On the characterization of models of $\mathcal{H}^*$

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Abstract

We give a characterization, with respect to a large class of models of untyped λ-calculus, of those models that are fully abstract for head-normalization, *i.e.*, whose equational theory is $\mathcal{H}^*$. An extensional K-model D is fully abstract if and only if it is hyperimmune, *i.e.*, non-well founded chains of elements of D cannot be captured by any recursive function.

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Introduction

The histories of full abstraction and denotational semantics are both rooted in four fundamental articles published in the course of a year.

In 1976, Hyland [22] and Wadsworth [36] independently proved the first full abstraction result of Scott's D_∞ for the untyped λ-calculus – two untyped λ-terms have the same interpretation in D_∞ iff they converge, with respect to head-normalization, in exactly the same contexts. The following year, Milner [28] and Plotkin [33] showed respectively that PCF (a Turing-complete extension of the simply typed λ-calculus) has a unique fully abstract model up to isomorphism and that this model is not in the category of Scott domains and continuous functions.

Later, various articles focused on circumventing Plotkin counter-example [2, 21] or investigating full abstraction results for other calculi [1, 25, 31]. But hardly anyone pointed out the fact that Milner's uniqueness theorem is specific to PCF, while $\mathcal{H}^*$ (the equatinal theory of observationally equivalent λ-terms) has various models that are fully abstract but not isomorphic.

The quest for a general characterization of the fully abstract models of head-normalization started by successive refinements of a sufficient but non necessary condition [13, 26, 38], improving the proof techniques from 1976 [22, 36]. While these results shed some light on various fully abstract semantics for $\mathcal{H}^*$, none of them could reach full characterization.

In this article we give the first full characterization of the full abstraction for a specific but large class of models. The class we choose is that of Krivine-models, or K-models [5, 24]. This class is essentially the subclass of Scott complete lattices (or filter models [9]) which are prime algebraic. We add two further conditions: extensionality (*i.e.*, η-equivalence) and commutativity with Böhm trees (Definition 5). The extensional K-models commuting with Böhm trees are the objects of our characterization and can be seen as a natural class of models obtained from models of linear logic [17]. Indeed, this class corresponds to the extensional reflexive objects commuting with Böhm trees of the co-Kleisli category associated with the exponential comonad of Ehrhard's ScottL category [14](Proposition 2).

We achieve the characterization of full abstraction in Theorem 1: a model D is fully abstract for $\mathcal{H}^*$ iff D is *hyperimmune* (Def. 6). Hyperimmunity is the key property that our paper introduces in denotational semantics. This property is reminiscent of Post's notion of hyperimmune sets in recursion theory. Hyperimmunity is not only undecidable, but also surprisingly high in the hierarchy of undecidable properties (it cannot be decided by a machine with an oracle deciding the halting problem) [30].

Roughly speaking, a model D is hyperimmune whenever the λ-terms can have access to only well-founded chains of elements of D. In other words, D might have non-well-founded chains $d_0 \geq d_1 \geq \cdots$, but these chains "grow" so fast (for a suitable notion of growth), that they cannot be contained in the interpretation of any λ-term.

The intuition that full abstraction of $\mathcal{H}^*$ is related with a kind of well-foundedeness can be found in the literature (*e.g.*, Hyland's [22], Gouy's [38] or Manzonetto's [26]). The contribution of our paper is to give, with hyperimmunity, a precise definition of this intuition, at least in the setting of K-models.

Incidentally, we obtain a significant corollary (also expressed in Theorem 1), stating that full abstraction coincides with inequational full abstraction (equivalence between observational and denotational orders). This is in contrast to what happens for other languages [15, 35].

In the literature, most of the proofs of full abstraction for λ-calculus are based on Nakajima trees [29] or some other notion of quotient of the space of all Böhm trees. This approach is too coarse because it considers arbitrary Böhm trees which are not necessarily present in the calculus (non computable ones). Thus we use here a different tool: the *calculi with tests* (Def. 9). These are syntactic extensions of the λ-calculus with operators defining compact elements of the given models. Since the model appears in the syntax, we are able to perform inductions (and co-inductions) directly on the reduction steps of actual terms, rather than on the construction of Böhm trees.

The idea of test mechanisms as syntactic extensions of the λ-calculus was first used by Bucciarelli *et al.* [8]. Even though it

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was mixed with a resource-sensitive extension, the idea was already used to define morphisms of the model. Nonetheless, we can notice that older notions like Wadsworth's labeled $\lambda\bot$-calculus [36] seem related to our calculi. The calculi with tests are not *ad hoc* tricks but powerful and general tools. One of the purposes of this article is to demonstrate their power in the study of the relations between denotational and operational semantics.

Outline: Section 1 introduces some technical preliminaries and notation on preorders and λ-calculus. Section 2 displays the class of models we are considering. We state our main result in Section 3. Section 4 introduces the λ-calculi with tests. Finally, the two remaining sections are devoted to the proof of the main theorem.

Contents

1. Preliminaries

1.1 Preorders

Given two partially ordered sets $D = (|D|, \leq_D)$ and $E = (|E|, \leq_E)$, we will denote:

- $D^{op} = (|D|, \geq_D)$ the reverse-ordered set.
- $D \times E = (|D| \times |E|, \leq_{D\times E})$ the Cartesian product endowed with the pointwise order: $(\delta, \epsilon) \leq_{D\times E} (\delta', \epsilon')$ iff $\delta \leq_D \delta'$ and $\epsilon \leq_E \epsilon'$.
- $\mathcal{A}_f(D) = (\mathcal{A}_f(|D|), \leq_{\mathcal{A}_f(D)})$ the set of finite antichains of D (*i.e.*, finite subsets whose elements are pairwise incomparable) endowed with the order :

$$a \leq_{\mathcal{A}_f(D)} b \ \Leftrightarrow\ \forall \alpha \in a, \exists \beta \in b, \alpha \leq_D \beta$$

In the following we will use D for $|D|$ when there is no ambiguity. Initial Greek letters α, β, γ... will vary on elements of ordered sets. Capital initial Latin letters A, B, C... will vary over subsets of ordered sets. And finally, initial Latin letters a, b, c... will denote finite antichains.

An *order isomorphism* between D and E is a bijection $\phi : |D| \to |E|$ such that ϕ and ϕ^{-1} are monotone.

Given a subset $A \subseteq |D|$, we will denote $\downarrow A = \{\alpha \mid \exists \beta \in A, \alpha {\leq} \beta\}$. We denote by $\mathcal{I}(D)$ the set of *initial segments of* D, that is $\mathcal{I}(D) = \{\downarrow A \mid A \subseteq |D|\}$. The set $\mathcal{I}(D)$ is a prime algebraic complete lattice with respect to the set-theoretical inclusion. The *glb* are given by the unions and the *prime elements* are the downward closure of the singletons. The *compact elements* are the downward closure of finite antichains.

The domain of a partial function f is denoted by $Dom(f)$. The *graph* of a Scott-continuous function $f : \mathcal{I}(D) \to \mathcal{I}(E)$ is

$$\text{graph}(f) = \{(a, \alpha) \in \mathcal{A}_f(D)^{op} {\times} E \mid \alpha \in f(\downarrow a)\} \qquad (1)$$

Notice that elements of $\mathcal{I}(\mathcal{A}_f(D)^{op}{\times}E)$ are in one-to-one corespondance with the graphs of a Scott-continuous functions from $\mathcal{I}(D)$ to $\mathcal{I}(E)$.

1.2 λ-calculus

The λ-terms are defined up to α-equivalence by the following grammar using notation "*à la* Barendregt" [4] (where variables are denoted by final Latin letters x, y, z...):

$$(\lambda\text{-terms}) \quad \mathbf{\Lambda} \quad M, N \quad ::= \quad x \quad | \quad \lambda x.M \quad | \quad M\,N$$

and are subject to the β-reduction (where $M[N/x]$ denotes the capture-free substitution of x by N):

$$(\beta) \qquad\qquad (\lambda x.M)\,N \ \to\ M[N/x]$$

A context C is a λ-term with possibly some occurrences of a hole, *i.e.*:

$$(\text{contexts}) \quad \mathbf{\Lambda}^{(\!|.|\!)} \quad C ::= \quad (\!|.|\!) \quad | \quad x \quad | \quad \lambda x.C \quad | \quad C_1\,C_2$$

The writing $C(\!|M|\!)$ denotes the term obtained by filling the holes of C by M. The small step reduction $\to$ is the closure of (β) by any context, and $\to_h$ is the closure of (β) by head contexts, *i.e.*, by the rules:

$$\frac{M \to_h M'}{\lambda x.M \to_h \lambda x.M'} \qquad \frac{M \to_h M' \qquad M \text{ is an application}}{M\,N \to_h M'\,N}$$

The transitive reduction $\to^*$ (resp $\to_h^*$) is the reflexive transitive closure of $\to$ (resp $\to_h$). The big step head reduction, denoted $M{\Downarrow}N$, is $M \to_h^* N$ for N in a head-normal form, *i.e.*, $N = \lambda x_1...x_k.y\ M_1 \cdots M_k$, for $M_1, ..., M_k$ any terms. We write $M{\Downarrow}$ for the convergence, *i.e.*, whenever there is N such that $M{\Downarrow}N$.

Other notions of convergence exit, but our study focuces on head-convergence, inducing the equational theory denoted by $\mathcal{H}^*$ (see Definition 7). Henforth, form now on, convergence of a λ-term means head convergence and full abstraction for λ-calculus means full abstraction for head-normalization (*i.e.*, $\mathcal{H}^*$).

We will use some syntactic sugar for usual λ-terms. We denote the n^{th} Church numeral by $\underline{n}$ and the successor function by $\boldsymbol{S}$, *i.e.*, $\underline{n} = \lambda f x.f(\overbrace{\cdots}^{}_{n\text{ times}}(f\ x)\cdots)$ and $\boldsymbol{S} = \lambda u f x.u\ f\ (f\ x)$. The bold symbols $\boldsymbol{I}, \mathbf{\Omega}$ and $\mathbf{\Theta}$ denote the identity, the looping term and the Turing fixedpoint combinator:

$$\boldsymbol{I} := \lambda x.x, \qquad\qquad \mathbf{\Omega} := (\lambda x.xx)\ (\lambda x.xx),$$
$$\mathbf{\Theta} := (\lambda uv.v\ (u\ u\ v))\ (\lambda uv.v\ (u\ u\ v)).$$

Moreover, we will abbreviate a nested abstraction $\lambda x_1...x_k.M$ into $\lambda \vec{x}^k M$, or, when k is irrelevant, into $\lambda \vec{x} M$.

Finally, we denote by $\mathbf{\Lambda}^0$ the subset of closed λ-terms and by $\text{FV}(M)$ the set of free variables of a λ-term M.

1.3 Böhm trees

Definition 1. *The* Böhm tree *(***BT** *for short) of a λ-term M is a co-inductive structure defined by*

- *If M head diverges, $\boldsymbol{BT}(M) = \Omega$,*
- *if $M \to_h^* \lambda x_1...x_n.y\ N_1 \cdots N_k$ then*

$$\boldsymbol{BT}(M) = \lambda x_1...x_n.y\ \boldsymbol{BT}(N_1) \cdots \boldsymbol{BT}(N_k).$$

Notice that a Böhm tree can be described as a finitely branching tree (of possibly infinite height) where nodes are labeled by a list of abstractions and by a head variable, or by a constant Ω. A **BT** *is finite if its height is finite.*

Capital final Latin letters U, V, W... will range over Böhm trees. The inclusion of Böhm trees $U \subseteq V$ (morally U is a prefix of V) is co-inductively defined by:

- $\Omega \subseteq V$ *for all* V
- *If for all* $i \leq k$, $U_i \subseteq V_i$, *then*

$$(\lambda x_1...x_n.y\ U_1 \cdots U_k) \subseteq (\lambda x_1...x_n.y\ V_1 \cdots V_k).$$

We denote by $\boldsymbol{BT_f}(M)$ *the set of finite Böhm trees* U *such that* $U \subseteq \boldsymbol{BT}(M)$.

Proposition 1 ([4])**.** *For every* $M, N \in \Lambda$ *such that* $M \to N$, $\boldsymbol{BT}(M) = \boldsymbol{BT}(N)$.

2. K-models

We display here the main object of our study: extensional K-models [24][5]. This class of models is a subclass of filter models [9] containing many extensional models from the continuous semantics, like Scott's D_∞ [34]. Extensional K-models correspond to the extensional reflexive Scott domains that are prime algebraic complete lattices and whose application embeds prime elements into prime elements [20, 37]. However we prefer to exhibit K-models as the extensional reflexive objects of the category ScottL$_!$ (Proposition 2).

This paper focuses on these models even if we believe that very similar results hold in other semantics (such as stable [16, 17], strongly stable [7] or relational semantics [14, 18]). The condition of prime algebraicity is used to simplify many proofs and does not impact much the result even if this excludes several known models [3, 19]. The condition of extensionality is quite natural since it is perfectly understood and necessary for full abstraction wrt $\mathcal{H}^*$.

Definition 2. *We define the Cartesian closed category* ScottL$_!$ *[14, 20, 37]:*

- *An* object *is a partially ordered set.*
- *A* morphism *from* D *to* E *is a Scott-continuous function from the complete lattice* $\mathcal{I}(D)$ *to* $\mathcal{I}(E)$.

The Cartesian product *is the disjoint sum of posets. The* terminal object $\mathbb{1}$ *is the empty poset. The* exponential object $D \Rightarrow E$ *is* $\mathcal{A}_f(D)^{op} \times E$. *Notice that an element of* $\mathcal{I}(D \Rightarrow E)$ *is the graph of a morphism from* D *to* E *(see Equation* (1)*). This construction provides a natural isomorphism between* $\mathcal{I}(D \Rightarrow E)$ *and the corresponding homset. Notice that if* $\simeq$ *denotes the isomorphism in* ScottL$_!$*:*

$$D \Rightarrow D \Rightarrow \cdots \Rightarrow D \simeq (\mathcal{A}_f(D)^{op})^n \times D. \tag{2}$$

For example $D \Rightarrow (D \Rightarrow D) \simeq \mathcal{A}_f(D)^{op} \times (\mathcal{A}_f(D)^{op} \times D) = (\mathcal{A}_f(D)^{op})^2 \times D$.

2.1 Total and partial K-models

Definition 3 ([24])**.** *An* extensional K-model *is a pair* (D, i_D) *where:*

- D *is a poset.*
- i_D *is an order isomorphism between* $D \Rightarrow D$ *and* D.

By abuse of notation we may denote the pair (D, i_D) simply by D when it is clear from the context we are referring to an extensional K-model.

Remark: In the literature (*e.g.* [5]), the exponential object $D \Rightarrow D$ is displayed by using finite subsets (or multisets) instead of the finite antichains. Our presentation is a quotient of the usual one (by the equivalence relation induced by the usual preorder). The two presentations are equivalent but our choice simplifies the definition of hyperimmunity (Definition 6).

Proposition 2. *Extensional K-models correspond exactly to extensional reflexive objects of* ScottL$_!$, i.e.*, an object* D *endowed with an isomorphism* $abs_D : (D \Rightarrow D) \to D$ *(and* $app_D := abs_D^{-1}$*).*

Proof. The left to right side is obtained by setting $abs_D(I) = \{i_D(a,\alpha) | (a,\alpha) \in I\}$. For the other side we verify that $abs_D(\downarrow(a,\alpha)) = \downarrow\beta$ for some β. □

In the following we will not distinguish between a K-model and its associated reflexive object, which is a model of the pure λ-calculus.

Definition 4. *An* extensional partial K-model *is a pair* (E, j_E) *where* E *is an object of* ScottL$_!$ *and* j_E *is a partial function from* $E \Rightarrow E$ *to* E *that is an order isomorphism between* Dom(j_E) *and* E.

$$E \quad \overset{j_E}{\longleftrightarrow} \quad \mathrm{Dom}(j_E) \quad \subseteq \quad (E \Rightarrow E)$$

Proposition 3. *Any extensional partial K-model* E *can be completed into the smallest extensional K-model* $\bar{E}$ *containing* E.

Remark: Any extensional K-model D is the extensional completion of itself: $D = \bar{D}$.

Example 1.

- *Scott's* D_∞ *[34] is the extensional completion of*

  $$|D| = \{*\}, \qquad \leq_D = id, \qquad j_D = \{(\emptyset, *) \mapsto *\};$$

  i.e.*, the completion is a triple* $(|D_\infty|, \leq_{D_\infty}, j_{D_\infty})$*; where* $|D_\infty|$ *is generated by:*

  $$|D_\infty| \qquad \alpha, \beta ::= \quad * \quad | \quad a \to \alpha$$
  $$|!D_\infty| \qquad a, b \in \quad \mathcal{A}_f(|D_\infty|)$$

  except that $\emptyset \to * \notin |D_\infty|$*;* j_{D_∞} *is defined by* $j_{D_\infty}(\emptyset, *) = *$ *and* $j_{D_\infty}(a, \alpha) = a \to \alpha$ *for* $(a, \alpha) \neq (\emptyset, *)$*; and* $\leq_{D_\infty}$ *is inductively defined using formulas of Section 1.1.*
- *Park's* P_∞ *[32] is the extensional completion of*

  $$|P| = \{*\}, \qquad \leq_P = id, \qquad j_P = \{(\{*\}, *) \mapsto *\};$$

  i.e.*,* $|P_\infty|$ *is defined by the previous grammar except that now we set that* $(\{*\} \to *) \notin |P_\infty|$ *while* $\emptyset \to * \in |P_\infty|$.
- Norm *or* D_∞^* *[10] is the extensional completion of*

  $$|E| = \{p, q\}, \qquad \leq_E = id \cup \{p < q\},$$
  $$j_E = \{(\{p\}, q) \mapsto q, (\{q\}, p) \mapsto p\}.$$

- *Well-stratified K-models [26] are the extensional completions of some* E *respecting* $\forall (a, \alpha) \in \mathrm{Dom}(j_E), a = \emptyset$.
- *The inductive* $\overline{\omega}$ *is the extensional completion of*

  $$|E| = \mathbb{N}, \qquad \leq_E = id, \qquad j_E = \{(\{k < n\}, n) \mapsto n \mid n \in \mathbb{N}\}.$$

- *The co-inductive* $\overline{\overline{\mathbb{Z}}}$ *is the extensional completion of*

  $$|E| = \mathbb{Z}, \qquad \leq_E = id \qquad , j_E = \{(\{n\}, n+1) \mapsto n+1 \mid n \in \mathbb{Z}\}.$$

- *Functionals* H^f *(given* $f : \mathbb{N} \to \mathbb{N}$*) are the extensional completions of*

  $$|E| = \{*\} \cup \{\alpha_j^n | n \geq 0,\ 1 \leq j \leq f(n)\}, \qquad \leq_E = id,$$
  $$\begin{aligned} j_E = \ & \{(\emptyset, *) \mapsto *\} \\ & \cup \{(\emptyset, \alpha_{j+1}^n) \mapsto \alpha_j^n \mid 1 \leq j < f(n)\} \\ & \cup \{(\{\alpha_1^{n+1}\}, *) \mapsto \alpha_{f(n)}^n \mid n \in \mathbb{N}\}. \end{aligned}$$

For the sake of simplicity, from now on we will work with a fixed extensional K-model D. Moreover, we will use the notation $a \to \alpha := i_D(a, \alpha)$. Notice that, due to the injectivity of i_D, any $\alpha \in D$ can be uniquely rewritten into $a \to \alpha'$, and more generally into $a_1 \to \cdots \to a_n \to \alpha_n$ for any n.

2.2 Interpretation of the λ-calculus

The Cartesian closed structure of ScottL$_!$ endowed with the isomorphism app_D and abs_D of the reflexive object induced by D (Proposition 2) defines a standard model of the λ-calculus [4].

A term M with at most n free variables $x_1,\dots,x_n$ is interpreted as the graph of a morphism $[\![M]\!]_D^{x_1\dots x_n}$ from D^n to D. By Equations (1) and (2) we have:

$$[\![M]\!]_D^{x_1\dots x_n} \subseteq (D \Rightarrow \dots \Rightarrow D \Rightarrow D) \simeq (\mathcal{A}_f(D)^{op})^n \times D.$$

In Figure 4a, we explicit the interpretation $[\![M]\!]_D^{x_1\dots x_n}$ by structural induction on M.

Example 2.

$$\begin{aligned} [\![\lambda x.y]\!]_D^y &= \{((a), b{\to}\alpha) \mid \alpha \leq_D \beta \in a\},\\ [\![\lambda x.x]\!]_D^y &= \{((a), b{\to}\alpha) \mid \alpha \leq_D \beta \in b\},\\ [\![\boldsymbol{I}]\!]_D &= \{a{\to}\alpha \mid \alpha \leq_D \beta \in a\},\\ [\![\underline{1}]\!]_D &= \{a{\to}b{\to}\alpha \mid c{\to}\alpha \leq_D \beta \in a,\ c \leq_{\mathcal{A}_f(D)} b\}. \end{aligned}$$

In the latter two cases we are interpreting the terms (which are closed) in an empty environment. We then omit the empty sequence associated with the empty environment, e.g.*, $a{\to}b{\to}\alpha$ stands for $((), a{\to}b{\to}\alpha)$.*
We can verify that extensionality indeed identifies $[\![\underline{1}]\!]_D$ and $[\![\boldsymbol{I}]\!]_D$ since $c{\to}\alpha \leq_D \beta \in a,\ c \leq_{\mathcal{A}_f(D)} b$ exactly says that $b{\to}\alpha \leq_D \beta \in a$ and since any element of D is a $\alpha{\to}\beta$ and conversely.

Definition 5. *A K-model D* commutes with Böhm trees *if it respects the approximation theorem for Böhm trees. Concretely it happens when the interpretation of any term M of the λ-calculus is the union of the interpretations of the finite prefixes of its Böhm tree:*

$$[\![M]\!]_D = \bigcup_{U \in \boldsymbol{BT}_f(M)} [\![U]\!]_D,$$

where the interpretation $[\![U]\!]_D$ of a finite **BT** *U is defined inductively as for terms of $\boldsymbol{\Lambda}$ after stating that $[\![\Omega]\!]_D = \emptyset$.*

Example 3. *Except Park's model P_∞, all the models of Example 1 commute with Böhm trees. Park's model does not commute since $[\![\boldsymbol{\Omega}]\!]_{P_\infty} \neq [\![\lambda x.\boldsymbol{\Omega}]\!]_{P_\infty}$ while the two terms share the same Böhm trees. In fact, Park's model is not even sensible.*

The commutation with Böhm trees is a very common property that holds in most known sensible K-models.[1]. Moreover, important classes of models universally accept this property, such that the relational models [27]. In fact it would not be presumptuous to conjecture that any fully abstract K-model commutes with Böhm trees.

3. The result

We state our main result (Theorem 1), showing an equivalence between hyperimmunity (Definition 6) and full abstraction (with respect to head-normalization).

Definition 6 (Hyperimmunity)**.** *A (possibly partial) extensional K-model D is said to be* hyperimmune *if for every sequence $(\alpha_n)_{n\geq 0} \in D^{\mathbb{N}}$, there is no recursive function $g : \mathbb{N}{\to}\mathbb{N}$ satisfying, for all $n{\geq}0$:*

$$\alpha_n = a_{n,1}{\to}\cdots{\to}a_{n,g(n)}{\to}\alpha'_n \qquad \text{and} \qquad \alpha_{n+1} \in \bigcup_{k \leq g(n)} a_{n,k}.$$

Notice, in the above definition, that the antichaines $a_{n,i}$ always exist and are uniquely determined by definition of the isomorphism between D and $D \Rightarrow D$.

[1] Indeed the usual proof by realisability used for of the sensibility are generelizable to the commutation with Böhm trees. We only know of one result on sensitivity that could not be generalised [11, 23].

The idea is the following. The sequence $(\alpha_n)_{n\geq 0}$ is morally describing a non well-founded chain of elements of D, through the isomorphism $D \simeq D \Rightarrow D$, allowing us to see any element α_i as an arrow (of any length):

$$\begin{array}{l} \alpha_1 = a_{1,1}{\to}\cdots\ a_{1,i_1}\cdots{\to}a_{1,g(1)}{\to}\alpha'_1 \\ \qquad\qquad \Cup \\ \qquad\qquad \alpha_2 = a_{2,1}{\to}\cdots\ a_{2,i_2}\cdots{\to}a_{2,g(2)}{\to}\alpha'_2 \\ \qquad\qquad\qquad\qquad \Cup \\ \qquad\qquad\qquad\qquad \alpha_3 = a_{3,1}{\to}\cdots\ a_{3,i_3}\cdots{\to}\ a_{3,g(3)}{\to}\alpha'_3 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots \end{array}$$

The growth rate $(i_n)_n$ of the chain $(\alpha_n)_n$ depends on how much long must be the arrow representing α_i (in order to see α_{i+1} as an element of an antecedent of the arrow). Now, the hyperimmunity means that if any such non-well founded chain $(\alpha_n)_n$ exists, then its growth rate $(i_n)_n$ cannot be bounded by any recursive function g.

Let us remark that it would not be sufficient to consider simply the function $n \mapsto i_n$ such that $\alpha_{n+1} \in a_{n,i_n}$ rather than g: indeed, it may be that $n \mapsto i_n$ is not recursive while g is.

Proposition 4. *For any extensional partial K-model E, the completion $\overline{E}$ is hyperimmune iff E is hyperimmune.*

Proof. By induction on the completion procedure of E. □

Example 4. *The well-stratified K-models of Example 1 (and in particular D_∞) are trivially hyperimmune since there is not even α_1, α_2 and n such that $\alpha_1 = a_1{\to}\cdots{\to}a_n{\to}\alpha'_1$ and $\alpha_2 \in a_n$ (since $a_n = \emptyset$).*

So is $\overline{\omega}$, indeed, any such $(\alpha_n)_n$ would respect $\alpha_{n+1} <_{\mathbb{N}} \alpha_n$ what is impossible by well-foundation on $\mathbb{N}$.

On the other hand the models P_∞, Norm and $\overline{\overline{\mathbb{Z}}}$ are not hyperimmune; indeed for all of them $g = (n \mapsto 1)$ satisfies the condition above, the respective non-well founded chains $(\alpha_i)_i$ being $(*, *, \dots)$, $(p, q, p, q, \dots)$, and $(0, -1, -2, \dots)$:

$$\begin{array}{lll} * = \{*\} \to * & p = \{q\} \to p & 0 = \{1\} \to 0 \\ \ \Cup & \ \Cup & \ \Cup \\ \quad * = \{*\} \to * & \quad q = \{p\} \to q & \quad 1 = \{2\} \to 1 \\ \quad\ \Cup & \quad\ \Cup & \quad\ \Cup \\ \qquad * = \{*\} \to * & \qquad p = \{q\} \to p & \qquad 2 = \{3\} \to 2 \\ \qquad\quad \ddots & \qquad\quad \ddots & \qquad\quad \ddots \end{array}$$

More interestingly, the model H^f is hyperimmune iff f is an hyperimmune function *[30],* i.e.*, iff there is no recursive $g : \mathbb{N} \to \mathbb{N}$ such that $f \leq g$; otherwise the corresponding sequence is $(\alpha_1^i)_i$.*

$$\begin{array}{l} \alpha_1^0 = \underbrace{\emptyset \to \cdots \emptyset \to}_{f(0)\ times}\{\alpha_1^1\} \to \emptyset \cdots \to * \\ \qquad\qquad \Cup \\ \qquad\qquad \alpha_1^1 = \underbrace{\emptyset \to \cdots \emptyset \to}_{f(1)\ times}\ \{\alpha_1^2\} \to \emptyset \cdots \to * \\ \qquad\qquad\qquad\qquad \Cup \\ \qquad\qquad\qquad\qquad \alpha_1^2 = \underbrace{\emptyset \to \cdots \emptyset \to}_{f(2)\ times}\{\alpha_1^3\} \to * \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \ddots \end{array}$$

The *observational preorder* is the usual one induced by convergence to a head-normal form:

Definition 7. *Given two λ-terms M and N:*

$$M \sqsubseteq_o N \quad \text{iff} \quad \forall C,\ C(\!|M|\!)\Downarrow \ \Rightarrow\ C(\!|N|\!)\Downarrow,$$
$$M \equiv_o N \quad \text{iff} \quad M \sqsubseteq_o N \text{ and } N \sqsubseteq_o M.$$

Definition 8 ([28, 33])**.** *We say that D is:*

- fully abstract *whenever:*
$$\forall M,N, \qquad M \equiv_o N \quad \text{iff} \quad [\![M]\!]_D = [\![N]\!]_D,$$
- inequationally fully abstract *whenever:*
$$\forall M,N, \qquad M \sqsubseteq_o N \quad \text{iff} \quad [\![M]\!]_D \subseteq [\![N]\!]_D.$$

Clearly, inequational full abstraction implies full abstraction, but the converse does not hold. Indeed a corollary of our result is that as long as a K-model commutes with Böhm trees, full abstraction and inequational full abstraction coincide.

Theorem 1 (**Main theorem**)**.** *For any extensional K-model D that commutes with Böhm trees, the following are equivalent:*

1. *D is hyperimmune,*
2. *D is inequationally fully abstract for Λ,*
3. *D is fully abstract for Λ.*

Example 5. *In Example 1, the models D_∞ and $\overline{\omega}$, and the well-stratified K-models are (inequationally) fully abstract as well as the H^f whose function f is hyperimmune. While the models P_∞, Norm, $\overline{\overline{\mathbb{Z}}}$ and the other H^f are not.*

The proof splits in two parts. Section 5 proves the implication (3⇒1), exhibiting a counter example to full abstraction when D is not hyperimmune (Theorem 4). Section 6 proves the implication (1⇒2), showing that the theory of D is $\mathcal{H}^*$ (and respect the order) when assuming the hyperimmunity (Theorem 8). The implication (2⇒3) is trivial.

The proofs of both implications use a crucial tool: the notion of λ-calculi with tests, providing a bridge between syntax and semantics. Section 4 is devoted to the introduction of this notion and the preliminary results needed in the remainder.

4. λ-calculi with D-tests

4.1 Syntax

The original idea of using *tests* for recovering full abstraction (via a theorem of definability) is due to Bucciarelli *et al.* [8]. We are displaying variants of Bucciarelli *et al.*'s calculus adapted to our models.

Directly depending on a given K-model D, the λ-calculus with D-tests $\mathbf{\Lambda}_{\tau(D)}$ is, to some extent, an internal language for D. We will in fact see that if D commutes with Böhm trees, it is fully abstract for this language.

The idea is to introduce tests as a new kind in the syntax. Tests $Q \in \boldsymbol{T}_{\tau(D)}$ are co-terms, in the sens that their interpretations are maps from the context to the dualizing object of ScottL ($\bot = \{*\}$):

$$[\![Q]\!]^{x_1 \ldots x_n} \in D^n \Rightarrow \bot$$

A test Q can be seen as a boolean that succeeds if there is convergence or fail if there is not.

The interaction between terms and tests is carried out by two groups of operations indexed by $\alpha \in D$:

$$\tau_\alpha : \mathbf{\Lambda}_{\tau(D)} \to \boldsymbol{T}_{\tau(D)} \quad \text{and} \quad \bar{\tau}_\alpha : \boldsymbol{T}_{\tau(D)} \to \mathbf{\Lambda}_{\tau(D)}.$$

The first operation, τ_α, will verify that its argument $M \in \mathbf{\Lambda}_{\tau(D)}$ has the point α in its interpretation. Intuitivelly, this is performed by recursively unfolding the Böhm tree of M and succeeds when α is in the interpretation of the finite unfolded Böhm tree. If $\alpha \notin [\![M]\!]$, the test $\tau_\alpha(M)$ will either diverge or refute (raising a 0 considered as an error).

The second operator, $\bar{\tau}_\alpha$, simply raises a term of interpretation $\downarrow \alpha$ if its argument succeeds and diverges otherwise. In addition to this operators, we use addition and multiplication as ways to introduce may (for the addition) and must (for the multiplication) non-determinism; in the spirit of the λ+‖-calculus [12]. Indeed, this two forms of non-determinism are necessary to explore the branching of Böhm trees.

Hereinafter, D denotes a fixed extensional K-model.

Definition 9. *The λ-*calculus with D-tests*, for short $\mathbf{\Lambda}_{\tau(D)}$, is given by the grammar in Figure 1.*
We denote the empty sum by $\mathbf{0}$, and the empty product by $\boldsymbol{\epsilon}$. Binary sums (resp. products) can be written with infix notation, e.g. $M+N$ *(resp $P{\cdot}Q$).*

Moreover, we use the notation $\bar{\boldsymbol{\epsilon}}_\alpha := \bar{\tau}_\alpha(\boldsymbol{\epsilon})$ and $\bar{\boldsymbol{\epsilon}}_a := \sum_{\alpha\in a} \bar{\boldsymbol{\epsilon}}_\alpha$; which are terms.

Sums and products are considered as multisets, in particular we suppose associativity, commutativity and neutrality with, respectively, $\mathbf{0}$ and $\boldsymbol{\epsilon}$.

The operational semantics is given by three sets of rules in Figure 2. The main rules *of Figure 2a are the effective rewriting rules. The* distributive rules *of Figure 2b implement the distribution of the sum over the test-operator and the product. The small step semantics $\to$ is the free contextual closure of the rules of Figures 2a and 2b. The* contextual rules *of Figure 2c implement the head reduction $\to_h$ that is the specific contextual extension we are considering.*

Let us notice that this calculus enjoys confluence, but we omit to state the theorem and to prove it here because confluence is not used in the sequel.

Example 6. *The operational behavior of D-tests depends on D. Recall the K-models of Example 1. In the case of Scott's D_∞ we have in $\mathbf{\Lambda}_{\tau(D_\infty)}$:*

$$\begin{aligned}
\tau_*((\lambda xy.x\, y)\ \bar{\boldsymbol{\epsilon}}_*) &\xrightarrow{\beta}_h \tau_*(\lambda y.\bar{\boldsymbol{\epsilon}}_*\ y) \xrightarrow{\tau}_h \tau_*(\bar{\boldsymbol{\epsilon}}_*\ \bar{\boldsymbol{\epsilon}}_\emptyset) \\
&\xrightarrow{\bar{\tau}}_h \tau_*(\bar{\boldsymbol{\epsilon}}_*) \ = \ \tau_*(\bar{\tau}_*(\boldsymbol{\epsilon})) \xrightarrow{\tau\bar{\tau}}_h \boldsymbol{\epsilon}, \\
\tau_*((\lambda xy.y\ x)\ \bar{\boldsymbol{\epsilon}}_*) &\xrightarrow{\beta}_h \tau_*(\lambda y.y\ \bar{\boldsymbol{\epsilon}}_*) \xrightarrow{\tau}_h \tau_*(\bar{\boldsymbol{\epsilon}}_\emptyset\ \bar{\boldsymbol{\epsilon}}_*) \\
&= \ \tau_*(\mathbf{0}\ \bar{\boldsymbol{\epsilon}}_*) \xrightarrow{\bar{\tau}}_h \tau_*(\mathbf{0}) \xrightarrow{\tau+}_h \mathbf{0}.
\end{aligned}$$

In the case of Park P_∞:

$$\tau_*(\lambda x.xx) \xrightarrow{\tau}_h \tau_*(\bar{\boldsymbol{\epsilon}}_*\ \bar{\boldsymbol{\epsilon}}_*) \xrightarrow{\bar{\tau}}_h \tau_*(\bar{\tau}_*(\tau_*(\bar{\boldsymbol{\epsilon}}_*))) \xrightarrow{\tau\bar{\tau}}_h \xrightarrow{\tau\bar{\tau}}_h \boldsymbol{\epsilon}.$$

In the case of Norm:

$$\tau_p(\lambda x.x) \xrightarrow{\tau}_h \tau_p(\bar{\boldsymbol{\epsilon}}_q) \xrightarrow{\tau\bar{\tau}}_h \boldsymbol{\epsilon}, \qquad \tau_q(\lambda x.x) \xrightarrow{\tau}_h \tau_q(\bar{\boldsymbol{\epsilon}}_p) \xrightarrow{\tau\bar{\tau}}_h \mathbf{0}.$$

Proposition 5. *A test is in* head-normal form *if it is in the following form:*

$$\Sigma_i \Pi_j \tau_{\alpha_{i,j}}(x_{i,j}\ M^1_{i,j} \cdots\ M^n_{i,j}),$$

with $M^k_{i,j}$ any term.
A term is in head-normal form *if it has one of the following shapes:*

$$\lambda x_1 \ldots. x_n.y\ M_1 \cdots\ M_k, \qquad or \qquad \lambda x_1 \ldots x_n.\Sigma_i \bar{\tau}_{\alpha_i}(Q_i),$$

with M_i any term, and every Q_i test in head-normal form without sums.

Proof. By structural induction on the grammar of $\mathbf{\Lambda}_{\tau(D)}$. In particular, notice that any test of the shape $\tau_\alpha(\lambda x.M)$ is not a head-normal form because i_D is surjective and thus $\alpha = a \to \beta$ for some a, β and we can apply Rule (τ) □

$$
\begin{array}{llllll}
\text{(term)} & \mathbf{\Lambda}_{\tau(D)} & M,N & ::= & x \mid \lambda x.M \mid M\,N \mid \sum_{i\le n}\bar\tau_{\alpha_i}(Q_i) & ,\forall(\alpha_i)_i\in D^n, n\ge 0\\
\text{(test)} & \boldsymbol{T}_{\tau(D)} & P,Q & ::= & \sum_{i\le n}P_i \mid \prod_{i\le n}P_i \mid \tau_\alpha(M) & ,\forall\alpha\in D, n\ge 0
\end{array}
$$

Figure 1: Grammar of the calculus with D-tests

$$
\begin{array}{llrcl}
(\beta) & & (\lambda x.M)\,N & \to & M[N/x]\\
(\tau) & \forall\beta=a\to\alpha, & \tau_\beta(\lambda x.M) & \to & \tau_\alpha(M[\bar\epsilon_a/x])\\
(\bar\tau) & \forall\beta_i=a_i\to\alpha_i, & (\Sigma_i\bar\tau_{\beta_i}(Q_i))\,M & \to & \Sigma_i\bar\tau_{\alpha_i}(Q_i\cdot\Pi_{\gamma\in a_i}\tau_\gamma(M))\\
(\tau\bar\tau) & \forall\alpha\le_D\beta, & \tau_\alpha(\bar\tau_\beta(Q)) & \to & Q\\
(\tau\bar\tau) & \forall\alpha\not\le_D\beta, & \tau_\alpha(\bar\tau_\beta(Q)) & \to & \mathbf{0}
\end{array}
$$

(a) Main rules

$$
\begin{array}{lrcl}
(\tau+) & \tau_\alpha(\Sigma_{i\le k}\bar\tau_{\beta_i}(Q_i)) & \to & \Sigma_{i\le k}\tau_\alpha(\bar\tau_{\beta_i}(Q_i))\\
(\cdot+) & \Pi_{i\le n}\Sigma_{j\le k_i}Q_{i,j} & \to & \Sigma_{j_1\le k_1,\dots,j_n\le k_n}\Pi_{i\le n}Q_{i,j_i}\\
(\bar\tau+) & \bar\tau_\alpha(\Sigma_iQ_i) & \to & \Sigma_i\bar\tau_\alpha(Q_i)
\end{array}
$$

(b) Distribution of the sum

$$
\frac{M\to_h M'}{\lambda x.M\to_h\lambda x.M'}\,(c\lambda)\qquad
\frac{M\to_h M'\qquad M\text{ is an application}}{M\,N\to_h M'\,N}\,(c@)\qquad
\frac{M\to_h M'\qquad M\text{ is an application}}{\tau_\alpha(M)\to_h\tau_\alpha(M')}\,(c\tau)
$$

$$
\frac{M\to_h M'}{M+N\to_h M'+N}\,(cs)\qquad
\frac{Q\to_h Q'}{Q+P\to_h Q'+P}\,(c+)\qquad
\frac{Q\to_h Q'\qquad Q\text{ is not a sum}}{Q\cdot P\to_h Q'\cdot P}\,(c\cdot)\qquad
\frac{Q\to_h Q'\qquad Q\text{ is not a sum}}{\bar\tau_\alpha(Q)\to_h\bar\tau_\alpha(Q')}\,(c\bar\tau)
$$

(c) Contextual rules

Figure 2: Operational semantics

Definition 10. *A term is* head converging *if it head reduces either to a head-normal form or to a term of the form*

$$\lambda x_1...x_n.(\bar\tau_\alpha(Q)+N)$$

with $\bar\tau_\alpha(Q)$ in head-normal form and N any term. This corresponds to a may-convergence *for the sum. The convergence of a term M will be denoted by $M\Downarrow$ and its divergence by $M\Uparrow$.*

Definition 11. *Grammars of* term-context $\mathbf{\Lambda}^{(\!|\cdot|\!)}_{\tau(D)}$ *and* test-context $\boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}$ *are given in Figure 3. A test-context is simply called a* context.

Definition 12. *The* observational preorder $\sqsubseteq_{\tau(D)}$ *on* $\mathbf{\Lambda}_{\tau(D)}$ *is defined by:*

$$M\sqsubseteq_{\tau(D)}N\ \text{ iff }\ (\forall K\in\boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)},\ K(\!|M|\!)\Downarrow\ \text{ implies }\ K(\!|N|\!)\Downarrow).$$

We denote by $\equiv_{\tau(D)}$ the observational equivalence, i.e., *the equivalence induced by $\sqsubseteq_{\tau(D)}$.*

4.2 Semantics

The standard interpretation of $\mathbf{\Lambda}$ into D (Figure 4a) can be extended to $\mathbf{\Lambda}_{\tau(D)}$ (Figure 4b).

Definition 13. *A term M with n free variables is* interpreted *as a morphism (Scott-continuous functions) from D^n to D and a test Q with n free variables as a morphism from D^n to the dualizing object $\{*\}$ (sigleton poset):*

$$
\begin{array}{lcl}
[\![M]\!]_D^{x_1,\dots,x_n} \subseteq (D\Rightarrow\cdots\Rightarrow D\Rightarrow D) & \simeq & (\mathcal{A}_f(D)^{op})^n\times D\\
[\![Q]\!]_D^{x_1,\dots,x_n} \subseteq (D\Rightarrow\cdots\Rightarrow D\Rightarrow\{*\}) & \simeq & (\mathcal{A}_f(D)^{op})^n
\end{array}
$$

This interpretation is given in Figure 4 by structural induction.

Theorem 2. *For any extensional K-model D, D is a model of the λ-calculus with D-tests,* i.e., *the interpretation is invariant under the reduction.*

Proof. The invariance under β-reduction is obtained, as usual, by the Cartesian closeness of ScottL$_!$. The other rules are easy to check directly. □

4.3 Characterization of the sensibility of tests

Definition 14. *D is* sensible for $\mathbf{\Lambda}_{\tau(D)}$ *whenever diverging terms (resp. tests) correspond exactly to terms (resp.tests) having empty interpretation,* i.e., *for all $M\in\mathbf{\Lambda}_{\tau(D)}$ and $Q\in\boldsymbol{T}_{\tau(D)}$:*

$$M\Uparrow\Leftrightarrow[\![M]\!]_D^{\vec x}=\emptyset\qquad\qquad Q\Uparrow\Leftrightarrow[\![Q]\!]_D^{\vec x}=\emptyset$$

Theorem 3. *If D commutes with Böhm Trees then D is sensible for $\mathbf{\Lambda}_{\tau(D)}$.*

Remark that the converse is also true but out of our scope.

Corollary 1. *If D commutes with Böhm Trees, then D is inequationally adequate for $\mathbf{\Lambda}_{\tau(D)}$,* i.e.*:*

$$[\![M]\!]_D^{\vec x}\subseteq[\![N]\!]_D^{\vec x}\ \Rightarrow\ \forall C\in\mathbf{\Lambda}^{(\!|\cdot|\!)}_{\tau(D)},(C(\!|M|\!)\Downarrow\Rightarrow C(\!|N|\!)\Downarrow)$$

5. Full abstraction implies hyperimmunity

5.1 Preliminaries on Böhm trees

Definition 15. *We write by $\ge_\eta$ the reflexive transitive closure of the η-reduction on Böhm trees. We write by $\ge_{\eta\infty}$ the co-inductive closure of $\ge_\eta$, that is $U\ge_{\eta\infty}V$ whenever one of the two following conditions holds:*

- *either $U=V=\Omega$,*
- *or $U=\lambda x_1...x_n.y\,U_1\cdots U_k$ and there is $\lambda x_1...x_n.y\,V_1\cdots V_k\succ_\eta V$ such that $U_i\ge_{\eta\infty}V_i$ for all $i\le k$.*

Given two λ-terms M and N, we say that M infinitely η-expands *N, written $M\ge_{\eta\infty}N$, if $\boldsymbol{BT}(M)\ge_{\eta\infty}\boldsymbol{BT}(N)$.*

Proposition 6 ([4, Theorem 19.2.9]). *For any terms $M,N\in\mathbf{\Lambda}$, $M\sqsubseteq_o N$ iff there exist two Böhm trees U,V such that:*

$$\boldsymbol{BT}(M)\le_{\eta_\infty}U\subseteq V\ge_{\eta\infty}\boldsymbol{BT}(N).$$

Corollary 2. *For all $M,N\in\mathbf{\Lambda}$,*

$$M\ge_{\eta\infty}N\ \Rightarrow\ M\equiv_o\ N.$$

(term-context)	$\mathbf{\Lambda}^{(\cdot)}_{\tau(D)}$	C	::=	$x \mid (\!\|.\|\!) \mid C\,C' \mid \lambda x.C \mid \sum_{i\leq n}\bar{\tau}_{\alpha_i}(K_i)$	$,\forall(\alpha_i)_i\in D^n, n\geq 0$
(test-context)	$\boldsymbol{T}^{(\cdot)}_{\tau(D)}$	K	::=	$\Sigma_{i\leq n}K_i \mid \Pi_{i\leq n}K_i \mid \tau_\alpha(C)$	$,\forall\alpha\in D, n\geq 0$

Figure 3: Grammar of the contexts in a calculus with D-tests

$$[\![x_i]\!]^{\vec{x}}_D = \{(\vec{a},\alpha)\mid \alpha\leq\beta\in a_i\}$$
$$[\![M\,N]\!]^{\vec{x}}_D = \{(\vec{a},\alpha)\mid \exists b, (\vec{a},(b\to\alpha))\in[\![M]\!]^{\vec{x}}_D \ \wedge \forall\beta\in b, (\vec{a},\beta)\in[\![N]\!]^{\vec{x}}_D\}$$
$$[\![\lambda y.M]\!]^{\vec{x}}_D = \{(\vec{a},(b\to\alpha))\mid (\vec{a}b,\alpha)\in[\![M]\!]^{\vec{x}y}_D\}$$

(a) Interpretation of $\mathbf{\Lambda}$

$$[\![\Sigma_{i\leq k}\bar{\tau}_{\alpha_i}(Q_i)]\!]^{\vec{x}}_D = \textstyle\bigcup_{i\leq k}\{(\vec{a},\beta)\mid \vec{a}\in[\![Q_i]\!]^{\vec{x}}_D \ \wedge\ \beta\leq_D \alpha_i\}$$
$$[\![\tau_\alpha(M)]\!]^{\vec{x}}_D = \{\vec{a}\mid(\vec{a},\alpha)\in[\![M]\!]^{\vec{x}}_D\}$$
$$[\![\Pi_{i\leq k}Q_i]\!]^{\vec{x}}_D = \textstyle\bigcap_{i\leq k}[\![Q_i]\!]^{\vec{x}}_D \qquad [\![\Sigma_{i\leq k}Q_i]\!]^{\vec{x}}_D = \bigcup_{i\leq k}[\![Q_i]\!]^{\vec{x}}_D$$

(b) Interpretation of tests extensions

Figure 4: Direct interpretation in D

Corollary 3. *Let $M = \lambda x_1...x_n.y\ M_1\cdots M_k$ and let $N = \lambda x_1...x_{n'}.y'\ N_1\cdots N_{k'}$ be such that $M \sqsubseteq_o N$. Then:*

1. *$n-k = n'-k'$,*
2. *$y = y'$,*
3. *if $i\leq n$ and $i\leq n'$ then $M_i\sqsubseteq_o N_i$,*
4. *if $i>n$ and $i\leq n'$ then $x_i\sqsubseteq_o N_{i-n}$,*
5. *if $i\leq n$ and $i>n'$ then $M_i\sqsubseteq_o x_{i-n'}$.*

5.2 The counter-example

Suppose that D commutes with BT but is not hyperimmune. By Definition 6 of hyperimmunity, there exists a recursive $g:(\mathbb{N}\to\mathbb{N})$ and a sequence $(\alpha_n)_{n\geq 0}\in D^{\mathbb{N}}$ such that $\alpha_n = a_{n,1}\to\cdots\to a_{n,g(n)}\to\alpha'_n$ with $\alpha_{n+1}\in\bigcup_{k\leq g(n)}a_{n,k}$. We will use the function g for defining a term $\boldsymbol{A}$ (Equation 5) such that $(\boldsymbol{A}\ \underline{0})$ is observationally equal to the identity in $\mathbf{\Lambda}$ (Lemma 2) but can be distinguished in $\mathbf{\Lambda}_{\tau(D)}$ (Lemma 3). From this latter statement and Corollary 1, we will obtain that $[\![\boldsymbol{A}\ \underline{0}]\!]_D\neq[\![\boldsymbol{I}]\!]_D$, and thus we conclude with Theorem 4.

Basically, $(\boldsymbol{A}\ \underline{0})$ is a generalization of the term $\boldsymbol{J}$ used in [10] to prove that the model D^*_∞ (Example 1) is not fully abstract. The idea is that $\boldsymbol{J}$ is the infinite η-expansion of the identity $\boldsymbol{I}$ where of each level of the Böhm tree we η-expand one variable. Our term $(\boldsymbol{A}\ \underline{0})$ is also an infinite η-expansion of $\boldsymbol{I}$, but now, at each level of the Böhm tree we η-expand $g(n)$ variables.[2]

Let $(\boldsymbol{G}_n)_{n\in\mathbb{N}}$ be the sequence of closed λ-terms defined by:

$$\boldsymbol{G_n} := \lambda uex_1...x_{g(n)}.e\ (u\ x_1)\ \cdots\ (u\ x_{g(n)}) \tag{3}$$

The recursivity of g implies that of the sequence $\boldsymbol{G}_n$. We can thus use the following proposition:

Proposition 7 ([4, Proposition 8.2.2] [3]**).**
Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of terms such that:

- *$\forall n\in\mathbb{N}, M_n\in\mathbf{\Lambda}^0$,*
- *$(n\mapsto M_n)$ is recursive,*

then there exists F such that:

$$\forall n, F\ \underline{n}\ \to^*\ M_n.$$

Hence, there exists $\boldsymbol{G}$ such that:

$$\boldsymbol{G}\ \underline{n}\to^*\boldsymbol{G_n}. \tag{4}$$

[2] In the article [6] of the same autor, the reader may also find another counter-example based on infinite behaviours of a fixedpoint.

[3] This is not the original statement. We remove the dependence on $\vec{x}$ that is empty in our case and we replace the β-equivalence by a reduction since the proof of Barendregt [4] works as well with this refinement.

Recall that $\boldsymbol{S}$ denotes the Church successor function and $\mathbf{\Theta}$ the Turing fixedpoint combinator. We define:

$$\boldsymbol{A} := \mathbf{\Theta}\ (\lambda uv.\boldsymbol{G}\ v\ (u\ (\boldsymbol{S}\ v))). \tag{5}$$

Then:

$$\boldsymbol{A}\ \underline{n}\to^*\boldsymbol{G_n}\ (\boldsymbol{A}\ \underline{n+1}). \tag{6}$$

Lemma 1. *For any terms $M,N\in\mathbf{\Lambda}$ and any fresh z:*

$$(M\ z\geq_{\eta\infty}N\ z)\quad\Rightarrow\quad(M\geq_{\eta\infty}N).$$

Proof. If M diverges then so does $M\ z$, then $N\ z\Uparrow$ and $N\Uparrow$. Otherwise we have $M\to^*_h\lambda x_1\dots x_n.y\ M_1\cdots M_k$.
If $n=0$ then $M\ z\to^*_h y\ M_1\cdots M_k\ z$ and $N\ z\to^*_h y\ N_1\cdots N_k\ z$ with $M_i\geq_{\eta\infty}N_i$, thus $M\geq_{\eta\infty}N$.
Either way $M\ z\ \to^*_h\ \lambda x_2\dots x_n.(y\ M_1\cdots M_k)[z/x_1]$ and $N\ z\to^*_h N'\leq_\eta\lambda x_2\dots x_n.y[z/x_1]\ N_1\cdots N_k$ with $M_i[z/x_1]\geq_{\eta\infty}N_i$, thus, since z is fresh,
$N\ \to^*_h\ \lambda x_1.N'[x_1/z]\ \leq_\eta\ \lambda x_2\dots x_n.y\ N_1[x_1/z]\cdots N_k[x_1/z]$ and $M_i\geq_{\eta\infty}N_i[x_1/z]$, so $M\geq_{\eta\infty}N$. □

Lemma 2. *We have $A\ \underline{0}\equiv_o\boldsymbol{I}$.*

Proof. We prove that $(\boldsymbol{A}\ \underline{n}\ z)\geq_{\eta\infty}z$ (where z is fresh) for every n, by co-induction and unfolding of $\boldsymbol{BT}(\boldsymbol{A}\ \underline{n}\ z)$ (using Proposition 1): $\boldsymbol{BT}(\boldsymbol{A}\ \underline{n}\ z)$ is equal to

$$\begin{aligned}
&\boldsymbol{BT}(\boldsymbol{G}_n\ (\boldsymbol{A}\ \underline{n+1}\ z)) && \text{by (6)}\\
&=\lambda\vec{x}^{g(n)}.z\ \boldsymbol{BT}(\boldsymbol{A}\ \underline{n+1}\ x_1)\cdots\boldsymbol{BT}(\boldsymbol{A}\ \underline{n+1}\ x_{g(n)}) && \text{by (3)}\\
&\geq_{\eta\infty}\lambda\vec{x}^{g(n)}.z\ x_1\cdots x_{g(n)} && \text{by co-Ind}\\
&\geq_\eta z
\end{aligned}$$

By applying Lemma 1, we know that $(\boldsymbol{A}\ \underline{n})\geq_{\eta\infty}\boldsymbol{I}$ and by Corollary 2 that $\boldsymbol{A}\ \underline{0}\equiv_o\boldsymbol{I}$. □

Lemma 3. *Recall that α_0 refers to the first element of the sequence $(\alpha_n)_n$, providing that D is not hyperimmune. We have $\tau_{\alpha_0}((\boldsymbol{A}\ \underline{0})\ \bar{\boldsymbol{\epsilon}}_{\alpha_0})\Uparrow$.*

Proof. Morally one proves that for every $n\in\mathbb{N}$ and test $Q\in\boldsymbol{T}_{\tau(D)}$, if $Q\ \to^*\ \tau_{\alpha_n}((\boldsymbol{A}\ \underline{n})\ \boldsymbol{\epsilon}_{\alpha_n})$, then there exists $P\in\boldsymbol{T}_{\tau(D)}$ such that $Q\ \to^+_h\ P+R$ and $P\ \to^*\ \tau_{\alpha_{n+1}}((\boldsymbol{A}\ \underline{n+1})\ \boldsymbol{\epsilon}_{\alpha_{n+1}})$ and R trivially diverges. □

Theorem 4. *If D is not hyperimmune, but commutes with BT, then it is not fully abstract for the λ-calculus.*

Proof. If D commutes with BT, we have seen in Theorem 3 that it is sensible for $\mathbf{\Lambda}_{\tau(D)}$. Thus, since $\tau_{\alpha_0}(\boldsymbol{I}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0}) \overset{\beta}{\to}_h \overset{\tau\bar{\tau}}{\to}_h \epsilon$, we have that $[\![\tau_{\alpha_0}(\boldsymbol{I}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0})]\!] \neq \emptyset$, and by Lemma 3 we have that $[\![\tau_{\alpha_0}(\boldsymbol{A}\ \underline{0}\ \bar{\boldsymbol{\epsilon}}_{\alpha_0})]\!] = \emptyset$, and thus $[\![\boldsymbol{A}\ \underline{0}]\!] \neq [\![\boldsymbol{I}]\!]$. Hence, by lemma 2, D is not fully abstract. □

6. Hyperimmunity implies full abstraction

In this section we show that if D commutes with BT and is hyperimmune, D is inequationally fully abstract for $\mathbf{\Lambda}$, that is Theorem 8. For this purpose we first show in Section 6.1 that D is (inequationally) fully abstract for $\mathbf{\Lambda}_{\tau(D)}$ (Theorem 6). We use this result (or rather its technical counterpart: Theorem 5) in order to express the problem in a purely syntactic form. Then, we prove Theorem 7 stating the (inequational) full abstraction of D for $\mathbf{\Lambda}$. This proof use Corollary 3, exhibiting non constructively contexts separating any M and N that are not denotationally equivalent. The purpose of Section 6.2 is to prove Lemma 8, which is a restricted version of Theorem 7 where M is a variable, this case being the key-point where the hypothesis of hyperimmunity is used.

6.1 Full abstraction with tests

Lemma 4.

$$(\vec{a}b, \alpha) \in [\![M]\!]^{\vec{y}x} \quad \Leftrightarrow \quad (\vec{a}, \alpha) \in [\![M[\bar{\boldsymbol{\epsilon}}_b/x]]\!]^{\vec{y}},$$

$$(\vec{a}, \alpha) \in [\![M]\!]^{\vec{y}} \quad \Leftrightarrow \quad \vec{a} \in [\![\tau_\alpha(M)]\!]^{\vec{y}}.$$

Proof. By structural induction on the rules of Figure 4. □

Theorem 5. *If D is sensible for $\mathbf{\Lambda}_{\tau(D)}$ then:*

$$(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}} \quad \Leftrightarrow \quad \tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])\Downarrow.$$

Proof. If $(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}}$ then $[\![\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])]\!]$ is not empty by lemma 4, thus it converges by sensibility. If $\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])\Downarrow$ then its interpretation is non empty, and since there is no free variable, necessarily $* \in [\![\tau_\alpha(M[(\bar{\boldsymbol{\epsilon}}_{a_i}/x_i)_{i\leq n}])]\!]$ (where $*$ denotes the only inhabitant of $\mathbb{1}$) and thus, by Lemma 4, $(\vec{a}, \alpha) \in [\![M]\!]^{\vec{x}}$. □

Theorem 6. *For any extensional K-model D, if D is sensible for $\mathbf{\Lambda}_{\tau(D)}$, then D is inequationally fully abstract for the calculus with D-tests:*

$$[\![M]\!] \subseteq [\![N]\!] \quad \Leftrightarrow \quad \forall C \in \mathbf{\Lambda}_{\tau(D)}, C(\!|M|\!)\Downarrow \Rightarrow C(\!|N|\!)\Downarrow.$$

Proof. The left-to-right implication is obtained with sensibility and stability of the interpretation through contexts. And the right-to-left is obtained with Theorem 5. □

6.2 The key-lemma

From now on, we consider an extensional K-model D that is hyperimmune and commutes with BT.

For technical purpose we must begin by stating an inversion lemma for $\mathbf{\Lambda}_{\tau(D)}$.

Lemma 5. *For all $M \in \mathbf{\Lambda}_{\tau(D)}$ and $\alpha = a_1 \to \cdots a_n \to \alpha'$, if $M \to_h^* \lambda x_1 ... x_n.y\ M_1\ \cdots M_m$ then*

$$\tau_\alpha(M) \to_h^* \tau_{\alpha'}((y\ M_1\ \cdots M_m)[\bar{\boldsymbol{\epsilon}}_{a_1}/x_1, \ldots, \bar{\boldsymbol{\epsilon}}_{a_n}/x_n]).$$

The following lemma is a key lemma that introduces the hyperimmunity in the picture:

Lemma 6 (Key-lemma). *Let $\alpha \in D$ and $a_0, \ldots, a_k \in \mathcal{A}_f(D)$ be such that $\alpha \in a_0$ and $k \geq 0$.*
Let $N \in \mathbf{\Lambda}$ and $x_0, \ldots, x_k$ be such that $\tau_\alpha(N[s])\Uparrow$ (with $[s] = [\bar{\boldsymbol{\epsilon}}_{a_0}/x_0, \ldots, \bar{\boldsymbol{\epsilon}}_{a_k}/x_k]$) and $N \sqsupseteq_o x_0$.
Let, for all n, $g_N(n)$ to be the maximal arity of the nodes of depth n in $\boldsymbol{BT}(N)$.
It holds that:

> *there exists $(\alpha_n)_{n\geq 0}$ with $\alpha_0 = \alpha$ and for all n, $\alpha_n = a_1^n \to \cdots a_{g_N(n)}^n \to \alpha_n'$ and $\alpha_{n+1} \in \bigcup_{i\leq g_N(n)} a_i^n$.*

Proof. We are constructiong $(\alpha_n)_n$ by co-induction, this is performed via the unfolding of $\boldsymbol{BT}(N)$:

- The case $\boldsymbol{BT}(N) = \Omega$ is imposible since $N \sqsupseteq_o x_0$.
- If $\boldsymbol{BT}(N) = \lambda y_1 ... y_n.z\ \boldsymbol{BT}(L_1) \cdots \boldsymbol{BT}(L_m)$ with $n \neq m$ or with $z \neq x_0$:
  By Items 1 and 2 of Corollary 3, we would have $N \not\sqsupseteq_o x_0$.
- If $\boldsymbol{BT}(N) = \lambda y_1 ... y_n.x_0\ \boldsymbol{BT}(L_1) \cdots \boldsymbol{BT}(L_n)$:
  Assuming $\alpha = b_1 \to \cdots \to b_n \to \alpha'$ and $a_0 = \{\alpha, \beta_1, \ldots, \beta_t\}$ with $\beta_i = c_1^i \to \cdots \to c_n^i \to \beta_i'$ then by lemma 5:
  $$\begin{aligned}\tau_\alpha(N[s]) &\to_h^* (\bar{\boldsymbol{\epsilon}}_\alpha + \Sigma_{i\leq t} \bar{\boldsymbol{\epsilon}}_{\beta_i})\ L_1[s, s'] \cdots L_m[s, s'] \\ &\to_h^* (\Pi_{m\leq n} \Pi_{\gamma \in b_m} \tau_\gamma(L_m[s, s'])\\ &\qquad + \Sigma_{i\leq t} \tau_{\alpha'}(\bar{\tau}_{\beta_i'}(\Pi_{m\leq n} \Pi_{\gamma\in c_m^i} \tau_\gamma(L_m[s, s'])))\end{aligned}$$
  with $s' = [\bar{\boldsymbol{\epsilon}}_{b_1}/y_1, \ldots, \bar{\boldsymbol{\epsilon}}_{b_n}/y_n]$.
  Since $\tau_\alpha(N[s])$ diverges, necessarily $\Pi_{m\leq n} \Pi_{\gamma\in b_m} \tau_\gamma(L_m[s, s'])$ diverges. In particular there is $m \leq n$ and $\gamma \in b_m$ such that $\tau_\gamma(L_m[s, s'])$ diverges.
  Then there is two cases:
  - either $L_m \not\sqsupseteq_o y_m$ but, according to Item 4 of Corollary 3, we would have $N \not\sqsupseteq_o x_0$.
  - or our co-induction hypothesis show the existance of $(\gamma_k)_k$ such that $\gamma_0 = \gamma$ and for all k, $\gamma_k = c_1^k \to \cdots c_{g_{M_m}(k)}^k \to \gamma_k'$ and $\gamma_{k+1} \in \bigcup_{i\leq g_{M_m}(k)} a_i^k$. In this case we can define $(\alpha_k)_k$ as follows:
    $$\alpha_0 = \alpha \qquad\qquad \forall k,\ \alpha_{k+1} = \gamma_k$$
    This is sufficient since:
    $$m \leq n = g_N(0)$$
    $$g_{M_m}(k) \leq sup_{j\leq n} g_{M_j}(k) = g_N(k+1)$$

□

One can see that this lemma basically corresponds to a negation of hyperimmunity when g_N is computable. In order to achive this we simply reject the case where $\boldsymbol{BT}(N)$ is non computable:

Lemma 7. *For any $N \sqsupseteq_o x$, its Böhm tree $\boldsymbol{BT}(N)$ has no leaf Ω.*

Proof. By Proposition 6 we would have two Böhm trees, U and V, such that $x \leq_{\eta\infty} U \subseteq V >_{\eta\infty} \boldsymbol{BT}(N)$. Since there is no leaf Ω in x, one can easily see that there is no leaf Ω, neither in U, nor in V nor in $\boldsymbol{BT}(N)$. □

Lemma 8. *Let $\alpha \in D$ and $a_0, \ldots, a_k \in \mathcal{A}_f(D)$ be such that $\alpha \in a_0$. Let $N \in \mathbf{\Lambda}$ and $x_0, \ldots, x_k$ be such that $\tau_\alpha(N[s])\Uparrow$ (with $s = [\bar{\boldsymbol{\epsilon}}_{a_0}/x_0, \ldots, \bar{\boldsymbol{\epsilon}}_{a_k}/x_k]$). Then:*

$$N \not\sqsupseteq_o x_0.$$

Proof. We define, for every n, $g_N(n)$ to be the maximal arity of the nodes of depth n in $\boldsymbol{BT}(N)$.
If we assume $N \not\sqsupseteq_o x_0$, according to Lemma 7,

> there exists $(\alpha_n)_n$ with $\alpha_0 = \alpha$ and for all n, $\alpha_n = a_1^n \rightarrow \cdots a_{g(n)}^n \rightarrow \alpha'_n$ and $\alpha_{n+1} \in \bigcup_{i \leq g(n)} a_i^n$.

Wich contradict hyperimmunity, indeed, Lemma 7 state that $\boldsymbol{BT}(N)$ has no leaf Ω and thus is totally computable, so g_N is recursive. □

6.3 Inequational completeness

Theorem 7. *For all $M, N \in \boldsymbol{\Lambda}$,*
if there exists $(a_0...a_k, \alpha) \in [\![M]\!]^{x_0 \cdots x_k}$ such that $(a_0...a_k, \alpha) \notin [\![N]\!]^{x_0 \cdots x_k}$, then $M \not\sqsubseteq_o N$.

Proof. We will prove the equivalent (by Theorem 5) statement:

> Let $\alpha \in D$ and $a_0, \ldots, a_k \in \mathcal{A}_f(D)$.
> Let a set of variables $\{x_0, \ldots, x_k\} \supseteq \mathrm{FV}(M)$,
> and let $[s] = [\bar{\boldsymbol{\epsilon}}_{a_0}/x_0 \cdots \bar{\boldsymbol{\epsilon}}_{a_k}/x_k]$.
> If $\tau_\alpha(M[s]) \Downarrow_n$ and $\tau_\alpha(N[s]) \Uparrow$ then $M \not\sqsubseteq_o N$.

The statement is proven by induction on the length n of the reduction $\tau_\alpha(M[s]) \Downarrow_n$:

- The case $n = 0$:
  M is of the form $(y\ M_1 \cdots M_m)$ with $y \neq x_i$ for all i. Moreover N is either a diverging term or a term with a principal head-normal form $\lambda z_1...z_n.y'\ N_1 \cdots N_{m'}$. The first case is trivial and the second yields (using Lemma 5) that $\tau_\alpha(\lambda z_1...z_n.y'\ N_1 \cdots N_{m'})[s]$ diverges, thus $y' \neq y$ (the only possible cases are $y' = z_i$ or $y' = x_i$) and by Item 2 of Corollary 3, $M \not\sqsubseteq_o N$.
- The case $n \geq 1$:
  Since $\tau_\alpha(M[s]) \Downarrow_n$, the interpretation of M is non empty, thus M is converging to a head-normal form $M \rightarrow_h^* \lambda y_1...y_n.z\, M_1 \cdots M_m$ (by Theorem 3). We can then make some assumptions:
  - We can assume that $N \rightarrow_h^* \lambda y_1...y_{n'}.z'\ N_1 \cdots N_{m'}$:
    If N does not converge then trivially $M \not\sqsubseteq_o N$.
  - We can assume that $n' \geq n$:
    If $n' < n$ then we can always define $N' = \lambda y_1...y_{n'} y_{n'+1}...y_n.z'\ N_1 \cdots N_{m'}\ y_{n'+1} \cdots y_n$, and we would have $N' \equiv_o N$ and $\tau_\alpha(N'[s]) \Uparrow$.
  - We can assume that n=0:
    Let $a_0 \rightarrow \cdots a_n \rightarrow \alpha' = \alpha$, $[s'] = [\bar{\boldsymbol{\epsilon}}_{a_0}/y_1, \ldots, \bar{\boldsymbol{\epsilon}}_{a_n}/y_n]$, $N' = \lambda y_{n+1}...y_{n'}.z'\ N_1 \cdots N_{m'}$ and $M' = z\ M_1 \cdots M_m$. Since $\tau_\alpha(M[s]) \rightarrow_h^* \tau_{\alpha'}(M'[s, s'])$ and $\tau_\alpha(N[s]) \rightarrow_h^* \tau_{\alpha'}(N'[s, s'])$ (by Lemma 5), we have $\tau_{\alpha'}(M'[s, s']) \Downarrow_n$ and $\tau_{\alpha'}(N'[s, s']) \Uparrow$. Moreover $M' \sqsubseteq_o N' \Leftrightarrow M \sqsubseteq_o N$.
  - We can assume that $z' = z = x_0$:
    Since $\{x_0 \ldots x_k\} \supseteq \mathrm{FV}(M)$, there is $j \leq k$ such that $z = x_j$, for simplicity we assume that $j = 0$. Then we can remark that by Item 2 of Corollary 3, either $M \not\sqsubseteq_o N$ or $z' = z = x_0$, we will thus continue with the second case.

  Altogether we have:

  $$M \rightarrow_h^* x_0\ M_1 \cdots M_m \qquad N \rightarrow_h^* \lambda y_1...y_{n'}.x_0\ N_1 \cdots N_{m'}$$

  The case $M = x_0$ corresponds exactly to the hypothesis of Lemma 8 that concludes by $M = x_0 \not\sqsubseteq_o N$.
  We are now assuming that $m \geq 1$.
  By Corollary 3, either $M \not\sqsubseteq_o N$ or the following holds:
  - $m = m' - n'$, and in particular $m \leq m'$
  - for $i \leq m$, $M_i \sqsubseteq_o N_i$
  - for $m < i \leq m'$, $y_{i-m} \sqsubseteq_o N_i$.

  We will assume that $m = m' - n'$ and then refute $M_i \sqsubseteq_o N_i$ or $y_{i-m} \sqsubseteq_o N_i$ for some i; we then conclude $M \not\sqsubseteq_o N$.

  Since $\tau_\alpha(N[s])$ diverges, assuming that $\alpha = b_1 \rightarrow \cdots \rightarrow b_{n'} \rightarrow \alpha'$ and $[s'] = [\bar{\boldsymbol{\epsilon}}_{b_1}/y_1 \ldots \bar{\boldsymbol{\epsilon}}_{b_{n'}}/y_{n'}]$, the following (given by Lemma 5) diverges:

  $$\tau_\alpha(N[s]) \rightarrow_h^* \tau_{\alpha'}((x_0\ N_1 \cdots N_{m'})[s, s'])$$

  If $a_0 = \{\beta_0 \ldots \beta_r\}$ with $\beta_t = c_1^t \rightarrow \cdots c_m^t \rightarrow \beta'_t$ and $\beta'_t = c_{m+1}^t \rightarrow \cdots c_{m'}^t \rightarrow \beta''_t$, we have (using Lemma 5):

  $$\tau_\alpha(N[s]) \rightarrow_h^* \tau_{\alpha'}(\bar{\boldsymbol{\epsilon}}_{a_0}\ N_1 \cdots N_{m'})[s, s']) \quad (7)$$

  $$\overset{\bar{\tau}}{\rightarrow}{}_h^* \Sigma_{t \leq r} \tau_{\alpha'}(\bar{\tau}_{\beta''_t}(\Pi_{i \leq m'} \Pi_{\gamma \in c_i^t} \tau_\gamma(N_i[s, s']))) \quad (8)$$

  $$\tau_\alpha(M[s]) \rightarrow_h^+ \Sigma_{t \leq r} \tau_\alpha(\bar{\tau}_{\beta'_t}(\Pi_{i \leq m'} \Pi_{\gamma \in c_i^t} \tau_\gamma(M_i[s]))) \quad (9)$$

  Observe that, since $m \geq 1$, Reduction (9) has at least one step. Then there exists $t \leq r$ such that $\tau_\alpha(\bar{\tau}_{\beta'_t}(\Pi_{i \leq m'} \Pi_{\gamma \in c_i^t} \tau_\gamma(M_i[s]))) \Downarrow_{n-1}$. Then since $\tau_{\alpha'}(\bar{\tau}_{\beta''_t}(\Pi_{i \leq m'} \Pi_{\gamma \in c_i^t} \tau_\gamma(N_i[s, s'])))$ diverges, there is two cases:
  - Either $\alpha' \nleq \beta''_t$: which is impossible since $\alpha \leq \beta'_t$ (by convergence of $\tau_\alpha(\bar{\tau}_{\beta'_t}(\Pi_{i \leq m'} \Pi_{\gamma \in c_i^t} \tau_\gamma(M_i[s]))))$.
  - Or there is $i \leq m'$ and $\gamma \in c_i^t$ such that $\tau_\gamma(N_i[s, s'])$ diverges.
    - Either $i \leq m$:
      Then since $\tau_\gamma(M_i[s, s']) = \tau_\gamma(M_i[s]) \Downarrow_{n-1}$, the induction hypothesis yields that $M_i \not\sqsubseteq_o N_i$.
    - Or $m < i$:
      Since $\alpha \leq \beta'_t$ we have $b_{i-m} \geq c_i^t$ and $\gamma \leq \gamma' \in b_{i-m}$. Moreover, using Theorem 5 and $\gamma \leq \gamma'$, we have that $\tau_{\gamma'}(N_i[s, s'])$ diverges. Thus we can apply Lemma 8 that results in $y_{i-m} \not\sqsubseteq_o N_i$.

□

Theorem 8. *Any extensional K-model D that is hyperimmune and commutes with Böhm trees is inequationally fully abstract for the pure λ-calculus.*

Proof. Inequational adequacy: inherited from the inequational adequacy of D for $\boldsymbol{\Lambda}_{\tau(D)}$ (Corollary 1).
Inequational completeness: for all $M, N \in \boldsymbol{\Lambda}$ such that $[\![M]\!]^{\bar{x}} \nsubseteq [\![N]\!]^{\bar{x}}$, there is $(\bar{a}, \alpha) \in [\![M]\!]^{\bar{x}} - [\![N]\!]^{\bar{x}}$, thus by Theorem 7, $M \not\sqsubseteq_o N$. □

Conclusion

This concludes the proof of the main theorem (Theorem 1):

For any extensional K-model D that commutes with Böhm trees, the following are equivalent:

1. *D is hyperimmune,*
2. *D is inequationally fully abstract for* Λ,
3. *D is fully abstract for* Λ.

The key role of the property of hyperimmunity (and the heart of our proofs) is to identify the inductive and the co-inductive interpretations of Böhm trees of actual terms:

$$\forall M \in \Lambda, [\![\boldsymbol{BT}(M)]\!]_{ind} = [\![\boldsymbol{BT}(M)]\!]_{co-ind}$$

Indeed Böhm trees, as infinite structures, can only be interpreted modulo the choice of a fixpoint. We can either take the smallest fixpoint, that is the inductive definition as the limit of the interpretations of its finite approximants $[\![U]\!]_{ind} = \bigcup_{V \subseteq_f U} [\![V]\!]$. Or we can take the largest fixpoint that is the co-inductive interpretation $[\![U]\!]_{co-ind}$, co-inductively defined by:

- $[\![\Omega]\!]_{co-ind} = \emptyset$
- $a_1 \rightarrow \cdots \rightarrow a_n \rightarrow \alpha \in [\![\lambda x_1...x_m.x_i\ U_1\ \cdots\ U_n]\!]_{co-ind}$ iff there exists $b_k \in [\![U_k]\!]_{co-ind}$ for all $k \leq n$, such that $b_1 \rightarrow \cdots \rightarrow b_n \rightarrow \alpha \in a_i$.

Our proof basically shows that hyperimmunity is equivalent to identify the inductive and co-inductive interpretations of Böhm trees. If you add the commutation with Böhm trees that basically says that the interpretation of terms is the inductive interpretation of their Böhm trees. Then you say that the interpretation of a term is the co-inductive interpretation of its Böhm tree, which assertion, together with the extensionality, corresponds to the usual characterization of $\mathcal{H}^*$ by Nakajima trees.

This presentation has not been chosen for matter of concision but will be developed further in the extended version.

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