not under the scope of a bang. This turns out to be the analogous of the head-reduction in the λ-calculus.

Definition 1 (*onf* and *monf*). *A term is in* outer-normal form*,* onf *for short, iff it has no redexes but under a* !*, that is a term of the form:*

$$\lambda x_1, \ldots, x_m.y\ [N^{(!)}_{1,1}, \ldots, N^{(!)}_{1,k_1}] \cdots [N^{(!)}_{n,1}, \ldots, N^{(!)}_{n,k_n}]$$

Where every $N^{(!)}_{i,j}$ are either banged or in outer-normal form*.*
A sum of terms is in may-outer-normal form*,* monf *for short, iff at least one of its elements is in* outer-normal form *(in particular* 0 *is not a monf).*

This notion generalizes the one of head-normal form of the untyped lambda calculus. Asking for linear terms of a bag to be in *monf* is a way of expressing that $x\ [\boldsymbol{\omega}\ [\boldsymbol{\omega}^!]]$ diverges while $x\ [(\boldsymbol{\omega}\ [\boldsymbol{\omega}^!])^!]$ is an onf. *Monf*'s correspond to *may-solvability* [**?**] in the same way as head-normal-forms correspond to solvability in untyped λ-calculus. From previous examples only contracta of (3.1), (4.1) and (4.2) are not *monf*, and only (3.2)'s redex is.
The restricted reduction leading to the (principal) *monf* of a term is the following:

Definition 2. *The* outer reduction*, denoted* $\to_o$ *is defined by the rules of Figure 2 but the rule* !*, which is omitted. We denote by* $\to^*$ *and* $\to_o^*$ *the reflexive and transitive closures of* $\to$ *and* $\to_o^*$*, respectively.*

In the Example 1, all reductions but the (3.2) are *outer reductions.*

Lemma 2 ([?]). *If* $M \to^* \mathbb{M}$ *and* $\mathbb{M}$ *is in* monf*, then there exists a* monf $\mathbb{N}$ *such that* $M \to_o^* \mathbb{N} \to^* \mathbb{M}$*. Thus the convergence to a* monf *and the* outer convergence *to a* monf *coincides.*

We will write $M \Downarrow_n$ if there exists a *monf* $\mathbb{M}$ and an *outer reduction* sequence from M to $\mathbb{M}$ of length at most n. We will write $M \Downarrow$ if there exists n such that $M \Downarrow_n$ and say that M outer converges.
The two rules s_1 and s_2 of Figure 2 allow the followings:

Theorem 1. *The* outer reduction *of the* $\partial\lambda$*-calculus is strongly confluent.*

Corollary 1. *If* $M \Downarrow_{n+1}$ *and* $M \to_o \mathbb{N}$ *then exists* $N \in \mathbb{N}$ *such that* $N \Downarrow_n$*.*

2.3 Taylor Expansion

A natural restriction of the $\partial\lambda$-calculus is the fragment $\partial\lambda^\ell$ which is obtained by removing the bang construction $[M^!]$ in Figure 1. This restriction has a very limited computational power, due to the following theorem:

Theorem 2 ([Folklore]). *The reduction* $\to$ *in* $\partial\lambda^\ell$ *is strongly normalizing.*

Proof. We set an order $\sqsubseteq$ on the finite multisets of terms generated by $\mathbb{M} \sqsubseteq \mathbb{N}$ if $\mathbb{M} = \mathbb{M}'+\mathbb{L}$, $\mathbb{N} = \mathbb{N}'+\mathbb{L}$ and there exists $N \in \mathbb{N}'$ such that for all $M \in \mathbb{M}'$, $|M| \leq |N|$ (where $|M|$ is the structural size of M). Then, $\to$ is strictly decreasing in this well founded order. $\square$

The main interest of $\partial\lambda^\ell$ comes with the Taylor expansion. The Taylor expansion of a λ-term M has been developed in [**?**,**?**] and it recalls the usual decomposition of an analytic function:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n(f)(0)x^n$$

In this paper, we are interested only in the support of the Taylor expansion of a $\partial\lambda$-term M defined in [**?**,**?**], *i.e.* in the set M^o of the $\partial\lambda^\ell$-terms appearing in the Taylor expansion of M with non-null coefficient. Such a set can be defined as follows.

Definition 3. *The Taylor expansion E^o of an expression E is a (possibly infinite) set of linear expressions defined by structural induction:*

$$(\lambda x.M)^o := \{\lambda x.M' | M' \in M^o\} \qquad (M\ P)^o := \{M'P' | M' \in M^o,\ P' \in P^o\}$$
$$[M]^o := \{[M'] | M' \in M^o\} \qquad (P{\cdot}Q)^o := \{P'{\cdot}Q' | P' \in P^o,\ Q' \in Q^o\}$$
$$[M^!]^o := \{[M_1, \ldots, M_n] | n \geq 0,\ M_1, \ldots, M_n \in M^o\} \quad 1^o := \{1\} \quad x^o := \{x\}$$

Lemma 3 ([?]). *For any term M and for any $N \in M^o$, if $N \to^* \mathbb{N}$ then there exists $\mathbb{M}$ such that $M \to^* \mathbb{M}$ and $\mathrm{supp}(\mathbb{N}) \subseteq \bigcup\{M'^o | M' \in \mathbb{M}\}$ where $\mathrm{supp}(\mathbb{N})$ denotes the support of the multiset $\mathbb{N}$.*

3 Model

3.1 Categorical model

We recall the interpretation of the $\partial\lambda$-calculus into the reflexive object $\mathcal{M}_\infty$ of MRel. MRel is the Cartesian closed category resulting from the co-Kleisli construction associated with the exponential comonad of the category Rel of sets and relations, which is a well-known model of Linear Logic (and Diferential Linear Logic). We refer to [?] for detailed exposition, here we briefly present MRel and the object $\mathcal{M}_\infty$.

The objects of MRel are the sets. Its morphisms from A to B are the relations from the set of the finite multi-sets of A, namely $\mathbb{N}\langle A\rangle$, and B; *i.e.* $\mathbf{MRel}(A, B) := \mathcal{P}(\mathbb{N}\langle A\rangle \times B)$. The composition of $f \in \mathbf{MRel}(A,B)$ and $g \in \mathbf{MRel}(B,C)$ is given by
$f;g = \{(a, \gamma) \in \mathbb{N}\langle A\rangle \times C \mid \exists (a_1,\beta_1),...,(a_n,\beta_n) \in f,\ a = \Sigma_i a_i \text{ and } ([\beta_1, \ldots, \beta_n], \gamma) \in g\}$
The identities are $\mathrm{id}_A := \{[\alpha], \alpha | \alpha \in A\}$. Given a family $(A_i)_{i\in I}$, its Cartesian product is $\&_{i\in I} A_i := \{(i, \alpha) | i \in I, \alpha \in A_i\}$;with the projections $\pi_i := \{([(i, \alpha)], \alpha) | \alpha \in A_i\}$. The terminal object is the empty set. And the exponential object internalizing $\mathbf{MRel}(A, B)$ is $A \Rightarrow B := \mathbb{N}\langle A\rangle \times B$. Then the adjuction $\mathbf{MRel}(A \& B, C) \simeq \mathbf{MRel}(A, B \Rightarrow C)$ holds since $\mathbb{N}\langle \& A_i\rangle \simeq \prod_i \mathbb{N}\langle A_i\rangle$.

The reflexive object we choose is the simplest stratified object[2] of [?]. It can be recursively defined by (see [?]):

$$\mathcal{M}_0 := \emptyset \qquad \mathcal{M}_{n+1} := \mathbb{N}\langle \mathcal{M}_n\rangle^{(\omega)} \qquad \mathcal{M}_\infty := \bigcup_n \mathcal{M}_n$$

Where $\mathbb{N}\langle M\rangle^{(\omega)}$ is the list of quasi everywhere empty multisets over M. Its element can be generated by:

(points)	$\mathcal{M}_\infty$:	$\alpha, \beta, \gamma ::=$	$* \mid a{::}\alpha$
(multisets)	$\mathcal{M}^b_\infty$:	$a, b, c ::=$	$[\alpha_1, \ldots, \alpha_n]$

[2] The others stratified object sharing the element $*$ will also be subject to the counter-example.

Where $*$, the unique element of $\mathcal{M}_1$, namely the infinite list of empty multisets, respect the equation:

$$* = []::*$$

The morphisms app $\in$ $\mathbf{MRel}(\mathcal{M}_\infty, \mathcal{M}_\infty \Rightarrow \mathcal{M}_\infty)$ and abs $\in$ $\mathbf{MRel}(\mathcal{M}_\infty \Rightarrow \mathcal{M}_\infty, \mathcal{M}_\infty)$ are defined by:

$$\mathit{app} := \{([a::\alpha], (a, \alpha)) | (a, \alpha) \in \mathcal{M}_\infty\} \qquad \mathit{abs} := \{([(a, \alpha)], a::\alpha) | a, \alpha \in \mathcal{M}_\infty\}$$

One can easily check that $\mathit{abs}; \mathit{app} = \mathrm{Id}_{\mathcal{M}_\infty \Rightarrow \mathcal{M}_\infty}$ (and even $\mathit{app}; \mathit{abs} = \mathrm{Id}_{\mathcal{M}_\infty}$).

We could have interpreted the terms of the $\partial\lambda$-calculus by using the categorical structure of MRel. However, we prefer to give a description of such an interpretation, using a non-idempotent intersection type system, following [?].

The grammar of the types correspond exactly to the grammar of $\mathcal{M}_\infty$ where the cons operator represent the usual arrow and the multisets notation replaces the intersection notation.

To increase readability, we present the interpretation of terms in the model through this type system and the multisets of $\mathcal{M}^b_\infty$ will be denoted multiplicatively. A *typing context* is a finite partial function from variables into multisets in $\mathcal{M}^b_\infty$, we denote $(x_i : a_i)_{i \in I}$ the context associating x_i to a_i for $i \in I$. We have two kinds of typing judgements, depending whether we type terms or bags: the former are typed by points in $\mathcal{M}_\infty$ and the latter by multisets in $\mathcal{M}^b_\infty$.

$$\frac{\Gamma \vdash M : \alpha}{x : [], \Gamma \vdash M : \alpha} \qquad \frac{\Gamma \vdash P : a}{x : [], \Gamma \vdash P : a} \qquad \frac{}{x : [\alpha] \vdash x : \alpha} \qquad \frac{\Gamma \vdash M : \alpha}{\Gamma \vdash M + \mathbb{M} : \alpha}$$

$$\frac{\Gamma, x : a \vdash M : \alpha}{\Gamma \vdash \lambda x.M : a::\alpha} \qquad \frac{(x_i : a_i)_{i \in I} \vdash M : b::\alpha \qquad (x_i : a'_i)_{i \in I} \vdash P : b}{(x_i : a_i \cdot a'_i)_{i \in I} \vdash M\ P : \alpha}$$

$$\frac{}{\vdash 1 : []} \qquad \frac{(x_i : a_i)_{i \in I} \vdash P : b \qquad (x_i : a'_i)_{i \in I} \vdash Q : c}{(x_i : a_i \cdot a'_i)_{i \in I} \vdash P \cdot Q : b \cdot c}$$

$$\frac{(x_i : a_i)_{i \in I} \vdash L : \beta}{(x_i : a_i)_{i \in I} \vdash [L] : [\beta]} \qquad \frac{(x_i : a^j_i)_{i \in I} \vdash L : \beta_j \qquad \text{for } j \leq m}{(x_i : \Pi_{j \leq m} a^j_i)_{i \in I} \vdash [L^!] : [\beta_1, \ldots, \beta_m]}$$

The usual presentation of the interpretation can be recovered with:

$$[\![\mathbb{M}]\!]^{x_1,\ldots,x_n} := \{((a_1, \ldots, a_n), \beta) | (x_i : a_i)_i \vdash M : \beta\} \quad \in \mathbf{MRel}(\mathop{\&}_{i=1}^{n} \mathcal{M}_\infty, \mathcal{M}_\infty)$$

$$[\![\mathbb{P}]\!]^{x_1,\ldots,x_n} := \{((a_1, \ldots, a_n), b) | (x_i : a_i)_i \vdash P : b\} \quad \in \mathbf{MRel}(\mathop{\&}_{i=1}^{n} \mathcal{M}_\infty, \mathcal{M}^b_\infty)$$

Theorem 3. *If* $\mathbb{M} \to \mathbb{N}$ *then* $[\![\mathbb{M}]\!]^{x_1,\ldots,x_n} = [\![\mathbb{N}]\!]^{x_1,\ldots,x_n}$

An important characteristic of this model that seems to make it particularly suitable for our original purpose is that it models the Taylor expansion:

Theorem 4 ([?]). *For any term* M, $[\![M]\!]^{\bar{x}} = \bigcup_{N \in M^\circ} [\![N]\!]^{\bar{x}}$

3.2 Observational order and adequacy

A first important result relating syntax and semantics is the sensibility theorem, a corollary of [?], but here reproved focussing on the role of the Taylor expansion.

Theorem 5. *$\mathcal{M}_\infty$ is sensible for* may-outer-convergence *of the $\partial\lambda$-calculus,* i.e.

$$\forall M, \quad M \Downarrow \;\Leftrightarrow\; [\![M]\!] \neq \emptyset$$

Proof. The left-to-right side is trivial since any *monf* has a non-empty interpretation.
Conversely, assume $(\bar{a}, \alpha) \in [\![M]\!]$, by Theorem 4 there exists $N \in M^o$ such that $(\bar{a}, \alpha) \in [\![N]\!]$. Any term of $\partial\lambda^\ell$-calculus converges either to 0 or to a normal form $N_0 + \mathbb{N}$ (by Theorem 2). Since $[\![0]\!] = \emptyset$, N converges into a normal form. By applying Lemma 3, we thus have $M \to^* M_0 + \mathbb{M}$ with $N_0 \in M_0^o$. Since the Taylor expansion conserves every redexes, M_0 is *outer-normal* and M is *may-outer converging.* □

Corollary 2. *A term is may-outer converging iff one of the elements of its Taylor expansion is may-outer converging: $M{\Downarrow} \Leftrightarrow \exists N{\in}M^o; N{\Downarrow}$. Equivalently, a term is* may-outer diverging *iff any element of its Taylor expansion reduces to 0*

Proof. For any closed term M, using Theorems 4 and 5:
$M{\Downarrow} \;\Leftrightarrow_{\text{th5}}\; [\![M]\!]{\neq}\emptyset \;\Leftrightarrow_{\text{th4}}\; \exists N{\in}M^o; [\![N]\!]{\neq}\emptyset \;\Leftrightarrow_{\text{th5}}\; \exists N{\in}M^o; N{\Downarrow}$ □

In the following we use contexts that are terms with holes that will be filled by terms. Contexts can be described by the grammar:

(contexts)	$\Lambda(\!\vert.\vert\!)$:	$C(\!\vert.\vert\!)$ ::=	$(\!\vert.\vert\!) \mid M \mid \lambda x.C(\!\vert.\vert\!) \mid C(\!\vert.\vert\!)\; P(\!\vert.\vert\!)$
(bag-contexts)	$\Lambda^b(\!\vert.\vert\!)$:	$P(\!\vert.\vert\!)$ ::=	$[C_1(\!\vert.\vert\!)^{(!)}, \ldots, C_n(\!\vert.\vert\!)^{(!)}]$

We define the notions of observational pre-order and equivalence using as basic observation the *may-outer-convergence* of terms. This is not the only possibility (*must* or *inner* declensions); we discuss this issue in the conclusion.

Definition 4. *For two terms M and N, we say that M is observationally below N (denoted $M \leq_o N$), if for all context $C(\!|.|\!)$:*

$$C(\!|M|\!) \Downarrow \;\Rightarrow\; C(\!|N|\!) \Downarrow$$

They are observationally equivalent (denoted $M \equiv_o N$) if $M \leq_o N$ and $N \leq_o M$

Using sensibility we thus assert our adequation.

Theorem 6. *$\mathcal{M}_\infty$ is inequationally adequate for $\partial\lambda$-calculus,*

$$\forall M, N, \quad [\![M]\!] \subseteq [\![N]\!] \;\Rightarrow\; M \leq_o N$$

Proof. Assume that $[\![M]\!] \subseteq [\![N]\!]$ and $C(\!|M|\!) \Downarrow$. Then since $[\![.]\!]$ is defined by structural induction we have $[\![C(\!|N|\!)]\!] \supseteq [\![C(\!|M|\!)]\!] \neq \emptyset$ and $C(\!|N|\!) \Downarrow$. □

4 Failure of the full abstraction

The main result of this paper is the refutation of the full abstraction conjecture:

Conjecture 1 ([?]). $\mathcal{M}_\infty$ is fully abstract for $\partial\lambda$-calculus. *i.e.* the denotational and the observational equivalences are identical:

$$\forall M, N, \quad [\![M]\!] = [\![N]\!] \quad \Leftrightarrow \quad M \equiv_o N$$

Its refutation (Theorem 8) proceeds as follows. First, we define a term $\boldsymbol{A}$ (Equation 7) and we prove that $\boldsymbol{I} \leq_o \boldsymbol{A}$ (Lemma 7, which uses a context lemma: Theorem 7), but $[\![\boldsymbol{I}]\!] \not\subseteq [\![\boldsymbol{A}]\!]$ (Lemma 9). This result in the refutation of the stronger conjecture:

Conjecture 2 ([?]). $\mathcal{M}_\infty$ is inequationally fully abstract for $\partial\lambda$-calculus. *i.e.* the denotational and the observational orders are identical:

$$\forall M, N, \quad [\![M]\!] \subseteq [\![N]\!] \quad \Leftrightarrow \quad M \leq_o N$$

Then we consider the term $\boldsymbol{A'} := \boldsymbol{I}\ [\boldsymbol{A}^!, \boldsymbol{I}^!]$ and we prove that $\boldsymbol{A'}$ and $\boldsymbol{A}$ yield a counter-example to Conjecture 1 (Theorem 8).

4.1 Context lemma

Definition 5. Linear contexts *are contexts with one and only one hole and with this hole in linear position:*

(linear contexts) $\Lambda(\!|.|\!)_l :\ \ D(\!|.|\!) ::= \ (\!|.|\!) \ \mid\ \lambda x.D(\!|.|\!) \ \mid\ D(\!|.|\!)\ P \ \mid\ M\ [D(\!|.|\!)]\cdot P$

The applicative contexts *are particular linear contexts of the form* $K(\!|.|\!) = (\lambda x_1 \ldots x_n.(\!|.|\!))\ P_1 \cdots\ P_k$

Lemma 4. *For any term* M *and any bags* P, Q*, there exists a decomposition* $P = P^{l_1}\cdot P^{l_2}\cdot P^e$ *such that* $P^{l_1}\cdot P^{l_2}$ *is linear (*i.e. *is a multiset of non-banged terms),* P^e *exponential (*i.e. *is a multiset of banged terms) and if* $(M\ Q)\langle\!\langle P/x\rangle\!\rangle \Downarrow_n$ *then* $M\langle\!\langle P^{l_1}\cdot P^e/x\rangle\!\rangle\ Q\langle\!\langle P^{l_2}\cdot P^e/x\rangle\!\rangle \Downarrow_n$

Proof. By definition of the may convergence, since $(M\ Q)\langle\!\langle P/x\rangle\!\rangle = \Sigma_{P=P^{l_1}.P^{l_2}.P^e} M\langle\!\langle P^{l_1}\cdot P^e/x\rangle\!\rangle\ Q\langle\!\langle P^{l_2}\cdot P^e/x\rangle\!\rangle$ □

Lemma 5 (Linear context lemma). *For any terms* M *and* N*, if there is a linear context* $D(\!|.|\!)$ *such that* $D(\!|M|\!) \Downarrow$ *and* $D(\!|N|\!) \Uparrow$ *then there is an applicative context that does the same.*

Proof. We will prove the following stronger property:
For every terms M, N, every bags $P_1, \ldots, P_{p+q}$, and every variables $x_1, \ldots, x_p \notin \bigcup_{1\leq i\leq p+q} \mathrm{FV}(P_i)$, if $\langle\!\langle s\rangle\!\rangle := \langle\!\langle P_1/x_1; \ldots; P_p/x_p\rangle\!\rangle$ and if a linear context $D(\!|.|\!)$ is such that $(D(\!|M|\!)\langle\!\langle s\rangle\!\rangle\ P_{p+1}\ \cdots\ P_{p+q}) \Downarrow_n$ and $(D(\!|N|\!)\langle\!\langle s\rangle\!\rangle\ P_{p+1}\ \cdots\ P_{p+q}) \Uparrow$ then there exists an applicative context $K(\!|.|\!) = (\lambda \bar{y}.(\!|.|\!))\ Q_1\ \cdots\ Q_n$ such that $K(\!|M|\!) \Downarrow$ and $K(\!|N|\!) \Uparrow$.
By cases, making induction on the lexicographically ordered pair $(n, D(\!|.|\!))$:

– If $D(\!|.|\!) = (\!|.|\!)$:
$K(\!|.|\!) = (\lambda x_1, ..., x_p(\!|.|\!))\ P_1\ \cdots\ P_{p+q}$
– If $D(\!|.|\!) = \lambda z.D'(\!|.|\!)$:
  - If $q = 0$:
    The hypothesis gives $D'(\!|M|\!)\langle\!\langle s\rangle\!\rangle \Downarrow_n$ and $D'(\!|N|\!)\langle\!\langle s\rangle\!\rangle \Uparrow$, thus we can directly apply our induction hypothesis on $D'(\!|.|\!)$. That gives the required result.
  - Otherwise:
    By assuming that z does not appear in $P_{p+2}, ..., P_{p+q}$:
    The hypothesis and Corollary 1 gives $(D'(\!|M|\!)\langle\!\langle P_1/z; s\rangle\!\rangle\ P_{p+2} \cdots P_{p+q}) \Downarrow_{n-1}$. Moreover $(D'(\!|N|\!)\langle\!\langle P_1/z; s\rangle\!\rangle\ P_2 \cdots P_q) \Uparrow$.
    Then the induction hypothesis gives the required result.
– If $D(\!|.|\!) = L\ [D'(\!|.|\!)]{\cdot}Q$:
By assuming that $x_i \notin \mathrm{FV}(P_j)$ for $i \leq j$ and by Lemma 4, there exists, for all $i \leq p$, a decomposition $P_i = P_i^{l_1}{\cdot}P_i^{l_2}{\cdot}P_i^e$ such that if $\langle\!\langle s_j\rangle\!\rangle := \langle\!\langle P_1^{l_j}{\cdot}P_1^e/x_1; ...; P_p^{l_j}{\cdot}P_p^e/x_p\rangle\!\rangle$, there is $L' \in L\langle\!\langle s_1\rangle\!\rangle$ with $(L'\ ([D'(\!|M|\!)]{\cdot}Q)\langle\!\langle s_2\rangle\!\rangle\ P_1 \cdots\ P_q) \Downarrow_n$ and $(L'\ ([D'(\!|N|\!)]{\cdot}Q)\langle\!\langle s_2\rangle\!\rangle\ P_1 \cdots\ P_q) \Uparrow$.
Then there are two cases. Either $L' \to_o \mathbb{L}$ and there is $L'' \in \mathbb{L}$ such that $((L''\ [D'(\!|M|\!)]{\cdot}Q)\langle\!\langle s_2\rangle\!\rangle\ P_1 \cdots\ P_q) \Downarrow_{n-1}$ (using Corollary 1) that allow us to apply the induction hypothesis. Or L' is in *outer-normal form*:
  - if $L' \to^* \lambda z.L''$:
    Let $D''(\!|.|\!) = L''\langle\!\langle [D'(\!|.|\!)]{\cdot}Q/z\rangle\!\rangle$.
    We have $(D''(\!|M|\!)\langle\!\langle s_2\rangle\!\rangle\ P_1 \cdots\ P_q) \Downarrow_{n-1}$ and $(D''(\!|N|\!)\langle\!\langle s_2\rangle\!\rangle\ P_1 \cdots\ P_q) \Uparrow$.
    Then we can apply our induction hypothesis on $D'(\!|.|\!)$ that is still a linear context since $D'(\!|.|\!)$ was not under a bang. This results exactly in the required applicative context.
  - if $L' \to^* y\ Q_1 \cdots Q_r$ with $y \neq x_i$ for all i:
    There exists, for all $i \leq p$, a multiset $P_i^{l_3} \subseteq P_i^{l_2}$ such that $D'(\!|M|\!)\langle\!\langle P_1^{l_3}{\cdot}P_1^e/x_1; ...; P_p^{l_3}{\cdot}P_p^e/x_p\rangle\!\rangle \Downarrow_n$ and $D'(\!|N|\!)\langle\!\langle P_1^{l_3}{\cdot}P_1^e/x_1; ...; P_p^{l_3}{\cdot}P_p^e/x_p\rangle\!\rangle \Uparrow$. Then we can apply the induction hypothesis on $D'(\!|.|\!)$.
– If $D(\!|.|\!) = D'(\!|.|\!)\ Q$:
By Lemma 4, there exists $P_i^{\ell_1}{\cdot}P_i^{\ell_1}{\cdot}P_i^e = P_i$ such that, if we denote $\langle\!\langle s_j\rangle\!\rangle := \langle\!\langle P_1^{\ell_i}{\cdot}P_1^e/x_1; ...; P_p^{\ell_i}{\cdot}P_p^e/x_p\rangle\!\rangle$:

$$(D'(\!|M|\!)\langle\!\langle s_1\rangle\!\rangle\ Q\langle\!\langle s_2\rangle\!\rangle\ P_{p+1}\ \cdots\ P_{p+q}) \Downarrow \quad (D'(\!|N|\!)\langle\!\langle s_1\rangle\!\rangle\ Q\langle\!\langle s_2\rangle\!\rangle\ P_{p+1}\ \cdots\ P_{p+q}) \Uparrow$$

The induction hypothesis on $D'(\!|.|\!)$ (with $Q\langle\!\langle s_2\rangle\!\rangle$ seen as one of the P_i's) results in the required applicative context.

□

Theorem 7 (Context lemma). *For any terms M and N, if there is a context $C(\!|.|\!)$ such that $C(\!|M|\!) \Downarrow$ and $C(\!|N|\!) \Uparrow$ then there is an applicative context that does the same.*

Proof. Let $C(\!|.|\!)$ be such a context.
Let $\{x_1, ..., x_n\} = \mathrm{FV}(M) \cup \mathrm{FV}(N)$ be the free variables of M and N.
Let $L = \lambda u.C(\!|u\ [x_1^!]\ \cdots\ [x_n^!]|\!)$, $D(\!|.|\!) = \lambda x_1...x_n(\!|.|\!)$ and $C'(\!|.|\!) = L\ [D(\!|.|\!)^!]$.
Notice that $C'(\!|M|\!) \to^* C(\!|M|\!)$ and $C'(\!|N|\!) \to^* C(\!|N|\!)$. Hence, the hypothesis and Lemma2 gives $C'(\!|M|\!) \Downarrow$ and $C'(\!|N|\!) \Uparrow$. Moreover, we have that $C'(\!|M|\!) = \bigcup_{n\geq 0}(L\ [D(\!|M|\!)^n)^o$; thus, by applying twice Corollary 2 we have an $n \in \mathcal{N}$ such that $L\ [D(\!|M|\!)^n] \Downarrow$. Also, since $(L\ [D(\!|N|\!)^n])^o \subseteq C'(\!|N|\!)^o$, the same corollary and the hypothesis $C'(\!|N|\!) \Uparrow$ gives $L\ [D(\!|N|\!)^n] \Uparrow$
Now there trivially exists $k<n$ such that $L\ [D\langle\!|N|\!\rangle^k, D\langle\!|M|\!\rangle^{n-k}]$ is converging and $L\ [D\langle\!|N|\!\rangle^{k+1}, D\langle\!|M|\!\rangle^{n-k-1}]$ is diverging. Thus by applying Lemma 5 on the *linear context* $C''(\!|.|\!) = L\ [D\langle\!|N|\!\rangle^k, D\langle\!|M|\!\rangle^{n-k-1}, D(\!|.|\!)]$ we can conclude. □

4.2 Counter example

We first exhibit a term $\boldsymbol{A}$ that is observationally above the identity $\boldsymbol{I}$, but whose interpretation will not contain $[*]::*$ in order to break Conjecture 2. We would like to have $\boldsymbol{A}$ somehow respecting:

$$\boldsymbol{A} \simeq \Sigma_{n\geq 1}\boldsymbol{B}_n \quad \text{with for } n \geq 1: \quad \boldsymbol{B}_n = \lambda v_1 \dots v_n w.w\ [\boldsymbol{I}\ [v_1^!]\ [v_2^!]\ \cdots\ [v_n^!]]$$

This term will converge on any applicative context that converges on the identity (take $\boldsymbol{B}_n$ with n greater than the number of applications), and thus is observationally above the identity. On the other side, its semantic will be independent to the semantics of the identity since none of the $[\![B_i]\!]$ contains $[*]::* \in [\![\boldsymbol{I}]\!]$.
Such an infinite sum $\Sigma_{n\geq 1}\boldsymbol{B}_n$ does not exists in our syntax so we have to represent it by using a fix point combinator and a bag of linear and non-linear resources. We define:

$$\boldsymbol{A} := \boldsymbol{\Theta}\ [\boldsymbol{G}, \boldsymbol{F}^!] \tag{7}$$

where $\boldsymbol{G}$ and $\boldsymbol{F}$ are defined by:

$$\boldsymbol{G} := \lambda uvw.w\ [\boldsymbol{I}\ [v^!]] \qquad\qquad \boldsymbol{F} := \lambda uv_1v_2.u\ [\boldsymbol{I}\ [v_1^!]\ [v_2^!]]$$

$\boldsymbol{A}$ seems quite complex, but, it can be seen as a non deterministic *while* that recursively apply $\boldsymbol{F}$ until it chooses (non-deterministically) to apply $\boldsymbol{G}$, giving one of the $\boldsymbol{B}_i$:

Lemma 6.

1. $\boldsymbol{G}[x^!] \to_o^* \boldsymbol{B}_1$
2. *For all* i, $\boldsymbol{F}\ \boldsymbol{B}_i \to_o^* \boldsymbol{B}_{i+1}$
3. $\boldsymbol{A} \equiv_\beta \boldsymbol{B}_1 + \boldsymbol{F}[\boldsymbol{A}]$

In particular, for every $i \geq 1$, *we have* $\boldsymbol{A} \equiv_\beta \boldsymbol{F}^i[\boldsymbol{A}] + \Sigma_{j=1}^{i-1}\boldsymbol{B}_j$,
where $\boldsymbol{F}^1[\boldsymbol{A}] := \boldsymbol{F}[\boldsymbol{A}]$ *and* $\boldsymbol{F}^{i+1} := \boldsymbol{F}^i[\boldsymbol{F}[\boldsymbol{A}]]$

Proof. Item 1 is trivial. Item 2 is just a one-step unfolding of $\boldsymbol{\Theta}$. Item 3 is obtained via the reduction $\boldsymbol{A} \to^* (\boldsymbol{G}[(\theta[\boldsymbol{F}^!])^!]) + (\boldsymbol{F}[\boldsymbol{A},(\boldsymbol{\Theta}[\boldsymbol{F}^!])^!]) \equiv_\beta \boldsymbol{B}_1 + (\boldsymbol{F}[\boldsymbol{A}])$ the last step using the linearity of $\boldsymbol{F}$ on its first variable (thus in a context of the kind $[U, V^!]$ only U matters). □

Lemma 7. *For all context $C(\!|.|\!)$ of the $\partial\lambda$-calculus, if $C(\!|\boldsymbol{I}|\!)$ converges then $C(\!|\boldsymbol{A}|\!)$ converges,* i.e. $\boldsymbol{I} \leq_o \boldsymbol{A}$

Proof. Let $C(\!|.|\!)$ be a context that converges on $\boldsymbol{I}$.
With the context lemma (Theorem 7), and since neither $\boldsymbol{I}$ nor $\boldsymbol{A}$ has free variables, we can assume that $C(\!|.|\!) = (\!|.|\!)\ P_1 \ \cdots \ P_k$ (where $P_1, ..., P_k$ are bags). Thus by Lemma 6, we have $\boldsymbol{A} \to^* \boldsymbol{C}_k + \boldsymbol{B}_k$ with $\boldsymbol{C}_k := \boldsymbol{F}^k[A^!]+\Sigma_{j=1}^{k-1} B_j$ and the following converges:

$$C(\!|\boldsymbol{A}|\!) \to^* C(\!|\boldsymbol{C}_k|\!) + \lambda w.w\ [\boldsymbol{I}\ P_1\ \cdots\ P_k] = C(\!|\boldsymbol{C}_k|\!) + \lambda w.w\ [C(\!|\boldsymbol{I}|\!)]$$

□

Let us now comparing $\boldsymbol{A}$ and $\boldsymbol{I}$ at the denotational level.

Lemma 8. *We have*

$$[\![\boldsymbol{A}]\!] = \bigcup_i [\![\boldsymbol{B}_i]\!]$$

Proof. $[\![\boldsymbol{A}]\!] \supseteq \bigcup_i [\![\boldsymbol{B}_i]\!]$ is a corollary of the previous lemma (the interpretation is stable by reduction), so we have to prove that $[\![\boldsymbol{A}]\!] \subseteq \bigcup_i [\![\boldsymbol{B}_i]\!]$:
Let $\alpha \in [\![\boldsymbol{A}]\!]$. By Lemma 4, there exists $M \in \boldsymbol{A}^o$ such that $\alpha \in [\![M]\!]$. By Theorem 2: $M \to^* \mathbb{N}$, with every element of $\mathbb{N}$ outer-normal. And trivially there is $N \in \mathbb{N}$ such that $\alpha \in [\![N]\!]$. By application of Lemma 3, there exists L such that $\boldsymbol{A} \to^* L + \mathbb{L}$ and $N \in L^o$ (thus $\alpha \in [\![L]\!]$). Since the Taylor expansion conserves all *outer-redexes*, necessary L is outer-normal. We conclude by Lemma 6 that one of the $\boldsymbol{B}_i$ is reducing to L. □

Lemma 9. $[*]{::}* \notin [\![\boldsymbol{A}]\!]$, *while* $[*]{::}* \in [\![\boldsymbol{I}]\!]$

Proof. Because of Lemma 8, we just have to prove that $[*]{::}*$ is not in any $\boldsymbol{B}_i$, which is trivial since the elements of $[\![\boldsymbol{B}_i]\!]$ must be of the form $a_1{::}\cdots{::}a_i{::}[a_1{::}\cdots{::}a_i{::}\alpha]{::}\alpha$, for $i \geq 1$. □

Hence, we have refuted the Conjecture 2 concerning the equality between the observational and denotational orders. We will now refute the Conjecture 1:

Theorem 8. *$\mathcal{M}_\infty$ is not fully abstract for the λ-calculus with resources.*
In particular $\boldsymbol{A'} := \boldsymbol{I}\ [\boldsymbol{A}^!, \boldsymbol{I}^!] \equiv_o \boldsymbol{A}$ *but* $[*]{::}* \in [\![\boldsymbol{A'}]\!]$ *and* $[*]{::}* \notin [\![\boldsymbol{A}]\!]$

Proof. Since $\boldsymbol{A'} \to \boldsymbol{A}+\boldsymbol{I}$, we have $\boldsymbol{A'} \geq_o \boldsymbol{A}$ and $\boldsymbol{A'} \leq_o \boldsymbol{A}+\boldsymbol{A} = \boldsymbol{A}$. But in the same time $[\![\boldsymbol{A'}]\!] = [\![\boldsymbol{A}]\!] \cup [\![I]\!] \ni [*]{::}*$ □

5 Conclusion

Literature on resource sensitive natural constructions from Linear Logic are especially focussing on two objects, one in the semantical world, $\mathcal{M}_\infty$, and the other in the syntactical one, $\partial\lambda$-calculus. But they appeared not to respect full abstraction.

This unexpected result leads to questions on its generalization. For example, the idea can be applied to refute the full abstraction of $\mathcal{M}_\infty$ for the may-non-deterministic λ-calculus (an extension with a non deterministic operator endowed with a may-convergence operational semantic). Indeed, we can set $\boldsymbol{A}_0 = \lambda x.\boldsymbol{\Theta}\ (\lambda xy.x + \lambda xy.y)$ playing the role of $\boldsymbol{A}$. Such an $\boldsymbol{A}_0$ behaves as the infinite sum $\Sigma_{i=1}^{\infty}\lambda x_1...x_n y.y$, that is a top in its observational order but whose interpretation is not above the identity.

It can even be extended to other models since we can refute the full abstraction of Scott's $\mathcal{D}_\infty$ for the same may-non-deterministic λ-calculus (restriction of $\partial\lambda$-calculus to terms with only banged bags) or the may-must-non-deterministic λ-calculus (λ-calculus with both a may and a must non determinism), using A' in the same way. One can notice that the last case refutes a conjecture of [**?**].

More generally this counter-example describe the ill-behaved interaction between fixpoints and may-non-determinism that can tests any non-adequation between the sights of the observation and of the model. We can thus conclude by giving the four keypoints that leads to this kind of counter-examples:

- **short-sightedness of the contexts:** Calculi that offer control operators behaving as infinite applicative contexts like the resource λ-calculus with tests [**?**] are free of these considerations. This traduce the importance of the context lemma in our proof.
- **good sight of the model:** It is our better hope to find a fully abstract model for $\partial\lambda$-calculus but no known interesting algebraic models seems to break this property. Models tend indeed to approximate the condition "for any contexts of any size" into "for any infinite contexts".
- **Untyped fixpoints:** It is the first constructor that is necessary to construct a term that have a non bounded range. Thus, calculi with no fixpoints like the bang-free fragment of $\partial\lambda$-calculus will not suffer such troubles. But those calculi have limited expressive power.
- **may-non-determinism:** The second constructor, that is the most important part and the most interesting one since it can change our view of this calculi. To get ride of this problem without loosing the non determinism one can imagine a finer observation that discriminate the non idempotence of the sum, like the one provided by a probabilistic calculus.

Finally one may be disappointed by the "magic" resolution of Lemma 9. It was unclear, seeing $\boldsymbol{A}$, that this result would arise, and it needed quite a number of untrivial lemmas. In this point lies a relation with tests mechanisms of [**?**], in this system $\tau(\llparenthesis . \rrparenthesis\ \bar{\tau}(\epsilon))$ *outer*-converges on $\boldsymbol{I}$ but not on $\boldsymbol{A}$, the calculus being inequationally fully abstract this gives Lemma 9 for free. That remark was the base of the previous (unpublished but cited) version of this article [**?**]. From our point of view the relation with tests is even deeper and essential. Indeed the counter-example was discovered naturally from a trial to prove full abstraction from reducing the one from the calculus with tests into the calculus without. This will be subject to an incoming paper.