Refining Properties of Filter Models: Sensibility, Approximability and Reducibility

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In this paper, we study the tedious link between the properties of sensibility and approximability of models of untyped λ-calculus. Approximability is known to be a slightly, but strictly stronger property that sensibility. However, we will see that so far, each and every (filter) model that have been proven sensible are in fact approximable. We explain this result as a weakness of the sole known approach of sensibility: the Tait reducibility candidates and its realizability variants.

In fact, we will reduce the approximability of a filter model D for the λ-calculus to the sensibility of D but for an extension of the λ-calculus that we call λ-calculus with D-tests. Then we show that traditional proofs of sensibility of D for the λ-calculus are smoothly extendable for this λ-calculus with D-tests.

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INTRODUCTION

Sensibility. It is the ability, for a model, to distinguish non terminating programms from meaningfull ones by collapsing their interpretation (Def. 1.7). Through Curry-Howard isomorphism, it also corresponds to the consistence of the internal theory of the model. This shows the importance in understanding sensibility, but also the undecidability of such a property.

Such profund but undecidable results are often targets for classification of more or less easy subclasses, serving as grinding stone for proof technics. Here we take an unorthodox approach consisting in classifying sensible models by using as discriminator a slightly stronger property called "Approximability". To our surprise, we found out that available methodes to prove sensibility (reducibility) where not powerfull enought to distinguish sensibility from approximablity.

Approximability. The approximation theorem (Def. 1.10) is an important concept when considering denotational models of the head reduction. In order to study head reduction, λ-calculists systematically use Böhm trees, which are basically normal forms of a degenerated λ-calculus using an error symbol (Def. 1.9). Such objects are able to approximate terms, the same way as partial evaluations are approximating the notion of evaluation. The approximability simply says that the model reflects the fact that a term is the limit its finite Böhm approximations.

This notion has been extensively studied [1, Section III.17.3] and this article presents a new sufficient condition for approximability, the *weak positivity* by far encompassing any previous results on approximability (of filter models). As a property on models, approximability is supposed to be striclty stronger that sensibility. Indeed, approximability implies that the interpretation of any diverging terms (and only those) are collapsed into the interpretation of the error symbol $\perp$. This inclusion is supposed to be strict as, for exemple, approximable models

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are not able to distinguish the Turring fixedpoint from the Church fixedpoint. In fact, there is a continuity of sensible but non-approximable λ-theories, it is surprising that we are not able to model any of those.

Reducibility. In this title, "Reducibility" refers to Tait reducibility methods [**?**] and its modern extensions (including realisability). These methods used to prove structural properties of types systems and models such as sensibility and approximability but also more practical properties [**?**]. For types systems, it consists in constructing saturated sets of terms with the wanted property by induction on types, then in proving that any typable term has been included. For denotational models, the method is more subtle due to the structure not being inductive : one must find a fixpoint to be able to apply the method, but the fixpoint does not need to be computable or constructive in any way.

In Section 4, we use the sensiblility and the approximability as a grinding stone to perform yet a new disection of those reducibiliy/realisability methodes. We try to be as general as possible until the last moment in order to get the the coarsest possible characterization, but also in order to point over the specific weaknesses of the methode. We will discuss in the conclusion and along the paper why we were not able to fill the gap between approximability and sensibility. In particular, we insist on the link between this blockage and the difficulty to performe fixpoints on non-monotonous functions.

Filter Models. Introduced in the 80s using the notion of type as the elementary brick for their construction, filter models [10] (Def.1.1) are extracted from a type theory with simple types enlarged by intersection types and subtyping. Formally, the interpretation of a λ-term is the filter generated by the set of its types. Variations on the intersection type theory induce different filter models. The resulting class essentially corresponds to the class of Scott complete lattices.

Filter models (and domains) form one of the classes of models of untyped λ-calculus that have been the more broadly studied, but properties such as sensibility and approximability are yet to be understood perfectly. In particular, a simple bibliographical analysis show that that the theoretically huge gap between sensible and approximable models have never been filed by any model. The best advancements toward this direction are covered by the third part of the recent book "Lambda-calculus with types" of Barendregt, Dekkers and Statman [1].

λ-calculi with tests. In order to exhibit the link between sensibility and approximability, we are using λ-calculi with tests of Section 2. These are syntactic extensions of the untyped λ-calculus with operators defining types of the underlying intersection type system. We will see (Sec. 3) that the approximability of a filter model D is equivalent to the sensibility of the same model D for the λ-calculus with D-tests $\Lambda_{\tau(D)}$ (with respect to a notion of head convergence). This theorem brings together the notions of sensibility and approximability in a very novel way!

The calculi with tests played a central role in this paper. The idea of test mechanisms as syntactic extensions of the λ-calculus was first used by Bucciarelli *et al.* [8] and developed further by the author in [4] and [5] for Krivine-models. The one presented in this paper is yet an other generalization to the broader (extensional and distributive) filter models. Originally inspired from Wadsworth's labeled $\lambda\perp$-calculus [23] and Girard experiments [12, 15], they are syntactic extensions of the λ-calculus with operators defining compact elements of the given models. Since the model appears in the syntax, we are able to perform inductions (and co-inductions) directly on the reduction steps of actual terms, rather than on the construction of Böhm trees.

Content. Section 1 will focuss on preliminarries, with mostly standard presentations of the untyped lambda-calculus, the filder models and the Böms trees. In Section 2, we present the λ-calculi with tests, mostly following previous works of the author [4]; we give their syntaxe, their interpretation in filter models, and finaly their main properties. Section 3 is short but central in this paper: we present here the collapse of the notions of approximability and sensibility at the level of test extensions.

In a Section 4, we will present a standard proof of sensibility by reducibility adapted to λ-calculi with tests. Using our new equivalence between sensibility for this calculus with tests and approximability, this *a priori* standard proof of sensibility becomes a non-standard proof of approximability! This allows us to describe a condition for approximability that encompasses every known sensible extensional filter models, bringing these two properties closer than we believed them to be.

1 PRELIMINARIES

1.1 The λ-calculus

In this paper, we only consider the minimal untyped λ-calculus with the contextual and/or the head reduction, in the pure tradition of Barendregt book [2].

The λ-terms are defined up to α-equivalence by the following grammar using notation "*à la* Barendregt" (where variables are denoted by final Latin letters x, y, z...):

$$(\lambda\text{-terms}) \qquad \Lambda \qquad\qquad M, N \ ::= \ x \ \mid \ \lambda x.M \ \mid \ M\,N$$

We denote by $\mathrm{FV}(M)$ the set of free variables of a λ-term M. Moreover, we denote by $M[N/x]$ the capture-free substitution of x by N.
The λ-terms are subject to the β-reduction:

$$(\beta) \qquad\qquad (\lambda x.M)\,N \xrightarrow{\beta} M[N/x]$$

A context C is a λ-term with possibly some occurrences of a hole, *i.e.*:

$$(\text{contexts}) \qquad \Lambda^{(\!|.|\!)} \qquad\qquad C \ ::= \ (\!|.|\!) \ \mid \ x \ \mid \ \lambda x.C \ \mid \ C_1\,C_2$$

The writing $C(\!|M|\!)$ denotes the term obtained by filling the holes of C by M. The small step reduction $\to$ is the closure of (β) by any context, and $\to_h$ is the closure of (β) by the rules:

$$\frac{M \to_h M'}{\lambda x.M \to_h \lambda x.M'} \qquad \frac{M \to_h M' \qquad M \text{ is an application}}{M\,N \to_h M'\,N}$$

The transitive reduction $\to^*$ (resp $\to_h^*$) is the reflexive transitive closure of $\to$ (resp $\to_h$).
The big step head reduction, denoted $M \Downarrow^{\mathrm{h}} N$, is $M \to_h^* N$ for N in a *head-normal form*, *i.e.*, of the form

$$\lambda x_1 ... x_k.y\ M_1 \cdots M_k\ , \qquad \text{for } M_1, ..., M_k \text{ any terms.}$$

We write $M\Downarrow^{\mathrm{h}}$ for the (*head*) *convergence*, *i.e.*, whenever there is N such that $M \Downarrow^{\mathrm{h}} N$.

Other notions of convergence exist (strong, lazy, call by value...), but our study focuses on head convergence.

1.2 Filter Models

Def of sensibility

We introduce here the main semantic object of this article: distributive and extensional filter models (of DEFiM).

Despite corresponding to reflexive complete lattices (endowed with continuous functions), we are not using this presentation to describe filter models, but rather its dual representation by Stone duality: the sup-lattice of compact elements. The following presentation is rather standard, and the notations can be find here [9] for example. This presentation has the advantage to match the representation of the interpretation of terms as intersection types derivations, as we will see in Proposition 1.

The models consists of a set D of "types" (or compact elements), and two operations: the intersection $\wedge$ (characterising the induced order) and the functional arrow $\to$ (characterising the reflexive embeding). Moreover, we will consider extensionality, which means the the η-conversion is viable, which we represent by the existence of a specific function $\mathbf{ext}_D : D \to \mathcal{P}_f(D{\times}D)$.

Definition 1.1 ([10]). A *filter model* is a triple $(D, \wedge, \to)$ where:

- $D = (|D|, \wedge)$ is a pointed join-semilattice; we denote ω the top element and $\geq_D$ the induced order :

$$\alpha \wedge \alpha = \alpha \qquad \alpha \wedge \beta = \beta \wedge \alpha \qquad \alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma \qquad \alpha \wedge \omega = \alpha \qquad (\alpha \geq_D \beta \;\Leftrightarrow\; \alpha \wedge \beta = \beta)$$

- $\rightarrow$ is a binary operation on D such that

$$\gamma \rightarrow \delta \geq_D \bigwedge_i \alpha_i \rightarrow \beta_i \qquad \Leftrightarrow \qquad \delta \geq_D \bigwedge_{\{i \mid \gamma \leq \alpha_i\}} \beta_i,$$

  in particular, $\gamma \rightarrow \delta = \omega$ iff $\delta = \omega$.

A filter model is *extensional* whenever there is a function $\mathbf{ext}_D : D \rightarrow \mathcal{P}_f(D \times D)$ that associates to each $\alpha \in D$ a finite subset $\mathbf{ext}_D(\alpha) \subseteq D \times D$ such that:

$$\alpha = \bigwedge_{(\beta,\gamma) \in \mathbf{ext}_D(\alpha)} \beta \rightarrow \gamma$$

It is free to consider that the image of $\mathbf{ext}_D(\alpha)$ by $\rightarrow$ is an anti-chain in the sens that for any pair $(\beta, \gamma) \in \mathbf{ext}_D(\alpha)$ and any non empty subset $I \subseteq \mathbf{ext}_D(\alpha)$:

$$\bigwedge_{(\beta',\gamma') \in I} (\beta' \rightarrow \gamma') \notin Im(\rightarrow) \qquad \text{and} \qquad \beta \rightarrow \gamma \not\geq_D \bigwedge_{(\beta',\gamma') \in \mathbf{ext}_D(\alpha) - (\beta,\gamma)} \beta' \rightarrow \gamma'$$

In particular $(\beta, \omega) \in \mathbf{ext}_D(\alpha)$ implies $\alpha = \omega$, moreover $\mathbf{ext}_D(\omega) = \{(\beta, \omega)\}$ for some arbitrary β since $\beta \rightarrow \omega = \omega$. Unfortunately, the function $\mathbf{ext}_D$ is generally not unique or even canonical. in order to make it canonical, we restrict our study to *distributive* filter models. A filter model D is distributive whenever any $\alpha \geq \beta \wedge \gamma$ is accessible in the sens that there exists a decomposition $\alpha = \beta' \wedge \gamma'$ such that $\beta' \geq_D \beta$ and $\gamma' \geq_D \gamma$.
For short, we call *DEFiM* the distributive and extensional filter models.

By abuse of notation we may denote the quadruple $(D, \wedge, \rightarrow, \mathbf{ext}_D)$ simply by D when it is clear from the context that we are referring to a DEFiM.

Creating a DEFiM from scratch is often heavy, as they have to satisfy complex rules even forcing the model to be an infinite object. Fortunately, there is a way to automatically infer the required properties from a smaller (often finite) core object. This core object is a partial DEFiM which is a basically a subset of a DEFiM.

Definition 1.2. An *partial filter model* is a triple $(E, \wedge, \rightarrow)$ satisfying the axioms of filter models except that $\rightarrow$ is partially defined and for any $\alpha, (\beta_i)_{i \leq n} \in E^{n+1}$:

$$(\forall i \leq n, \alpha \rightarrow \beta_i \text{ defined }) \qquad \Rightarrow \qquad \alpha \rightarrow \bigwedge_i \beta_i \text{ defined as } \bigwedge_i (\alpha \rightarrow \beta_i)$$

It is a *partial DEFiM* if $\mathbf{ext}_E$ is defined and E satisfies the other axioms of distributive extensional filter models.

Definition 1.3. The *completion of a partial DEFiM* $(E, \wedge, \rightarrow, \mathbf{ext}_E)$ is the union

$$\bar{E} \;:=\; \left(\bigcup_{n \in \mathbb{N}} E_n \,,\, \bigcup_{n \in \mathbb{N}} (\wedge_n) \,,\, \bigcup_{n \in \mathbb{N}} (\rightarrow_n) \,,\, \bigcup_{n \in \mathbb{N}} \mathbf{ext}_{E_n} \right)$$

of partial completions $(E_n, \wedge_n, \rightarrow_n, \mathbf{ext}_{E_n})$ that are partial DEFiM defined by induction on n:
The initialization $(E_0, \wedge_0, \rightarrow_0, \mathbf{ext}_{E_0}) := (E, \wedge, \rightarrow, \mathbf{ext}_E)$ is performed by the partial DEFiM, and we continue by completing:

- $|E'_{n+1}| := \mathcal{P}_f(|E_n| \uplus (|E_n|^2 - Dom(\rightarrow_n)))$, for readability, use a, b.. for elements of $|E'_{n+1}|$ and we write $\alpha \rightarrow_* \beta$ for (α, β) in the second component,
- $\rightarrow'_{n+1}$ is defined only over $|E_n|^2 \subseteq |E'_{n+1}|^2$ by $\{\alpha\} \rightarrow'_{n+1} \{\beta\} := \{\alpha \rightarrow_n \beta\}$ whenever $(\alpha, \beta) \in Dom(\rightarrow_n)$ and by $\{\alpha\} \rightarrow'_{n+1} \{\beta\} := \{\alpha \rightarrow_* \beta\}$ whenever $(\alpha, \beta) \in |E_n|^2 - Dom(\rightarrow_n)$,
- $\mathbf{ext}'_{n+1}$ is defined over $|E'_{n+1}|$ by $\mathbf{ext}'_{n+1}(a) = \{\mathbf{ext}_n(\alpha) \mid \alpha \in E_n \cap a\} \wedge \{(\alpha, \beta) \mid \alpha \rightarrow_* \beta \in a\}$.

$$[\![x_i]\!]_D^{\vec{x}} = \{(\vec{\alpha},\beta) \mid \beta \geq \alpha_i\} \qquad [\![\lambda y.M]\!]_D^{\vec{x}} = \{(\vec{\alpha}, \textstyle\bigwedge_i(\beta_i \to \gamma_i)) \mid \forall i,\ (\vec{\alpha}\beta_i, \gamma_i) \in [\![M]\!]_D^{\vec{x}y}\}$$

$$[\![M\ N]\!]_D^{\vec{x}} = \{(\vec{\alpha}, \textstyle\bigwedge_i \beta_i) \mid \exists \vec{\gamma}_i, (\vec{\alpha}, \bigwedge_i(\gamma_i \to \beta_i)) \in [\![M]\!]_D^{\vec{x}}\ \wedge (\vec{\alpha}, \bigwedge_i \gamma_i) \in [\![N]\!]_D^{\vec{x}}\}$$

Fig. 1. Direct interpretation of Λ in the model D

- $|E_{n+1}| := |E'_{n+1}|/_{\equiv}$ is the quotient of $|E'_{n+1}|$ by the equivalence $a \equiv b$ whenever:

$$\forall(\alpha,\gamma) \in \mathbf{ext}'_{n+1}(a), \quad \gamma \geq \bigwedge_{\{(\beta,\delta)\in\mathbf{ext}_{n+1}(b)\mid\alpha\leq\beta\}} \delta \qquad\qquad \forall(\beta,\delta) \in \mathbf{ext}'_{n+1}(b), \quad \delta \geq \bigwedge_{\{(\alpha,\gamma)\in\mathbf{ext}_{n+1}(a)\mid\beta\leq\alpha\}} \gamma$$

- $\wedge_{n+1}$, $\to_{n+1}$ and $\mathbf{ext}_{n+1}$ are the quotients of $\wedge$, $\to'_{n+1}$ and $\mathbf{ext}'_{n+1}$ by $\equiv$ (notice that $\to'_{n+1}$ only need to be defined for one element equivalent class for $\to'_{n+1}$ to be defined).

We consider that $E_n \subseteq E_{n+1}$ since for each $\alpha \in |E_n|$, $\{\alpha\}$ is in a different equivalence class.

Remark 1. *The completion of a partial filter model $(E, \to, \mathbf{ext}_E)$ is well defined and corresponds to the coarsest DEFiM $\bar{E}$ containing E. In particular, any DEFiM model D is the completion of itself: $D = \bar{D}$.*

Example 1.4. Most filter models found in the literature can in fact be given as extensional completions of extremely simple partial filter models. Here are some example, the three first one are from the literatures and the two last one are fully expressing the power of the extensional completion:

(1) *Scott's D_∞* [21] is the completion of

$$|D| := \{\omega, *\}, \qquad * \wedge \omega := *, \qquad \omega \to * := * \qquad \mathbf{ext}_D(*) := \{(\omega, *)\}.$$

Notice, that $* \to *$ is undefined in D so that we need the completion.

(2) *Park's P_∞* [20] is the completion of

$$|P| := \{\omega, *\}, \qquad \omega \wedge * := *, \qquad * \to * := * \qquad \mathbf{ext}_P(*) := \{(*, *)\}.$$

(3) *Norm* or D^*_∞ [11] is the completion of

$$|D^*| := \{\omega, p, q\}, \qquad \omega \wedge p := p \qquad \omega \wedge q := q \qquad p \wedge q := q$$
$$p \to q := q \qquad q \to p := p \qquad \mathbf{ext}_{D^*}(q) := \{(p, q)\} \qquad \mathbf{ext}_{D^*}(p) := \{(q, p)\},$$

(4) Z_∞ is the completion of

$$|Z| := \{\underline{n} \mid n \geq 0\}, \qquad n \wedge \omega := n, \qquad \omega \to \underline{n+1} := \underline{n} \qquad \mathbf{ext}_D(n) := \{(\omega, \underline{n+1})\}.$$

(5) U_∞ is the completion of

$$|U| := \{\underline{n} \mid n \geq 0\}, \qquad n \wedge \omega := n, \qquad \underline{n+1} \to \underline{n+1} := \underline{n} \qquad \mathbf{ext}_D(n) := \{(\underline{n+1}, \underline{n+1})\}.$$

Remark 2. *The completion of a partial filter model is in fact the free completion in the sens that for any partial DEFiM $E \subseteq D$ contains in a DEFiM D, there is a function $\phi : \bar{E} \to D$ stable in E such that $[\![.]\!]_{\bar{E}} \subseteq [\![.]\!]_D$, where $[\![.]\!]_{\bar{E}}$ (resp. $[\![.]\!]_D$) is the interpretation of the λ-calculus into $\bar{E}$ (resp. D) as defined below.*

It is folklore that the interpretation of the λ-calculus into a given D can be equivalently characterized by a specific *intersection type system* whose types are elements $\alpha \in D$, with $\wedge$ modeling the intersection and $\to$ the logical implication.

Definition 1.5 (Interpretation of λ-terms). In Figure 1, we give the interpretation of M into a filter model D. The interpretation $[\![M]\!]_D^{x_1\ldots x_n}$ of M is suppose to be a morphism (Scott-continuous function) from $D^{\vec{x}}$ to D where $\vec{x}$ is a superset of the free variables of M. Concretely, we use the Cartesian closeness do define $[\![M]\!]_D^{x_1\ldots x_n}$ as a downward-close subsets of $(D^{op})^{\vec{x}} \times D$.

$$\frac{}{x:\alpha \vdash x:\alpha} \qquad \frac{\Gamma \vdash M:\alpha}{\Gamma, x:\beta \vdash M:\alpha} \qquad \frac{\Gamma \vdash M:\beta \qquad \alpha \geq \beta}{\Gamma \vdash M:\alpha}$$

$$\frac{\Gamma, x:\alpha \vdash M:\beta}{\Gamma \vdash \lambda x.M:\alpha{\to}\beta} \qquad \frac{\Gamma \vdash M:\alpha{\to}\beta \qquad \Gamma \vdash N:\alpha}{\Gamma \vdash M\,N:\beta} \qquad \frac{\Gamma \vdash M:\alpha \qquad \Gamma \vdash M:\beta}{\Gamma \vdash M:\alpha\wedge\beta}$$

Fig. 2. Intersection types for the λ-calculus in D

In Figure 7a, we give the intersection-type assignment corresponding to D. Notice that we can infer typing sequents for the form $\Gamma \vdash M : \alpha$ for $\Gamma = (x_1 : \alpha_1, ..., x_n : \alpha_n)$ an environment defined (at least) over all free variables of M.

Example 1.6.

$$[\![\lambda x.y]\!]_D^y = \left\{((\alpha), \bigwedge_i (\beta_i{\to}\alpha'_i)) \;\middle|\; \forall i,\ \alpha'_i \geq_D \alpha\right\},$$

$$[\![\lambda x.x]\!]_D^y = \left\{((\alpha), \bigwedge_i (\beta_i{\to}\beta'_i)) \;\middle|\; \forall i,\ \beta'_i \geq_D \beta_i\right\},$$

$$[\![\boldsymbol{I}]\!]_D = \left\{\bigwedge_i (\alpha_i{\to}\alpha'_i) \;\middle|\; \forall i,\ \alpha'_i \geq_D \alpha_i\right\},$$

$$[\![\underline{1}]\!]_D = \left\{\bigwedge_i (\alpha_i{\to}\alpha'_i) \;\middle|\; \exists \vec{\beta}', \vec{\gamma}', \vec{\beta}, \vec{\gamma},\quad \begin{array}{ll} \bigwedge_i \alpha'_i = \bigwedge_j (\beta'_j{\to}\gamma'_j), & \bigwedge_j \gamma'_j = \bigwedge_k \gamma'_k, \\ \bigwedge_i \alpha_i \leq \bigwedge_k (\beta_k{\to}\gamma_k), & \bigwedge_j \beta'_j \leq \bigwedge_k \beta_k, \end{array}\right\}.$$

In the last two cases, terms are interpreted in an empty environment. We, then, omit the empty sequence associated with the empty environment, *e.g.*, $\alpha{\to}\beta{\to}\alpha$ stands for $((), \alpha{\to}\beta{\to}\alpha)$.
We can verify that extensionality holds, indeed $[\![\underline{1}]\!]_D = [\![\boldsymbol{I}]\!]_D$. To prove it we use $\mathbf{ext}_D$ as the witness function for both existential.

Proposition 1. *Let M be a term of Λ and D a filter model, the following statements are equivalent representations of the interpretation of M in D:*

- $(\vec{\alpha},\beta) \in [\![M]\!]_D^{\vec{x}}$ *for the interpretation defined in Figure 1,*
- *the type judgment $\vec{x} : \alpha \vdash M : \beta$ is derivable by the rules of Figure 7a.*

Proof. By structural induction on the grammar of Λ. □

Definition 1.7 (Sensibility). A filter model D is sensible for the untyped λ-calculus if diverging terms corresponds exactly to those of empty interpretation:

$$M{\Downarrow}^{\mathrm{h}} \quad \Leftrightarrow \quad [\![M]\!]^{\vec{x}} \neq \emptyset\,.$$

1.3 Böhm Approximants

The Böhm approximants (or finite Böhm trees) are the normal forms of a λ-calculus extended with a constant $\perp$ and an additional reduction $\to_\perp$.

A *λ-term* M is a λ-term possibly containing occurrences of the constant $\perp$. The set λ of all *λ-terms* is generated by the grammar:

$$\lambda : \quad M,N \ ::= \ x \ \mid \ \lambda x.M \ \mid \ MN \ \mid \ \perp$$

Similarly a *(single hole) λ-context* is a (single hole) context $C(\!|\,|\!)-$ possibly containing occurrences of $\bot$. The $\bot$*-reduction* $\to_\bot$ is defined as the $\lambda\bot$-contextual closure of the rules:

$$(\bot) \qquad\qquad \lambda x.\bot \to \bot \qquad\qquad \bot M \to \bot$$

The β-reduction is extended to λ-terms in the obvious way. We write $\boldsymbol{B}$ for the set of λ-terms in $\beta\bot$-normal forms and we denote its elements by $s, t, u, \ldots$

The following characterization of $\beta\bot$-normal forms is well known.

Lemma 1.8. *Let $M \in \lambda$. We have $M \in \boldsymbol{B}$ if and only if either $M = \bot$ or M has shape $\lambda x_1 \ldots x_n.x_i M_1 \cdots M_k$ (for some $n, k \geq 0$) and each M_i is $\beta\bot$-normal.*

The set of all Böhm approximants of M can be obtained by calculating the direct approximants of all λ-terms β-convertible with M.

Definition 1.9. Let $M \in \lambda$.

(1) The *direct approximant of M*, written $\boldsymbol{ap}(M)$, is the λ-term defined as:
 - $\boldsymbol{ap}(M) = \bot$ if $M = \lambda x_1 \ldots x_k.(\lambda y.M')NM_1 \cdots M_k$,
 - $\boldsymbol{ap}(M) = \lambda x_1 \ldots x_n.x_i\boldsymbol{ap}(M_1) \cdots \boldsymbol{ap}(M_k)$ if $M = \lambda x_1 \ldots x_n.x_i M_1 \cdots M_k$,

(2) The *set of finite approximants of M* is defined by:

$$\boldsymbol{B}(M) = \{\boldsymbol{ap}(M') \mid M \to_h^* M'\}$$

We can now fully describe the property of approximability for a filter model:

Definition 1.10. A filter model is approximable iff:

$$[\![M]\!]^{\vec{x}} \quad = \bigcup_{N \in \boldsymbol{B}(M)} [\![N]\!]^{\vec{x}},$$

where the interpretation of Böhm approximants is the immediate extension of the interpretation of terms plus the minimal interpretation given to the bottom: $[\![\bot]\!] := \{(\vec{\alpha}, \bot) \mid \forall \vec{\alpha}\}$.

2 λ-CALCULI WITH D-TESTS

2.1 Syntax

The original idea of using *tests* to recover full abstraction (via a theorem of definability) is due to Bucciarelli *et al.* [8]. Here we define variants of Bucciarelli *et al.*'s calculus adapted to DEFiMs.

Directly dependent on a given DEFiM D, the λ-calculus with D-tests $\Lambda_{\tau(D)}$ is, to some extent, an internal calculus for D. In fact, we will see that, for D to be fully abstract for $\Lambda_{\tau(D)}$, it is sufficient to be sensible (Th. 2.13).

The idea is to introduce tests as a new kind in the syntax. Tests $Q \in \boldsymbol{T}_{\tau(D)}$ are sort of co-terms, in the sense that their interpretations are maps from the context to the dualizing object of the linear category ScottL ($\omega = \{*\}$):

$$[\![Q]\!]^{x_1 \ldots x_n} \in D^n \Rightarrow \omega$$

The type ω is the unit type, having only one value representing the convergence of the evaluation, seen as a success.[1]

The interaction between terms and tests is carried out by two groups of operations indexed by the elements $\alpha \in D$:

$$\tau_\alpha : \Lambda_{\tau(D)} \to \boldsymbol{T}_{\tau(D)} \quad \text{and} \quad \bar{\tau}_\alpha : \boldsymbol{T}_{\tau(D)} \to \Lambda_{\tau(D)}.$$

[1]We will see in Remark 3 that in a polarized context, the behavior of test does not correspond to co-term (or stack), but to commands (or processes), *i.e.*, to interactions between usual terms and fictive co-terms extracted from the semantics.

$$
\begin{array}{lllllll}
\text{(term)} & \Lambda_{\tau(D)} & M,N & ::= & x \mid \lambda x.M \mid M\,N \mid \sum_{i\leq n} \bar{\tau}_{\alpha_i}(Q_i) & & ,\forall(\alpha_i)_i \in (D-\omega)^n, n\geq 0 \\
\text{(test)} & \boldsymbol{T}_{\tau(D)} & P,Q & ::= & \sum_{i\leq n} P_i \mid \prod_{i\leq n} P_i \mid \tau_\alpha(M) & & ,\forall\alpha \in (D-\omega), n\geq 0
\end{array}
$$

Fig. 3. Grammar of the calculus with D-tests

The first operation, τ_α, will verify that its argument $M \in \Lambda_{\tau(D)}$ has the point α in its interpretation. Intuitively, this is performed by recursively unfolding the Böhm tree of M and succeeding (*i.e.*, converging) when α is in the interpretation of the finite unfolded Böhm tree. If $\alpha \notin [\![M]\!]$, the test $\tau_\alpha(M)$ will either diverge or refute (raising a $\mathbf{0}$ considered as an error). Concretely, it is an infinite application that feeds its argument with empty $\bar{\tau}$ operators.

The second operator, $\bar{\tau}_\alpha$, simply constructs a term of interpretation $\downarrow\alpha$ if its argument succeeds and diverges otherwise. Concretely, it is an infinite abstraction that runs its test argument, but also tests each of its applicants using τ operators.

In addition to these operators, we use *sums* and *products* as ways to introduce may (for the addition) and must (for the multiplication) non-determinism; in the spirit of the λ+||-calculus [13]. Indeed, these two forms of non-determinism are necessary to explore the branching of Böhm trees.

The idea of these two operators is to use the parametricity of our terms toward their intersection types. As a result, $\bar{\tau}_\alpha(\epsilon)$ (further on denoted $\bar{\boldsymbol{\epsilon}}_\alpha$), that transfers the always succeeding test ϵ into a term of interpretation $\downarrow\alpha$, constitutes the canonical term of type α; its behavior is exactly the common behavior of every term of type α. Symmetrically, the test $\tau_\alpha(M)$ will verify whether M behaves like a term of type α.

Hereafter, D denotes a fixed DEFiM.

Definition 2.1. The *λ-calculus with D-tests*, for short $\Lambda_{\tau(D)}$, is given by the grammar in Figure 3. We denote the empty sum by $\mathbf{0}$, and the empty product by $\boldsymbol{\epsilon}$. Binary sums (resp. products) can be written with infix notation, *e.g.* $P+Q$ (resp $P{\cdot}Q$).

Moreover, we use the notation $\bar{\boldsymbol{\epsilon}}_\alpha := \bar{\tau}_\alpha(\boldsymbol{\epsilon})$ and $\bar{\boldsymbol{\epsilon}}_a := \sum_{\alpha\in a} \bar{\boldsymbol{\epsilon}}_\alpha$; which are terms.

Sums and products are considered as multisets, in particular we suppose associativity, commutativity and neutrality with, respectively, $\mathbf{0}$ and $\boldsymbol{\epsilon}$.

In the following, an *abstraction* can refer either to a λ-abstraction or to a sum of $\bar{\tau}$ operators. This notation is justified by the behavior of $\Sigma_i \bar{\tau}_{\alpha_i}(Q_i)$ that mimics an infinite abstraction.

The operational semantics is given by three sets of rules in Figure 4. The *main rules* of Figure 5a are the effective rewriting rules. The *distributive rules* of Figure 5b implement the distribution of the sum over the test-operators and the product. The small step semantics $\to$ is the free contextual closure (*i.e.*, by the rules of Figure 5d) of the rules of Figures 5a and 5b. The *contextual rules* of Figure 5c implement the *head reduction* $\to_h$ that is the specific contextual extension we are considering.

Example 2.2. The operational behavior of D-tests depends on D. Recall the DEFiMs of Example 1.4. In the case of Scott's D_∞ we have in $\Lambda_{\tau(D_\infty)}$:

$$
\tau_*(\underline{(\lambda xy.x\ y)\ \bar{\boldsymbol{\epsilon}}_*}) \overset{\beta}{\to}_h \underline{\tau_*(\lambda y}.\bar{\boldsymbol{\epsilon}}_*\ y) \overset{\tau}{\to}_h \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_*\ \mathbf{0}}) \overset{\bar{\tau}}{\to}_h \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_*}) = \underline{\tau_*(\bar{\tau}_*}(\boldsymbol{\epsilon})) \overset{\tau\bar{\tau}}{\to}_h \boldsymbol{\epsilon},
$$

$$
\tau_*(\underline{(\lambda xy.y\ x)\ \bar{\boldsymbol{\epsilon}}_*}) \overset{\beta}{\to}_h \underline{\tau_*(\lambda y}.y\ \bar{\boldsymbol{\epsilon}}_*) \overset{\tau}{\to}_h \tau_*(\underline{\mathbf{0}}\ \bar{\boldsymbol{\epsilon}}_*) \overset{\bar{\tau}}{\to}_h \underline{\tau_*(\mathbf{0})} \overset{\tau\bar{\tau}}{\to}_h \mathbf{0}.
$$

In the case of Park P_∞:

$$
\underline{\tau_*(\lambda x}.xx) \overset{\tau}{\to}_h \tau_*(\underline{\bar{\boldsymbol{\epsilon}}_*\ \bar{\boldsymbol{\epsilon}}_*}) \overset{\bar{\tau}}{\to}_h \underline{\tau_*(\bar{\tau}_*}(\underline{\tau_*(\bar{\boldsymbol{\epsilon}}_*}))) \overset{\tau\bar{\tau}}{\to}_h \overset{\tau\bar{\tau}}{\to}_h \boldsymbol{\epsilon}.
$$

$$
\begin{array}{llcl}
(\beta) & (\lambda x.M)\, N & \to & M[N/x] \\
(\bar\tau) & (\sum_i \bar\tau_{\alpha_i}(Q_i))\, N & \to & \sum_i \sum_{(\beta,\gamma)\in \mathbf{ext}_D(\alpha_i)} \bar\tau_\gamma(Q_i \cdot \tau_\beta(N)) \\
(\tau) & \tau_\alpha(\lambda x.M) & \to & \prod_{(\beta,\gamma)\in \mathbf{ext}_D(\alpha)} \tau_\gamma(M[\bar{\boldsymbol\epsilon}_\beta/x]) \\
(\tau\bar\tau) & \tau_\alpha(\sum_{i\in I} \bar\tau_{\beta_i}(Q_i)) & \to & \sum_{\{I'\subseteq I \mid \alpha \geq \bigwedge_{i\in I'} \beta_i\}} \prod_{i\in I'} Q_i
\end{array}
$$

(a) Main rules

$$
\begin{array}{lrcl}
(\cdot +) & \Pi_{i\leq n}\Sigma_{j\leq k_i} Q_{i,j} & \to & \Sigma_{j_1\leq k_1,\dots,j_n\leq k_n} \Pi_{i\leq n} Q_{i,j_i} \\
(\bar\tau +) & \bar\tau_\alpha(\Sigma_i Q_i) & \to & \Sigma_i \bar\tau_\alpha(Q_i)
\end{array}
$$

(b) Distribution of the sum

$$
\frac{M \to_h M'}{\lambda x.M \to_h \lambda x.M'}\ (h\text{-}c\lambda) \qquad \frac{M \to_h M' \qquad M \text{ is an application}}{M\, N \to_h M'\, N}\ (h\text{-}c@)
$$

$$
\frac{M \to_h M' \qquad M \text{ is an application}}{\tau_\alpha(M) \to_h \tau_\alpha(M')}\ (h\text{-}c\tau) \qquad \frac{Q \to_h Q' \qquad Q \text{ is not a sum}}{\bar\tau_\alpha(Q) \to_h \bar\tau_\alpha(Q')}\ (h\text{-}c\bar\tau)
$$

$$
\frac{M \to_h M'}{M + N \to_h M' + N}\ (h\text{-}cs) \qquad \frac{Q \to_h Q'}{Q + P \to_h Q' + P}\ (h\text{-}c+) \qquad \frac{Q \to_h Q' \qquad Q \text{ is not a sum}}{Q\cdot P \to_h Q'\cdot P}\ (h\text{-}c\cdot)
$$

(c) Contextual rules for the head reduction

$$
\frac{M \to M'}{\lambda x.M \to \lambda x.M'}\ (c\lambda) \qquad \frac{M \to M'}{M\, N \to M'\, N}\ (c@L) \qquad \frac{N \to N'}{M\, N \to M\, N'}\ (c@R)
$$

$$
\frac{M \to M'}{\tau_\alpha(M) \to \tau_\alpha(M')}\ (c\tau) \qquad \frac{Q \to Q'}{\bar\tau_\alpha(Q) \to \bar\tau_\alpha(Q')}\ (c\bar\tau)
$$

$$
\frac{M \to M'}{M + N \to M' + N}\ (cs) \qquad \frac{Q \to Q'}{Q + P \to Q' + P}\ (c+) \qquad \frac{Q \to Q'}{Q\cdot P \to Q'\cdot P}\ (c\cdot)
$$

(d) Contextual rules for the full reduction

Fig. 4. Operational semantics of the calculus with D-tests.
In rules $\bar\tau$ and τ, notice that we use the notations $\tau_\omega(M) := \boldsymbol\epsilon$ and $\bar{\boldsymbol\epsilon}_\omega := \mathbf{0}$ in order to keep the rule simpler.

In the case of Norm:

$$\underline{\tau_p(\lambda x.x)} \quad \overset{\tau}{\to}_h \quad \underline{\tau_p(\bar{\boldsymbol{\epsilon}}_q)} \quad \overset{\tau\bar{\tau}}{\to}_h \quad \boldsymbol{\epsilon}, \qquad \text{and} \qquad \underline{\tau_q(\lambda x.x)} \quad \overset{\tau}{\to}_h \quad \underline{\tau_q(\bar{\boldsymbol{\epsilon}}_p)} \quad \overset{\tau\bar{\tau}}{\to}_h \quad \mathbf{0}.$$

In the case of Z_∞:

$$\tau_{\underline{n+2}}(\bar{\boldsymbol{\epsilon}}_{\underline{n}}\ M) \overset{\bar{\tau}}{\to}_h \tau_{\underline{n+2}}(\bar{\boldsymbol{\epsilon}}_{n+1}) \overset{\tau\bar{\tau}}{\to}_h \mathbf{0}\ , \quad \tau_{\underline{n+2}}(\bar{\boldsymbol{\epsilon}}_{\underline{n}}\ M\ N) \overset{\bar{\tau}}{\to}{}_h^2\ \tau_{\underline{n+2}}(\bar{\boldsymbol{\epsilon}}_{\underline{n+2}}) \overset{\tau\bar{\tau}}{\to}_h \boldsymbol{\epsilon}\ , \quad \tau_{\underline{n+2}}(\bar{\boldsymbol{\epsilon}}_{\underline{n}}\ M\ N\ L) \overset{\bar{\tau}}{\to}{}_h^3\ \tau_{\underline{n+2}}(\bar{\boldsymbol{\epsilon}}_{\underline{n+3}}) \overset{\tau\bar{\tau}}{\to}_h \mathbf{0}$$

Remark 3. *In a polarized (or classical) framework with explicit co-terms (or stacks) as the framework presented in [19], tests would correspond to commands (or processes), or, more exactly, to conjunctions and disjunctions of commands. Indeed, a test $\tau_\alpha(M)$ is nothing else than the command $\langle M \mid \pi_\alpha\rangle$ where π_α would be the canonical co-term of interpretation $\uparrow\alpha$, the same way that $\bar{\boldsymbol{\epsilon}}_\alpha$ is the canonical term of interpretation $\downarrow\alpha$. Similarly, the term $\bar{\tau}(Q)$ can be seen as the canonical term $\bar{\boldsymbol{\epsilon}}_\alpha$ endowed with a parallel composition referring to the set of commands Q. To resume, we have:*

$$\tau_\alpha(M) \ \simeq\ \langle M \mid \uparrow\alpha\rangle \qquad\qquad \langle\bar{\tau}_\alpha(Q) \mid \pi\rangle \ \simeq\ \langle\downarrow\alpha \mid \pi\rangle\cdot Q$$

Definition 2.3. A test is in *may-head-normal form* iff it has the following shape:

$$\Pi_i \tau_{\alpha_i}(x_i\ M_i^1 \cdots\ M_i^n) + Q,$$

with $i \geq 0$ and M_i^k any term.
A term is in *may-head-normal form* if it has one of the following shapes:

$$\lambda x_1....x_n.y\ M_1 \cdots\ M_m, \qquad\qquad \text{or} \qquad\qquad \lambda x_1...x_n.\bar{\tau}_\alpha(Q) + N,$$

where $m, n \geq 0$, $\alpha \in (D - \omega)$, M_i and N any terms, and Q any test in head-normal form without sums.

Coherently with the head convergence in λ-calculus, the convergence will be denoted by $\Downarrow^{\mathrm{h}}$ and the divergence by $\Uparrow^{\mathrm{h}}$.

Example 2.4. For any $n \in \mathbb{N}$, the term $\underline{n}\ (\lambda x.\bar{\tau}_\alpha(\tau_\alpha(x)+\tau_\beta(x)))\ \bar{\boldsymbol{\epsilon}}_\alpha$ may-head-converges.

Let us notice that this calculus enjoys the properties of confluence and standardization as shown in the Breuvart's thesis [5, Thm. 2.3.1.26 and 2.3.1.29]. It also enjoys a very nice property stating that tests-reductions can always be postponed until the very end:

Theorem 2.5. *Let D a DEFiM and $M, N \in \Lambda_{\tau(D)}$.
For any reduction $M \to^* N$, there exists $M', N' \in \Lambda_{\tau(D)}$ such that $M \to_\beta^* M'$ with only β-reductions, $M' \to_{\not\beta}^*$ with only tests reductions, and $N \to^* N$.*

$$\begin{array}{ccc} M & \to^* & N \\ {}_\beta\!\downarrow_* & \wr & \downarrow_* \\ M' & \to_{\not\beta}^* & N' \end{array}$$

In particular, M is may-head converging iff there is a sequence of β-reductions $M \to_\beta^* L$ with L that is may-head converging without any β-reduction.

Definition 2.6. Grammars of *term-contexts* $\Lambda_{\tau(D)}^{(\!|\cdot|\!)}$ and *test-contexts* $\boldsymbol{T}_{\tau(D)}^{(\!|\cdot|\!)}$ are given in Figure 5.

Definition 2.7. The *observational preorder* $\sqsubseteq_{\tau(D)}$ of $\Lambda_{\tau(D)}$ is defined by:

$$M \sqsubseteq_{\tau(D)} N \ \text{ iff } \ (\forall K \in \boldsymbol{T}_{\tau(D)}^{(\!|\cdot|\!)},\ K(\!|M|\!)\Downarrow^{\mathrm{h}} \ \text{ implies } \ K(\!|N|\!)\Downarrow^{\mathrm{h}}).$$

We denote by $\equiv_{\tau(D)}$ the *observational equivalence*, *i.e.*, the equivalence induced by $\sqsubseteq_{\tau(D)}$.

(term-context)	$\Lambda_{\tau(D)}^{(\!|\cdot|\!)}$	$C ::= x \mid (\!|.|\!) \mid C\,C' \mid \lambda x.C \mid \sum_{i\leq n} \bar{\tau}_{\alpha_i}(K_i) \quad, \forall(\alpha_i)_i \in D^n, n \geq 0$
(test-context)	$\boldsymbol{T}_{\tau(D)}^{(\!|\cdot|\!)}$	$K ::= \sum_{i\leq n} K_i \mid \prod_{i\leq n} K_i \mid \tau_\alpha(C) \quad, \forall \alpha \in D, n \geq 0$

Fig. 5. Grammar of the contexts in a calculus with D-tests

$$[\![x_i]\!]_D^{\vec{x}} = \{(\vec{\alpha},\beta) \mid \beta \geq \alpha_i\} \qquad [\![\lambda y.M]\!]_D^{\vec{x}} = \{(\vec{\alpha}, \textstyle\bigwedge_i(\beta_i \to \gamma_i)) \mid \forall i,\ (\vec{\alpha}\beta_i, \gamma_i) \in [\![M]\!]_D^{\vec{x}y}\}$$

$$[\![M\,N]\!]_D^{\vec{x}} = \{(\vec{\alpha}, \textstyle\bigwedge_i \beta_i) \mid \exists \vec{\gamma}_i, (\vec{\alpha}, \bigwedge_i(\gamma_i \to \beta_i)) \in [\![M]\!]_D^{\vec{x}} \ \wedge (\vec{\alpha}, \bigwedge_i \gamma_i) \in [\![N]\!]_D^{\vec{x}}\}$$

(a) Interpretation of Λ

$$[\![\Sigma_{i\in J}\bar{\tau}_{\alpha_i}(Q_i)]\!]_D^{\vec{x}} = \{(\vec{\beta},\gamma) \mid \exists I \subseteq J, \vec{\beta} \in \textstyle\bigcap_{i\in I}[\![Q_i]\!]_D^{\vec{x}} \ \wedge\ \gamma \geq_D \bigwedge_{i\in I} \alpha_i\}$$

$$[\![\mathbf{0}]\!]_D^{\vec{x}} = \{(\vec{\alpha},\omega)\} \qquad [\![\tau_\alpha(M)]\!]_D^{\vec{x}} = \{\vec{\beta} \mid (\vec{\beta},\alpha) \in [\![M]\!]_D^{\vec{x}}\}$$

$$[\![\Pi_{i\leq k} Q_i]\!]_D^{\vec{x}} = \textstyle\bigcap_{i\leq k}[\![Q_i]\!]_D^{\vec{x}} \qquad [\![\boldsymbol{\epsilon}]\!]_D^{\vec{x}} = D^{\vec{x}} \qquad [\![\Sigma_{i\leq k} Q_i]\!]_D^{\vec{x}} = \bigcup_{i\leq k}[\![Q_i]\!]_D^{\vec{x}} \qquad [\![\mathbf{0}]\!]_D^{\vec{x}} = \emptyset$$

(b) Interpretation of tests extensions

Fig. 6. Direct interpretation in D

$$\frac{}{x:\alpha \vdash x:\alpha} \qquad \frac{\Gamma \vdash M:\alpha}{\Gamma, x:\beta \vdash M:\alpha} \qquad \frac{\Gamma \vdash M:\beta \qquad \alpha \geq \beta}{\Gamma \vdash M:\alpha}$$

$$\frac{\Gamma, x:\alpha \vdash M:\beta}{\Gamma \vdash \lambda x.M:\alpha\to\beta} \qquad \frac{\Gamma \vdash M:\alpha\to\beta \qquad \Gamma \vdash N:\alpha}{\Gamma \vdash M\,N:\beta} \qquad \frac{\Gamma \vdash M:\alpha \qquad \Gamma \vdash M:\beta}{\Gamma \vdash M:\alpha\wedge\beta}$$

(a) Intersection types for the λ-calculus in D

$$\frac{\Gamma \vdash M:\alpha}{\Gamma \vdash \tau_\alpha(M)} \qquad \frac{\Gamma \vdash Q_j}{\Gamma \vdash \sum_{i\in I} \bar{\tau}_{\alpha_i}(Q_i):\alpha_j} \qquad \frac{\Gamma \vdash Q_j}{\Gamma \vdash \sum_{i\in I} Q_i} \qquad \frac{\forall i \in I,\ \Gamma \vdash Q_i}{\Gamma \vdash \prod_{i\in I} Q_i}$$

(b) Intersection types for the D-tests extension in D

Fig. 7. Intersection type system computing the interpretation in D

Remark 4. *The observational preorder could have been defined using term-contexts rather than test-contexts, but this appears to be equivalent and test-contexts are easier to manipulate (because normal forms for tests are simpler).*

can be removed

2.2 Semantics

The standard interpretation of Λ into D can be extended to $\Lambda_{\tau(D)}$ (Fig. 6b).

Definition 2.8. A term M with n free variables is *interpreted* as a morphism (Scott-continuous function) from D^n to D and a test Q with n free variables as a morphism from D^n to the dualizing object $\{*\}$. Concretely, we use the Cartesian closeness to define $[\![M]\!]_D^{\vec{x}}$ as a downward-close sets of $(D^{op})^{\vec{x}} \times D$ and $[\![Q]\!]_D^{\vec{x}}$ as a downward-close subsets of $(D^{op})^{\vec{x}}$.

This interpretation is given in Figure 6 by structural induction.

Proposition 2. *Any DEFiM D is a model for its own test extension (the λ-calculus with D-tests), in the sens that the interpretation is contextual and invariant under reduction.*

Proof. The invariance under β-reduction is obtained, as usual, by the Cartesian closedness of ScottL$_!$. The other rules are easy to check directly. □

The idea of intersection types can be generalized to to tests as shown in Figure 7b. Notice that tests have no type: a test does not carry any behavior, and under a specific environment it can only be succeeding (and typable) or diverging (untypable).

Theorem 2.9 (Intersection types). *Let D be a DEFiM and M a term of $\Lambda_{\tau(D)}$ (resp. Q a test of $\boldsymbol{T}_{\tau(D)}$), the following statements are equivalent:*

- *$(\vec{\alpha},\beta) \in [\![M]\!]^{\vec{x}}_D$ (resp. $\vec{\alpha} \in [\![Q]\!]^{\vec{x}}_D$) in the interpretation of Figure 6,*
- *the type judgment $\vec{x} : \vec{\alpha} \vdash M : \beta$ (resp. $\vec{x} : \vec{\alpha} \vdash Q$) is derivable by the rules of Figure 7.*

Proof. By structural induction on the grammar of $\Lambda_{\tau(D)}$. □

ref to a forthcoming article?

2.2.1 Full abstraction and sensibility for tests.

The main interest of the full abstraction with tests is to be fully abstract as soon as it is sensible (Theorem 2.13). The sensibility is a very commune property saying that diverging terms are collapsed together and separated from non-diverging terms. In other worlds, such a model is able to give meaning to terminating terms and those only. The full abstraction, however, is a much stronger property stating that the equality in the model corresponds exactly to the observational equality (for the head-convergence). Collapsing those two properties gives the real meaning of tests: they are syntactical representation of "reasonable" domains. Where "reasonable" means extensional and (as we will see later on) approximable domains.

Definition 2.10. A DEFiM D is *sensible for* $\Lambda_{\tau(D)}$ whenever diverging terms (resp. tests) correspond exactly to the terms (resp. tests) having empty interpretation, *i.e.*, for all $M \in \Lambda_{\tau(D)}$ and $Q \in \boldsymbol{T}_{\tau(D)}$:

$$M\Uparrow^{\mathrm{h}} \quad \Leftrightarrow \quad [\![M]\!]^{\vec{x}}_D = \{(\vec{\alpha},\omega) \mid \forall \vec{\alpha}\} \qquad\qquad Q\Uparrow^{\mathrm{h}} \quad \Leftrightarrow \quad [\![Q]\!]^{\vec{x}}_D = \emptyset$$

The following is an immediate lemma:

Lemma 2.11. *If D is sensible for $\Lambda_{\tau(D)}$ then:*

$$(\vec{\alpha}\beta,\gamma) \in [\![M]\!]^{\vec{x}y} \quad \Leftrightarrow \quad (\vec{\alpha},\gamma) \in [\![M[\bar{\epsilon}_\beta/y]]\!]^{\vec{x}},$$
$$(\vec{\alpha},\gamma) \in [\![M]\!]^{\vec{x}} \quad \Leftrightarrow \quad \vec{\alpha} \in [\![\tau_\gamma(M)]\!]^{\vec{x}}.$$

Theorem 2.12 (Definability). *If D is sensible for $\Lambda_{\tau(D)}$ then:*

$$(\vec{\alpha},\beta) \in [\![M]\!]^{\vec{x}} \quad \Leftrightarrow \quad \tau_\beta(M[(\bar{\epsilon}_{\alpha_i}/x_i)_{i\leq n}])\Downarrow^{\mathrm{h}}.$$

Proof. If $(\vec{\alpha},\beta) \in [\![M]\!]^{\vec{x}}$ then $[\![\tau_\beta(M[(\bar{\epsilon}_{\alpha_i}/x_i)_{i\leq n}])]\!]$ is not empty by Lemma 2.11, thus it converges by sensibility. Conversely, if $\tau_\beta(M[(\bar{\epsilon}_{\alpha_i}/x_i)_{i\leq n}])\Downarrow^{\mathrm{h}}$, since it has no variable, its interpretation is either empty or $\{()\}$, it has to be the second by sensibility, which means $(\vec{\alpha},\beta) \in [\![M]\!]^{\vec{x}}$ (by Lemma 2.11). □

Theorem 2.13 (full abstraction). *For any DEFiM D, if D is sensible for $\Lambda_{\tau(D)}$, then D is inequationally fully abstract for the observational preorder of $\Lambda_{\tau(D)}$:*

$$[\![M]\!] \subseteq [\![N]\!] \quad \Leftrightarrow \quad \forall C \in \boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}, C(\!|M|\!)\Downarrow^{\mathrm{h}} \Rightarrow C(\!|N|\!)\Downarrow^{\mathrm{h}}.$$

change in the thesis!!!

Proof. Let $[\![M]\!] \subseteq [\![N]\!]$ and $C(\!|M|\!)\Downarrow^{\mathrm{h}}$. Then by sensibility we have that $[\![C(\!|M|\!)]\!]$ is non-empty. Moreover, by Proposition 2 we have that $[\![C(\!|M|\!)]\!] \subseteq [\![C(\!|N|\!)]\!]$. Thus $[\![C(\!|N|\!)]\!]$ is non-empty and by sensibility, $C(\!|N|\!)\Downarrow^{\mathrm{h}}$.
Conversely, suppose that for all context $C \in \boldsymbol{T}^{(\!|\cdot|\!)}_{\tau(D)}$, $C(\!|M|\!)\Downarrow^{\mathrm{h}} \Rightarrow C(\!|N|\!)\Downarrow^{\mathrm{h}}$ and let $(\vec{\alpha},\beta) \in [\![M]\!]^{\vec{x}}$:
Then by Theorem 2.12, $\tau_\beta(M[(\bar{\boldsymbol{\epsilon}}_{\alpha_i}/x_i)_{i\leq n}])\Downarrow^{\mathrm{h}}$ where n is the length of $\vec{\alpha}$. Thus, after stating the context $C = \tau_\beta((\lambda x_1...x_n.(\!|.|\!))\ \bar{\boldsymbol{\epsilon}}_{\alpha_1} \cdots \bar{\boldsymbol{\epsilon}}_{\alpha_n})$, we have $C(\!|M|\!) \rightarrow^n_h \tau_\beta(M[(\bar{\boldsymbol{\epsilon}}_{\alpha_i}/x_i)_{i\leq n}])\Downarrow^{\mathrm{h}}$ which implies that $C(\!|N|\!)\Downarrow^{\mathrm{h}}$. However, there is no choice for the n first head reductions of $C(\!|N|\!)$, those are forced to be $C(\!|N|\!) \rightarrow^n_h \tau_\beta(N[(\bar{\boldsymbol{\epsilon}}_{\alpha_i}/x_i)_{i\leq n}])$ so that this term is also head-converging. Then by applying the reverse implication of Theorem 2.12 we conclude $(\vec{\alpha},\beta) \in [\![N]\!]^{\vec{x}}$. □

3 COLLAPSING SENSIBILITY AND APPROXIMABILITY FOR TESTS

Once we have said that sensibility and full abstraction are equivalent properties for test, it should not surprise the reader to learn that approximability is also equivalent to those properties. Indeed, approximability usually corresponds to the adequation of the Böhm-tree's equality, which is a property between sensibility and full abstraction. However, the situation is a bit mere subtle: if the properties of sensibility and full abstraction for $\Lambda_{\tau(D)}$ strongly refer to tests mechanisms, the property of approximability is defined independently from tests. This really means that D-tests will behave well exactly whenever D is approximable.

First we extend the languages of approximants with tests (or rather the language of tests with approximants):

Theorem 3.1. *Each of the above properties are still true when adding to the calculus with D-test the term $\perp$ and the rules:*

$$\lambda x.\perp \ \rightarrow\ \perp \qquad\qquad \perp M \ \rightarrow\ \perp \qquad\qquad \tau_\alpha(\perp)\ \rightarrow\ \mathbf{0}.$$

Proof. The term $\perp$ behave similarly to $\bar{\tau}_\alpha(\mathbf{0})$ (for any α). The only difference is that the rule $\bar{\tau}+$ is delayed. But this rule is cosmetic, one can work with or without it since it distributes with every other rules. □

We can now use the approximants of Definition 1.9 together with tests:

Lemma 3.2. *For any DEFiM D, any sequence $\vec{\alpha} \in D^{\vec{x}}$, any $\beta \in D{-}\{\omega\}$ and any $M \in \Lambda$ (with free variables $\vec{x}$), the following are equivalent:*

- *the test $\tau_\beta(M[\overline{\bar{\boldsymbol{\epsilon}}_\alpha}/\vec{x}])$ is may-head converging without β-reduction,*
- *the test with approximants $\tau_\beta(\boldsymbol{ap}(M)[\overline{\bar{\boldsymbol{\epsilon}}_\alpha}/\vec{x}])$ is may-head converging,*
- $(\vec{\alpha},\beta) \in [\![\boldsymbol{ap}(M)]\!]^{\vec{x}}$.

Proof. Considering that $\perp$ is a notation for $\bar{\tau}_\alpha(\mathbf{0})$, the second and third points are equivanent by Theorem 2.12. The equivalence between the two first points is obtained by induction on $\boldsymbol{ap}(M)$:

- Immediate when $\boldsymbol{ap}(M) = M = x_i$.
- When $\boldsymbol{ap}(M) = \lambda y.\boldsymbol{ap}(N)$ for $M = \lambda y.N$, we can use the induction hypothesis on N.
- When $\boldsymbol{ap}(M) = \perp$, this means that $\tau_\beta(M[\overline{\bar{\boldsymbol{\epsilon}}_\alpha}/\vec{x}]) \rightarrow^* \tau'_\beta((\lambda y.M')\ M_1 \cdots M_n)$ cannot converges without performing a β-reduction.
- Otherwise, $\boldsymbol{ap}(M) = x_i\boldsymbol{ap}(N_1)\cdots\boldsymbol{ap}(N_n)$ with $M = x_iN_1\cdots N_n$ thus the terms $\tau_\beta(M[\overline{\bar{\boldsymbol{\epsilon}}_\alpha}/\vec{x}])$ and $\tau_\beta(\boldsymbol{ap}(M)[\overline{\bar{\boldsymbol{\epsilon}}_\alpha}/\vec{x}])$ can performe the same sequence of $\bar{\tau}$-reductions followed by a $\tau\bar{\tau}$-reduction which results in a sum and product combination of tests behaving the same way by induction hypothesis.

□

This clearly shows that taking the approximants is an operation that distribute with the semantics. This is sufficient to get the approximation theorem whenever the extension with tests is sensible.

Theorem 3.3. *Any extensional filter model D, is approximable if and only if it is sensible for D-tests.*

Proof. Both implications are considered separatelly.

- If D is sensible for $\Lambda_{\tau(D)}$ then it is approximable:
  Let $\vec{\alpha} \in D^{\vec{x}}, \beta \in D{-}\bot$ and $M \in \Lambda$.
  - If $(\vec{\alpha},\beta) \in [\![\boldsymbol{ap}(N)]\!]_D^{\vec{x}}$ for some $M \to^* N$, then $\tau_\alpha(N[\overline{\boldsymbol{\epsilon}_\alpha}/\vec{x}]) \Downarrow^{\mathrm{h}}{}_h$ by Lemma 3.2, thus $\tau_\alpha(M[\overline{\boldsymbol{\epsilon}_\alpha}/\vec{x}]) \Downarrow^{\mathrm{h}}{}_h$ and $(\vec{\alpha},\beta) \in [\![\boldsymbol{ap}(M)]\!]_D^{\vec{x}}$.
  - If $(\vec{\alpha},\beta) \in [\![\boldsymbol{ap}(M)]\!]_D^{\vec{x}}$, then $\tau_\alpha(M[\overline{\boldsymbol{\epsilon}_\alpha}/\vec{x}]) \Downarrow^{\mathrm{h}}{}_h$. By Theorem 2.5, $M \to_\beta^* N$ with $\tau_\alpha(N[\overline{\boldsymbol{\epsilon}_\alpha}/\vec{x}])$ that may-head converges without β-reduction. Thus, $(\vec{\alpha},\beta) \in [\![\boldsymbol{ap}(N)]\!]^{\vec{x}}$ by Lemma 3.2.
- If D is approximable then it is sensible for $\Lambda_{\tau(D)}$:
  Let $\vec{\alpha} \in D^{\vec{x}}, \beta \in D{-}\bot$ and $M \in \Lambda$.
  - If $\tau_\alpha(M[\overline{\boldsymbol{\epsilon}_\alpha}/\vec{x}]) \Downarrow^{\mathrm{h}}{}_h$, then by Theorem 2.5, $M \to_\beta^* N$ with $\tau_\alpha(N[\overline{\boldsymbol{\epsilon}_\alpha}/\vec{x}])$ that may-head converges without β-reduction. Thus, by Lemma 3.2, $(\vec{\alpha},\beta) \in [\![\boldsymbol{ap}(N)]\!]^{\vec{x}}$, which is included in $[\![M]\!]^{\vec{x}}$ by approximability.
  - If $(\vec{\alpha},\beta) \in [\![\boldsymbol{ap}(M)]\!]_D^{\vec{x}}$, then there is $M \to^* N$ such that $(\vec{\alpha},\beta) \in [\![\boldsymbol{ap}(N)]\!]^{\vec{x}}$. By Lemma 3.2 we have $\tau_\beta(M[\overline{\boldsymbol{\epsilon}_\alpha}/\vec{x}])$ that is may-head converging so that $\tau_\alpha(M[\overline{\boldsymbol{\epsilon}_\alpha}/\vec{x}]) \Downarrow^{\mathrm{h}}{}_h$.

□

4 SUFFICIENT CONDITION FOR THE SENSIBILITY OF TESTS

So far we could not find a generic proof of the approximation property in the literature for standard filter model.[2] Hence, we give a sufficient condition (Def 4.11) for a filter model D to be approximable (Th. 4.16). Indeed, we use this condition in Example 4.20 for stating the approximation property of the models of Example 1.4 (save for P_∞) and Example **??**.

Here, we make a strong use of the equivalence between approximability and sensibility with tests (Th. 3.3) proven in the previous chapter. Indeed, if approximability is also proved using Tait reducibility methods [22], the process is not as well understood as in the proofs of sensibility. By directly relying on the connection with tests, we can get the more refined analysis of the theorem of approximation that we have ever find.

After our detailed analysis, we describe a sufficient, but not necessary, condition for the approximability. Generalizing the study of sensible models carried out by Berline [3] and her students (Kerth [16] in particular). In fact, we include (by far) all filter models proven sensible in the literature!

4.1 Realizers

In this section, we are adapting Tait proof of reducibility to the filter models and the calculi with tests. The adaptation for filter models (or restrictions) have already been extensively studied; for example, see Berline's survey [3]. However, the adaptation to tests is new and quite interesting. Indeed, we will see that the set of realizers we are looking into is much more coarser and refined, making the search more readable.

The first step is to defined what is a correct realizer:[3]

Definition 4.1. A *saturated set* is a set of term $G \subseteq \Lambda_{\tau(D)}$ that is close by backward reduction.
Given two saturated sets G, H, we denotes by $G \mapsto H$ the saturated set of terms M such that $(M\ N) \in H$ whenever $N \in G$.

[2]Save Chapter 17.3 of the book of Barendregt, Dekkers ans Statman [1] where this proof is done in parallel for several models of different classes.

[3]Notice that this notion of Realizers is not exactly what Tait call a realizer, but more like a D-indexed set of those. However, since we will look into a set of such D-indexed sets of Realizers; we changed the level of abstraction...

Definition 4.2. A *realizer* of D in Λ is a function $\Re$ from D to saturated subsets of $\Lambda_{\tau(D)}$ such that for all $\alpha,\beta \in D$, we have

$$\Re(\alpha \wedge \beta) = \Re(\alpha) \cap \Re(\beta) \qquad \Re(\alpha{\to}\beta) = \Re(\alpha) \mapsto \Re(\beta).$$

Given any D-indexed sequence S of saturated sets, a realizer $\Re$ of D in Λ is a S-realizer if for all α, $\Re(\alpha) \in R_\alpha$. This definition trivially is extended for a partial DEFiM J in place of D.

Intuitively, a S-realizer is a proof that a certain property represented by S is true for every typable terms. This "certain property represented by S" is basically the commune property of elements of S. In our case, we are looking for sensibility, this gives us following, the quite complex but also quite refined, set:

Definition 4.3. We denote, for all $\alpha \in D{-}\omega$:

- $\mathcal{N}^+_\alpha := \{M \in \Lambda_{\tau(D)} \mid \forall\beta \geq_D \alpha, \tau_\beta(M){\Downarrow}^{\mathrm{h}}\}$,
- $\mathcal{N}^-_\alpha := \{(\sum_i \bar{\boldsymbol{\epsilon}}_{\beta_i} + L \mid \bigwedge_i \beta_i \leq_D \alpha,\ L \in \Lambda_{\tau(D)}\}$,
- S_α is the set of $\tau(D)$-saturated subsets of $\mathcal{N}^+_\alpha$ that contains $\mathcal{N}^-_\alpha$,
- $S_\omega = \Lambda_{\tau(D)}$
- $S^D_{\tau(D)} = (S^\alpha_{\tau(D)})_{\alpha\in D}$ is the set of D-indexed collections of elements of $S^\alpha_{\tau(D)}$.

The definition is extended for partial models.

Lemma 4.4. *Let $\Re$ be a $S^D_{\tau(D)}$-realizer in D.*

$$\begin{array}{llllll} \textit{if} & (\vec{\alpha},\beta) \in [\![M]\!]^{\vec{x}} & \textit{and} & (\forall i, L_i \in \Re(\alpha_i)) & \textit{then} & M[\vec{L}/\vec{x}] \in \Re(\beta) \\ \textit{if} & \vec{\alpha} \in [\![Q]\!]^{\vec{x}} & \textit{and} & (\forall i, L_i \in \Re(\alpha_i)) & \textit{then} & Q[\vec{L}/\vec{x}] \to^* \boldsymbol{\epsilon} \end{array}$$

Proof. By induction on M and Q.:

- $M = x_i$: then $\alpha_i \leq_D \beta$. Thus if $L_i \in \Re(\alpha_i) \subseteq \Re(\beta)$, we have $M[\vec{L}/\vec{x}] = L_i \in \Re(\beta)$.
- $M = N_1\, N_2$: there exists $(\gamma_j,\beta_j)_{j\leq n}$ such that $\beta = \bigwedge_j \beta_j$, $(\vec{\alpha}, \bigwedge_j \gamma_j{\to}\beta_j) \in [\![N_1]\!]^{\vec{x}}$ and $(\vec{a}; \bigwedge_j \gamma_j) \in [\![N_2]\!]^{\vec{x}}$. Thus, by induction hypothesis, if for all i, $L_i \in \Re(\alpha_i)$, $N_1[\vec{L}/\vec{x}] \in (\bigcap_j(\Re(\gamma_j) \mapsto \Re(\beta_j)))$ and $N_2[\vec{L}/\vec{x}] \in \bigcap_j \Re(\gamma_j)$. We conclude by $(N_1N_2)[\vec{L}/\vec{x}] \in \bigcup_j \Re(\beta_i)$.
- $M = \lambda y.N$: then $\beta = \bigwedge_j \gamma_j{\to}\beta_j$ and $((\vec{\alpha}, \bigwedge_i \gamma_i); \bigwedge_i \beta_i) \in [\![N]\!]^{\vec{x}y}$. We want to show that whenever $\forall i \leq |\vec{x}|$, $L_i \in \Re(\alpha_i)$ and $j \leq n$, we have $\lambda y.N[\vec{L}/\vec{x}] \in \Re(\gamma_j) \mapsto \Re(\beta_j)$. But if $L \in \Re(\bigwedge_i \gamma_i)$ for all i, the induction hypothesis give us that for any j, $N[\vec{L}/\vec{x}][L/y] \in \Re(\beta_j)$.
- $M = \Sigma_{j\in J}\bar{\tau}_{\gamma_j}(Q_j)$: there is $J' \subseteq J$ such that $\beta \leq \bigwedge_{j\in J'} \gamma_j$ and $\vec{\alpha} \in \bigcap_{j\in J'} [\![Q_j]\!]^{\vec{x}}$.
  By induction hypothesis, when given $L_i \in \Re(\alpha_i)$ for each $i \leq |\vec{x}|$, we get $Q_j[\vec{L}/\vec{x}] \to^* \boldsymbol{\epsilon}$ for any $j \in J'$. Thus, for all $j \in J'$, $M[\vec{L}/\vec{x}] \to^* M' + \bar{\boldsymbol{\epsilon}}_{\gamma_j} \in \mathcal{N}^-_{\gamma_j} \subseteq \Re(\gamma_j)$, so that $M[\vec{L}/\vec{x}] \in \Re(\bigwedge_{j\in J'} \gamma_i) \subseteq \Re(\beta)$.
- $Q = \tau_\beta(M)$: we have $(\vec{\alpha},\beta) \in [\![M]\!]^{\vec{x}}$, and by induction hypothesis if $\forall i \leq |\vec{x}|$, $L_i \in \Re(\alpha_i)$ then $M[\vec{L}/\vec{x}] \in \Re(\beta) \subseteq \mathcal{N}^+_\beta$. Thus, by definition, $\tau_\alpha(M[\vec{L}/\vec{x}]) \to^* \boldsymbol{\epsilon}$
- $Q = Q_1{\cdot}Q_2$: then $\vec{\alpha} \in [\![Q_1]\!]^{\vec{x}} \cap [\![Q_2]\!]^{\vec{x}}$ and by induction hypothesis whenever $\forall i \leq |\vec{x}|$, $L_i \in \Re(\alpha_i)$, $Q_1[\vec{L}/\vec{x}] \to^* \boldsymbol{\epsilon}$ and $Q_2[\vec{L}/\vec{x}] \to^* \boldsymbol{\epsilon}$, thus trivially $Q_1{\cdot}Q_2 \to^* \boldsymbol{\epsilon}$
- $Q = Q_1 + Q_2$: then there is $j \in \{1,2\}$, $\vec{\alpha} \in [\![Q_j]\!]^{\vec{x}}$ and by induction hypothesis whenever $\forall i \leq |\vec{x}|$, $L_i \in \Re(\alpha_i)$, $Q_j[\vec{L}/\vec{x}] \to^* \boldsymbol{\epsilon}$, thus trivially $Q_1{\cdot}Q_2 \to^* \boldsymbol{\epsilon}$

□

Theorem 4.5. *A DEFiM D is sensible for $\Lambda_{\tau(D)}$ iff there is a $S^D_{\tau(D)}$-realizer in D.*

Proof. Let $\Re$ an S^D-realizer in D and $\vec{\alpha} \in [\![Q]\!]$. Since for all $i \leq n$, $\bar{\epsilon}_{\alpha_i} \in \mathcal{N}^-_{\alpha_i} \subseteq \Re(\alpha_i)$, by Lemma 4.4 there is $Q[\bar{\epsilon}_{\alpha_1}/x_1...\bar{\epsilon}_{\alpha_n}/x_n] \to^* \epsilon$. In particular Q is converging.
Conversely, if D is sensible for $\Lambda_{\tau(D)}$, then $\Re(\beta) := \{M \mid \exists \vec{\alpha}, (\vec{\alpha}, \beta) \in [\![M]\!]\}$ is a realizer. □

This means that all we have to do to prove the sensibility of a model is to look for a realizer! Unfortunately, finding such a realizer is equally difficult (which is not so surprising as both propositions are equivalent). However, if you consider that a realizer is an element of $S^D_{\tau(D)}$ respecting the two equations of Definition 4.2, then we can try to make a systematic research in this set. More exactly, it is quite tempting to find such a realizer by a fixpoint research. For this we have to turn this equations into function, but if the first one can be turned into a function using the extensionality, this is not feasible for the second one. Regardless, the second equation is natural as a structural equation and we can do our fixpoint research inside $S^{D\wedge}_{\tau(D)}$:

Lemma 4.6. *If we call $S^{D\wedge}_{\tau(D)}$ the subset of $S^D_{\tau(D)}$ of $\Re$ such that $\Re(\alpha \wedge \beta) = \Re(\alpha) \cap \Re(\beta)$, the following function is defined over $S^{D\wedge}_{\tau(D)}$:*

$$H(\Re)(\beta) := \bigcap_{(\gamma,\delta)\in \mathbf{ext}_D(\beta)} \big(\Re(\gamma) \mapsto \Re(\delta)\big).$$

Proof. if $\Re \in S^{D\wedge}_{\tau(D)}$, then:

- For all α, $H(\Re)(\alpha)$ is saturated since function spaces and intersections (even infinite) of saturated set are saturated,
- For all α, $\mathcal{N}^-_\alpha \subseteq H(\Re)(\alpha) \subseteq \mathcal{N}^+_\alpha$: idem,
- For all α, β, $H(\Re)(\alpha\wedge\beta) \subseteq H(\Re)(\alpha) \cap H(\Re)(\beta)$: Let $(\gamma, \delta) \in \mathbf{ext}_D(\alpha)$. Since $(\gamma \to \beta) \geq_D \bigwedge_{(\gamma',\beta')\in\mathbf{ext}_D(\alpha\wedge\beta)} (\gamma' \to \beta')$, we can use the distributivity to get a decomposition $\delta = \bigwedge_i \delta_j$ such that for all i, $\gamma \to \delta_i \geq_D \gamma'_i \to \delta'_i$ for some $(\gamma'_i, \delta'_i) \in \mathbf{ext}_D(\alpha \wedge \beta)$. This means that $\big(\Re(\gamma) \mapsto \Re(\delta)\big) = \bigcup_i \big(\Re(\gamma) \mapsto \Re(\delta_i)\big) \subseteq \bigcup_i \big(\Re(\gamma'_i) \mapsto \Re(\delta'_i)\big)$ since $\Re(\gamma) \supseteq \Re(\gamma'_i)$ and $\Re(\delta_i) \subseteq \Re(\delta'_i)$, we conclude since each $(\gamma'_i, \delta'_i) \in \mathbf{ext}_D(\alpha \wedge \beta)$.
- For all α, β, $H(\Re)(\alpha\wedge\beta) \supseteq H(\Re)(\alpha) \cap H(\Re)(\beta)$: Let $(\gamma, \delta) \in \mathbf{ext}_D(\alpha\wedge\beta)$. Since $(\gamma \to \beta) \geq_D \bigwedge_{(\gamma',\beta')\in\mathbf{ext}_D(\alpha)\cup\mathbf{ext}_D(\beta)} (\gamma' \to \beta')$, we can use the distributivity to get a decomposition $\delta = \bigwedge_i \delta_j$ such that for all i, $\gamma \to \delta_i \geq_D \gamma'_i \to \delta'_i$ for some $(\gamma'_i, \delta'_i) \in \mathbf{ext}_D(\alpha) \cup \mathbf{ext}_D(\beta)$. This means that $\big(\Re(\gamma) \mapsto \Re(\delta)\big) = \bigcup_i \big(\Re(\gamma) \mapsto \Re(\delta_i)\big) \subseteq \bigcup_i \big(\Re(\gamma'_i) \mapsto \Re(\delta'_i)\big)$ since $\Re(\gamma) \supseteq \Re(\gamma'_i)$ and $\Re(\delta_i) \subseteq \Re(\delta'_i)$, we conclude since each $(\gamma'_i, \delta'_i) \in \mathbf{ext}_D(\alpha) \cup \mathbf{ext}_D(\beta)$.

□

Now, all we need is to find a fixpoint... which easier said than done. In fact, interesting examples will have to be dealt using strong fixpoint theorems. Indeed, fixpoints *à la* Curry are not sufficient, even Tarski's fixpoints are often not enough. Among order theoretic fixpoint theorems, the following version is the most general that the author could find.[4]

Definition 4.7. The *lexicographic stratification* of a set S is a sequence $(\sqsubseteq_n)_{n\in\kappa}$ of preorders, such that:

- κ is a cardinal,
- $\bigcap(\equiv_n)$ is the equality in S,
- for any n in κ, $(\sqsubseteq_n) \subseteq (\equiv_{\downarrow n})$, where $(\equiv_{\downarrow n}) := \bigcap_{m<n} (\equiv_m)$,
- for all $X \in S_{/\equiv_{\downarrow n}}$, the poset $(X_{/\equiv_n}, \sqsubseteq_n)$ is a dcpo.

A function f on such a stratification is *lexicographically-monotonous* whenever:

- f respect the equivalences $(\equiv_{\downarrow n})$, *i.e.*, for any $n \in \kappa$ and any pair x, y:

$$(x \equiv_{\downarrow n} y) \quad \Rightarrow \quad (f(x) \equiv_{\downarrow n} f(y)) ,$$

[4] To the author knowledge, it is the first time it has been enunciated formally.

- f is $\downarrow n$-monotonous over $(\equiv_{\downarrow m})_{m<n}$-fixpoints, *i.e.*, for any $n \in \kappa$, any $X \in S_{/\equiv_{\downarrow n}}$ and any pair $x, y \in X$:

$$f(X) \equiv_{\downarrow n} X \quad \Rightarrow \quad (\ x \sqsubseteq_n y \quad \Rightarrow \quad f(x) \leq_n f(y)\ )\ .$$

Proposition 3. *Any lexicographically-monotonous function on a lexicographically-stratified set has a fixpoint.*

Proof. By induction on $n \in \kappa$. Suppose given $X_{\downarrow n} \in S_{/\equiv_{\downarrow n}}$ such that $f(X_{\downarrow n}) \equiv_{\downarrow n} X_{\downarrow n}$, then f make sens and is monotonous in the dcpo $(X_{\downarrow n/\equiv_n}, \sqsubseteq_n)$. Thus it has a least fixpoint X_n. Notice that $X_{\downarrow n} \supseteq X_n$ so that we can take limits. In the end, we get a fixpoint $X_{\downarrow\kappa} \in D_{/\bigcup \equiv_n} = D$. □

Now that we have our fixpoint theorem, we have to link it to the considered filter model and stratify $S^{D\wedge}_{\tau(D)}$. Since we are looking for a condition on the atoms (or the intersection types) of our model, it is only natural to try to stratify $S^{D\wedge}_{\tau(D)}$ along those. However, this may be a bit arbitrary, which in turn may be one of the reason of our ultimate incompleteness...

Definition 4.8. A preorder $(D, \preceq)$ is said *well founded* if the quotiented poset $(D, \preceq)_{/\simeq}$ over the induced equivalence $\simeq := (\preceq \cap \succeq)$ is well founded. It is said total if any two element are comparable.

Definition 4.9. A DEFiM D is said $S^{D}_{\tau(D)}$*-realizable by stratification* if

- for every $\alpha \in D$, there is a dcpo $(\subseteq_\alpha)$ over $S^{\alpha}_{\tau(D)}$,
- there is a total and well founded preorder $(S, \preceq)$ on D,
- $S^{D\wedge}_{\tau(D)}$ is lexicographically stratified by $(\sqsubseteq_a)_{a \in D_{/\simeq}}$ defined by:

$$\mathfrak{R} \sqsubseteq_{[\alpha]} \mathfrak{Q} \quad \text{iff} \quad \begin{cases} \forall \beta \prec \alpha, & \mathfrak{R}(\beta) = \mathfrak{Q}(\beta) \\ \forall \beta \simeq \alpha, & \mathfrak{R}(\beta) \subseteq_\beta \mathfrak{Q}(\beta) \end{cases}$$

- H is lexicographically-monotonous.

Remark 5.
- *Remark that H may not be monotonous, and will not be in general.*
- *More important, notice that for* $(\sqsubseteq_a)_{a \in D_{/\simeq}}$ *to be a stratification, we only need to prove the last condition;* i.e.*, that for all* $X \in S^{D\wedge}_{\tau(D)/\equiv_{\downarrow a}}$*, the poset* $(X_{/\equiv_a}, \sqsubseteq_a)$ *is a dcpo. This property says that for any sequence* $(\mathfrak{R}(\beta))_{\beta<\alpha} \in (S^{\beta}_{\tau(D)})_{\beta<\alpha}$ *that can be extended as an element of* $S^{D\wedge}_{\tau(D)}$*, the set of possible extensions for the class a forms a dcpo.*
- *Assuming the axiom of choice, the preorder* $\preceq$ *may not have to be total.*

Theorem 4.10. *Any DEFiM D that is* $S^{D}_{\tau(D)}$*-realizable by stratification has a* $S^{D}_{\tau(D)}$*-realizer in D.*

4.2 Positive stratification

The notion of "realizability by stratification" is still too abstract; it particular, it intrinsically refers to syntactical aspects of the considered calculi. We had like a property only referring to the internal structure of the type system without any syntactic notion.

In order to achieve this goal, we need yet another change of perspective, which in turn introduce yet an other source of arbitrary. Nonetheless, positive stratification include all filter models proven sensible in the literature. We will discuss at the end of those that are conjectured sensible but not proven by lake of adequate techniques.

Definition 4.11. A (partial) DEFiM D is *stratified positive* (SP for short) if there exist

- a valuation $\mathcal{V}$, called polarity, from $D - \{\omega\}$ in the Booleans $\{\mathit{tt}, \boldsymbol{f}\}$,
- a well founded and total preorder $\preceq$ in D with ω as a bottom,

such that for all $\gamma \in D$ and all $(\alpha,\beta) \in \mathbf{ext}_D(\gamma)$:

$$\gamma \geq \beta, \qquad \gamma \simeq \beta \;\Rightarrow\; \mathcal{V}(\gamma) = \mathcal{V}(\beta),$$
$$\gamma \geq \alpha, \qquad \gamma \simeq \alpha \;\Rightarrow\; \mathcal{V}(\gamma) \neq \mathcal{V}(\alpha),$$

(where $\simeq := (\preceq \cap \succeq)$ is the equivalence relation induced by the preorder)
and such that:

$$\alpha \wedge \beta \preceq \gamma \text{ for } \gamma = \alpha \text{ or for } \gamma = \beta \qquad (\alpha \wedge \beta) \prec \alpha \Rightarrow (\alpha \wedge \beta) = \beta$$

Moreover, we also require that the polarity is coherent with the intersections on $\simeq$-equivalence classes:

$$\alpha \simeq \beta \quad \Rightarrow \quad \mathcal{V}(\alpha \wedge \beta) = \mathcal{V}(\alpha) \wedge \mathcal{V}(\beta).$$

This condition can be seen as a stratification given by $\preceq$, where the quotient $D_{/\simeq}$ represents the different levels of the stratification, each level endowed with a positive polarity $\mathcal{V}$. This stratification improves the condition of [3] that only considers completions of positive partial DEFiM.[5] This condition is the invariant by completion, which simplify the proof of stratified positivity of DEFiMs of Example 1.4 (save for P_∞).

Proposition 4. *Assuming the axiom of choice, in the definition of stratified positive DEFiM, the preorder $\preceq$ can be taken non-total without lost of generality.*

Proposition 5. *A partial DEFiM E is stratified positive iff its completion $\bar{E}$ is stratified positive.*

Example 4.12. The models of Example 1.4 are stratified positive except P_∞ and U_∞:

- D_∞ is SP: The stratified positivity is given by $\mathcal{V}(*) = \boldsymbol{f}$ and $\omega \prec *$.
- D^*_∞ is SP: Idem, we set $\mathcal{V}(q) = \boldsymbol{t}$, $\mathcal{V}(p) = \boldsymbol{f}$ and $\omega \prec p \simeq q$.
- Z_∞ is SP: Idem, we set $\mathcal{V}(\underline{2n}) = \boldsymbol{t}$, $\mathcal{V}(\underline{2n+1}) = \boldsymbol{f}$ and $\omega \prec \underline{m} \simeq \underline{n}$ for all m and n.
- P_∞ is not SP: Since $*\rightarrow* = *$, they are $\preceq$-equivalent and with the same polarity, contradicting the second implication in Definition 4.11.
- U_∞ is not SP: Since $\underline{n} = \underline{n+1}\rightarrow\underline{n+1}$, we must have $\underline{n} \succ \underline{n+1}$, which create a non well-founded chain.

Example 4.13. The model of Example **??** is stratified positive. Morally, the preorder $\preceq$ represents the number of nested but non-mutually recursive ν-construction, while the polarity is only used inside a coinductive call and behave well due to positiveness.

Lemma 4.14. *Any stratified positive DEFiM D is $S^D_{\tau(D)}$-realizable by stratification.*

Proof. • For any $\alpha \in D$ we define the order $(\subseteq_\alpha) := (\subseteq_{\mathcal{V}(\alpha)})$ where $(\subseteq_{\boldsymbol{f}}) := (\subseteq)$ and $(\subseteq_{\boldsymbol{t}}) := (\supseteq)$, so that $(S^\alpha_{\tau(D)}, \subseteq_\alpha)$ is a dcpo.

- The equivalence classes $D_{/\simeq}$ forms a J-partition of D for J the cardinal of $D_{/\simeq}$.
- $S^{D\wedge}_{\tau(D)}$ is lexicographically stratified by $(\sqsubseteq_a)_{a\in D_{/\simeq}}$ defined by:

$$\mathfrak{R} \sqsubseteq_{[\alpha]} \mathfrak{Q} \quad \text{iff} \quad \begin{cases} \forall \beta \prec \alpha, & \mathfrak{R}(\beta) = \mathfrak{Q}(\beta) \\ \forall \beta \simeq \alpha, & \mathfrak{R}(\beta) \subseteq_\beta \mathfrak{Q}(\beta) \end{cases}$$

  We only need to prove the last condition, which corresponds to Lemma 4.15
- Remains to show that H is lexicographically-monotonous:
  - H respects the equivalences $(\equiv_{\downarrow c})$:
    Let $\alpha \in D$ and $\mathfrak{R} \equiv_{\downarrow[\alpha]} \mathfrak{Q}$. Let $\beta \geq_D \alpha$, we have $H(\mathfrak{R})(\beta) := \bigcap_{(\gamma,\delta)\in\mathbf{ext}_D(\beta)}\big(\mathfrak{R}(\gamma) \mapsto \mathfrak{R}(\delta)\big)$ and $H(\mathfrak{Q})(\beta) := \bigcap_{(\gamma,\delta)\in\mathbf{ext}_D(\beta)}\big(\mathfrak{Q}(\gamma) \mapsto \mathfrak{Q}(\delta)\big)$. It is sufficient to show that $\big(\mathfrak{R}(\gamma) \mapsto \mathfrak{R}(\delta)\big) = \big(\mathfrak{Q}(\gamma) \mapsto \mathfrak{Q}(\delta)\big)$ for any $(\gamma,\delta) \in \mathbf{ext}_D(\beta)$. But this is immediate since $\gamma, \delta \preceq \beta \preceq \alpha$ and $\mathfrak{R} \equiv_{\downarrow[\alpha]} \mathfrak{Q}$.

[5]More exactly it considers a subclass of DEFiM called K-models.

– H is $\downarrow[\alpha]$-monotonous over $(\equiv_{\downarrow[\beta]})_{\alpha\prec\beta}$-fixpoints:
Let $\alpha \in D$ and $\mathfrak{R} \sqsubseteq_{\downarrow[\alpha]} \mathfrak{Q}$ such that $H(\mathfrak{R})(\beta) = \mathfrak{R}(\beta)$ for all $\beta \prec \alpha$. For any $\beta \prec \alpha$, we have $H(\mathfrak{R})(\beta) = H(\mathfrak{Q})(\beta)$ since H respects the equivalences $(\equiv_{\downarrow c})$. Remains to show that for all $\beta \simeq \alpha$, $H(\mathfrak{R})(\beta) \subseteq_\beta H(\mathfrak{Q})(\beta)$. We will show that for any $(\gamma,\delta) \in \mathbf{ext}_D(\beta)$, $\big(\mathfrak{R}(\gamma) \mapsto \mathfrak{R}(\delta)\big) \subseteq_\beta \big(\mathfrak{Q}(\gamma) \mapsto \mathfrak{Q}(\delta)\big)$. We do the case where $\mathcal{V}(\beta) = \boldsymbol{f}$, the other is symmetric. Either $\gamma \prec \beta$ and $\mathfrak{R}(\gamma) = \mathfrak{Q}(\gamma)$ (since $\mathfrak{R} \equiv_{\downarrow[\gamma]} \mathfrak{Q}$) or $\gamma \simeq \beta$ has the polarity $\mathcal{V}(\gamma) = \boldsymbol{t}$ and $\mathfrak{R}(\gamma) \supseteq \mathfrak{Q}(\gamma)$, in any case, $\mathfrak{R}(\gamma) \supseteq \mathfrak{Q}(\gamma)$. Similarly, in any case $\mathfrak{R}(\delta) \subseteq \mathfrak{Q}(\delta)$, so that we have $\big(\mathfrak{R}(\gamma) \mapsto \mathfrak{R}(\delta)\big) \subseteq_\beta \big(\mathfrak{Q}(\gamma) \mapsto \mathfrak{Q}(\delta)\big)$.

□

Lemma 4.15. *Let $\mathfrak{R} \in (S^{\alpha}_{\tau(D)})_{\alpha\prec\delta}$ such that $\mathfrak{R}(\alpha \wedge \beta) = \mathfrak{R}(\alpha) \cap \mathfrak{R}(\beta)$ for $\alpha,\beta \prec \delta$.*
The set of extensions of $\mathfrak{R}$ to all $\alpha \preceq \delta$, ordered by $\sqsubseteq_{\mathcal{V}}$, is a dcpo with a sup $(\bigvee_i \mathfrak{R}_i)(\alpha)$ defined by induction on $\preceq$:

$$(\bigvee_i \mathfrak{R}_i)(\alpha) = \begin{cases} \mathcal{N}^-_\alpha \cup \bigcup\limits_{\gamma\leq_D\alpha,\gamma\prec\delta} \mathfrak{R}(\gamma) \cup \bigcup \mathfrak{R}_i(\alpha) & \text{whenever } \mathcal{V}(\alpha) = \boldsymbol{f}, \\ \mathcal{N}^+_\alpha \cap \bigcap\limits_{\gamma\geq_D\alpha,\gamma\prec\delta} \mathfrak{R}(\gamma) \cap \bigcap \mathfrak{R}_i(\alpha) & \text{whenever } \mathcal{V}(\alpha) = \boldsymbol{t}. \end{cases}$$

in particular, $(\bigvee_i \mathfrak{R}_i)(\alpha) = \mathfrak{R}(\alpha)$ for $\alpha \prec \delta$.

Proof. We first show that for all $\alpha \leq_D \beta$, then $(\bigvee_i \mathfrak{R}_i)(\alpha) \subseteq (\bigvee_i \mathfrak{R}_i)(\beta)$.

- if $\mathcal{V}(\beta) = \boldsymbol{f}$: The case where $\alpha \prec \delta$ is trivial (it is the second terms of the definition above). Otherwise, necessarily $\mathcal{V}(\alpha) = \boldsymbol{f}$: We have $\mathcal{V}(\beta) = \mathcal{V}(\alpha)$ thus we only have to check term to term. First, we have $\mathcal{N}^-_\alpha \subseteq \mathcal{N}^-_\beta$. For the second term, we have that $\{\gamma \mid \gamma \leq_D \alpha, \gamma \prec \delta\} \subseteq \{\gamma \mid \gamma \leq_D \beta, \gamma \prec \delta\}$, thus $\bigcup\limits_{\gamma\leq_D\alpha,\gamma\prec\delta} \mathfrak{R}(\gamma) \subseteq \bigcup\limits_{\gamma\leq_D\beta,\gamma\prec\delta} \mathfrak{R}(\gamma)$. And last, we have $\mathfrak{R}_i(\alpha) \subseteq \mathfrak{R}_i(\beta)$ for all i.
- if $\mathcal{V}(\alpha) = \boldsymbol{t}$: The case where $\beta \prec \delta$ is trivial (it is the second terms of the definition above). Otherwise, necessarily $\mathcal{V}(\beta) = \boldsymbol{t}$: We have $\mathcal{V}(\beta) = \mathcal{V}(\alpha)$ thus we only have to check term to term. First, we have $\mathcal{N}^+_\alpha \subseteq \mathcal{N}^+_\beta$. For the second term, we have that $\{\gamma \mid \gamma \geq_D \alpha, \gamma \prec \delta\} \supseteq \{\gamma \mid \gamma \geq_D \beta, \gamma \prec \delta\}$, thus $\bigcap\limits_{\gamma\geq_D\alpha,\gamma\prec\delta} \mathfrak{R}(\gamma) \subseteq \bigcap\limits_{\gamma\geq_D\beta,\gamma\prec\delta} \mathfrak{R}(\gamma)$. And last, we have $\mathfrak{R}_i(\alpha) \subseteq \mathfrak{R}_i(\beta)$ for all i.
- if $\mathcal{V}(\alpha) = \boldsymbol{f}$ and $\mathcal{V}(\beta) = \boldsymbol{t}$: We have $\mathcal{N}^-(\alpha) \subseteq \mathcal{N}^-_\beta \subseteq (\bigvee \mathfrak{R})(\beta)$. Similarly, $(\bigvee \mathfrak{R})(\alpha) \subseteq \mathcal{N}^+_\alpha \subseteq \mathcal{N}^+_\beta$. For any $\gamma_+, \gamma_- \prec \delta$ such that $\gamma_+ \leq_D \alpha \leq_D \beta \leq_D \gamma_-$, we have $\mathfrak{R}(\gamma_+) \subseteq \mathfrak{R}(\gamma_-)$. For any $\gamma \prec \delta$ such that $\gamma \leq_D \alpha \leq_D \beta$ and any $i \in I$, $\mathfrak{R}(\gamma) = \mathfrak{R}_i(\gamma) \subseteq \mathfrak{R}_i(\beta)$. Similarly, for any $\gamma \prec \delta$ such that $\gamma \geq_D \beta \geq_D \alpha$ and any $i \in I$, $\mathfrak{R}(\gamma) = \mathfrak{R}_i(\gamma) \supseteq \mathfrak{R}_i(\alpha)$. The only remaining case is for each $i, j \in I$, to prove that $\mathfrak{R}_i(\alpha) \subseteq \mathfrak{R}_j(\beta)$, but we know that $\mathfrak{R}_i(\alpha) \subseteq \mathfrak{R}_{i\vee j}(\alpha)$ since $\mathcal{V}(\alpha) = \boldsymbol{f}$, similarly, $\mathfrak{R}_{i\vee j}(\beta) \subseteq \mathfrak{R}_j(\beta)$ since $\mathcal{V}(\beta) = \boldsymbol{t}$, and we conclude by $\mathfrak{R}_{i\vee j}(\alpha) \subseteq \mathfrak{R}_{i\vee j}(\beta)$ since $\alpha \leq_D \beta$.

Now, we have to verify that all meets are conserved. One inclusion is already done, so that we have to show that $(\bigvee_i \mathfrak{R}_i)(\alpha) \cap (\bigvee_i \mathfrak{R}_i)(\beta) \subseteq (\bigvee_i \mathfrak{R}_i)(\alpha \wedge \gamma)$. Moreover, the cases where $\alpha,\beta \prec \delta$, $\alpha = (\alpha \wedge \beta)$ or $\beta = (\alpha \wedge \beta)$ are trivial, thus we assume that $\alpha \preceq \beta \simeq (\alpha \wedge \beta) \simeq \delta$:

- If $\mathcal{V}(\alpha \wedge \beta) = \boldsymbol{t}$:
Then necessarily $\mathcal{V}(\beta) = \boldsymbol{t}$. We have $\mathcal{N}^+_{\alpha\wedge\beta} \supseteq \mathcal{N}^+_\alpha \cap \mathcal{N}^+_\beta \supseteq (\bigvee \mathfrak{R})(\alpha) \cap (\bigvee \mathfrak{R})(\beta)$. Moreover, for any $\gamma \prec \delta$ such that $\gamma \geq_D \alpha \wedge \beta$, we have $\mathfrak{R}(\gamma) = (\bigvee \mathfrak{R})(\gamma) \supseteq (\bigvee \mathfrak{R})(\alpha \wedge \beta)$. Finally, we got the difficult case: let $i \in I$, we have $\mathfrak{R}_i(\alpha \wedge \beta) = \mathfrak{R}_i(\alpha) \cap \mathfrak{R}_i(\beta)$ since $\mathfrak{R}_i$ respect intersections, and we have $(\bigvee \mathfrak{R})(\beta) \subseteq \mathfrak{R}_i(\beta)$ since $\mathcal{V}(\beta) = \boldsymbol{t}$, we thus need to show that $(\bigvee \mathfrak{R})(\alpha) \subseteq \mathfrak{R}_i(\alpha)$ to get $\mathfrak{R}_i(\alpha \wedge \beta) \supseteq (\bigvee \mathfrak{R})(\alpha) \cap (\bigvee \mathfrak{R})(\beta)$; there is two cases:
  – either $\alpha \simeq \delta$: then necessarily $\mathcal{V}(\alpha) = \boldsymbol{t}$, so that $(\bigvee \mathfrak{R})(\alpha) \subseteq \mathfrak{R}_i(\alpha)$,

  - or $\alpha \prec \delta$: then $(\bigvee \Re)(\alpha) = \Re(\alpha) = \Re_i(\alpha)$.
- If $\mathcal{V}(\alpha \wedge \beta) = \boldsymbol{f}$:
  We can the consider that $\mathcal{V}(\beta) = \boldsymbol{f}$ without lost of generality.[6] Notice that $\Re(\alpha) \cap \mathcal{N}^-_\beta \subseteq \mathcal{N}^-_\alpha \cap \mathcal{N}^+_\beta$ which is included in the completion of $\mathcal{N}^-_{\alpha\wedge\beta}$ and thus in $(\bigvee \Re)(\alpha \wedge \beta)$.
  - If $\alpha \prec \delta$: Then for all $\gamma \prec \delta$ such that $\gamma \leq \beta$, $(\bigvee \Re)(\alpha) \cap \Re(\gamma) = \Re(\alpha) \cap \Re(\gamma) = \Re(\alpha \wedge \gamma) \subseteq (\bigvee \Re)(\alpha \wedge \beta)$ the last inclusion being because $\alpha \wedge \gamma \leq_D \alpha \wedge \beta$. Moreover, for any $i \in I$, $(\bigvee \Re)(\alpha) \cap \Re_i(\beta) = \Re_i(\alpha) \cap \Re_i(\beta) = \Re_i(\alpha \wedge \beta) \subseteq (\bigvee \Re)(\alpha \wedge \beta)$ the last inclusion being because $\mathcal{V}(\alpha \wedge \beta) = \boldsymbol{f}$.
  - If $\alpha \simeq \delta$ and $\mathcal{V}(\alpha) = \boldsymbol{f}$: For all $\gamma_1, \gamma_2 \prec \delta$ such that $\gamma_1 \leq_D \alpha$ and $\gamma_2 \leq_D \beta$, we have $\Re(\gamma_1) \cap \Re(\gamma_2) = \Re(\gamma_1 \wedge \gamma_2) \subseteq (\bigvee \Re)(\alpha \wedge \beta)$, the last inclusion being because $\alpha \wedge \gamma \leq_D \alpha \wedge \beta$. Moreover, for all $\gamma \prec \delta$ such that $\gamma \leq_D \alpha$ and all $i \in I$, $\Re(\gamma) \cap \Re_i(\beta) = \Re_i(\gamma) \cap \Re_i(\beta) = \Re_i(\gamma \wedge \beta) \subseteq \Re_i(\alpha \wedge \beta) \subseteq (\bigvee \Re)(\alpha \wedge \beta)$. Finally, for any $i, j \in I$, we have $\Re_i(\alpha) \cap \Re_j(\beta) \subseteq \Re_{i\vee j}(\alpha) \cap \Re_{i\vee j}(\beta) = \Re_{i\vee j}(\alpha \wedge \beta) \subseteq (\bigvee \Re)(\alpha \wedge \beta)$.
  - If $\alpha \simeq \delta$ and $\mathcal{V}(\alpha) = \boldsymbol{t}$: Then for all $i \in I$, $(\bigvee \Re)(\alpha) \cap \Re_i(\beta) \subseteq \Re_i(\alpha) \cap \Re_i(\beta) = \Re_i(\alpha \wedge \beta) \subseteq (\bigvee \Re)(\alpha \wedge \beta)$, the first inclusion being because $\mathcal{V}(\alpha) = \boldsymbol{t}$. Moreover, for any $\gamma \prec \delta$ such that $\gamma \leq_D \beta$, we have seen that $(\bigvee \Re)(\alpha) \cap (\bigvee \Re)(\gamma) \subseteq (\bigvee \Re)(\alpha \wedge \gamma) \subseteq (\bigvee \Re)(\alpha \wedge \beta)$.

□

Theorem 4.16. *Any stratified positive DEFiM D is sensible for $\Lambda_{\tau(D)}$ and approximable.*

Example 4.17. By Theorem 4.16 and Example 4.13, all the DEFiMs D of Examples 1.4 and **??** are approximable except for P_∞ and U_∞.

4.3 Further generalization

We strongly conjecture that this result does not fundamentally use the extensionality:

Conjecture 4.18. *Any stratified positive filter model D is approximable.*

This result should be obtained following the same way, but with a lot of technical hindrance. In particular the rules (τ) and $(\bar\tau)$ would become potentially infinitary:[7]

$$(\tau) \qquad \tau_\alpha(\lambda x.M) \to \sum_{A\in\mathcal{A}} \prod_{(\beta,\gamma)\in A} \tau_\gamma(M[\bar{\boldsymbol{\epsilon}}_\beta/x]) \qquad \text{where } \mathcal{A} = \left\{ A \subseteq_f D\times D \;\middle|\; \bigwedge_{(\beta,\gamma)\in A} (\beta\to\gamma) \leq \alpha \right\}$$

$$(\bar\tau) \qquad (\sum_i \bar\tau_{\alpha_i}(Q_i))N \to \sum_i \sum_{(\beta\to\gamma)\geq\alpha_i} \bar\tau_{\gamma_i}(Q\cdot\tau_{\beta_i}(N))$$

An other technical issue is the definition of the function H of Lemma 4.6 that would be no more a function, but just linear constraints.

This generalization is expected for the long version; especially because it surprisingly permit to weaken the condition positive stratification by dropping the well foundedness of the strata.

Proposition 4.19. *Let D a filter model satisfying all the conditions of stratified positiveness except for the well foundedness of the preorder $\preceq$.*
If Conjecture 4.18 is true, then D equates any terms with the same Böhm trees, and is in particular sensible.

Proof. Let M and N two terms with the same set of Böhm approximations and let $(\vec\alpha,\beta) \in [\![M]\!]^{\vec x}_D$. We will show that $(\vec\alpha,\beta) \in [\![N]\!]^{\vec x}_D$.

There exists a derivation π of $(x_i : \alpha_i)_i \vec M : \beta_i$ in the intersection type system of D. Since π is finite, there is only a finite set $F \subseteq_f D$ of elements of D appearing in the derivation.

[6] It is also possible that we only have $\alpha \simeq \beta$ and $\mathcal{V}(\alpha) = \boldsymbol{f}$, but we can the conclude by symmetry.
[7] In the sens that sum and product could be infinite.

Let $F^\wedge \subseteq D$ the $\wedge$-completion $F^\wedge := \{\bigwedge_i \gamma_i \mid \forall i, \gamma_i \in F\}$ of F. Let $\to_F$ partially defined by $\gamma \to_F \delta = \gamma \to \delta$ when it makes sense, *i.e.*, when $\gamma, \delta, (\gamma \to \delta) \in F^\wedge$.

Then $(F^\wedge, \wedge, \to_F)$ is a partial filter model that can be freely completed into $\overline{F}$. Moreover, $(F^\wedge, \wedge, \to_F)$ is stratified positive since it is finite and a subset of D; thus $\overline{F}$ is stratified positive.

Since π only use elements of F, it is also a derivation in $\overline{F}$, so that $(\vec{\alpha}, \beta) \in [\![M]\!]^{\vec{x}}_{\overline{F}}$. Since $\overline{F}$ is stratified positive and M and N have the same set of Böhm approximations, $(\vec{\alpha}, \beta) \in [\![N]\!]^{\vec{x}}_{\overline{F}}$. Moreover, since D and $\overline{F}$ are two completions of $(F^\wedge, \wedge, \to_F)$ but $\overline{F}$ is free, we have $[\![.]\!]_{\overline{F}} \subseteq [\![.]\!]_D$; so that $(\vec{\alpha}, \beta) \in [\![N]\!]^{\vec{x}}_D$. □

Remark 6. *Equating all terms with the same Böhm trees is a notion similar to approximability, but slightly weaker. This is a property that says that the interpretation of a term is characterized by the interpretations of its Böhm trees; but it may not be the union that is considered. Morally, however, this is a kind of approximation theorem where the "limit" of the interpretations can be arbitrary (and not just the union).*

Example 4.20. Assuming Conjecture 4.18, the filter model U_∞ of Examples 1.4 equates any terms with the same Böhm trees.

Related Works

The quest for sensibility and approximability of different filter models was very important in the 90's. A survey of this quest can be found in the book "Lambda calculus with types" [1, Chapter 17].

To this survey, we only have one reference to add to their survey, this is the works of Berline [3] and her students Guy [24], Kerth [17] and Manzonetto [18]. They performed deep studies on the limits and classification of the traditional classes of models. In that aspect, they follow an approach very similar to ours.

As a systematic study of a specific property in a large class of models, this article also follows recent works of Breuvart, Manzonetto and Ruopolo [4, 6, 7] that are rather studying the property of full abstraction for different reduction strategies.

Indirectly, the (relatively) recent results of Ehrhard on the extensional collapse [14] are also linked with our result as the target of the described extensional collapse are automatically approximable (because the source is a class containing only approximable models). This gives yet a different and modern approach of approximability.

Further Works

One may ponder the generality of our work considering the restriction taken on our class of model. First, the choice of filter models over usual Scott domains seems relatively safe as a Scott domain can be turned into a filter model by adding a top element; in the other side not having to consider the existence of an intersection is before all a comfort for the reader. Moreover, switching to Scott domains would make heavier the definition of tests, similarly for the others enforced restrictions: the extensionality and the distributivity. We strongly believe that the detour by tests mechanism can be removed, removing these unnatural restrictions. Nonetheless, we choose to stick with tests as they illustrate the link between sensibility and approximability in a very readable manner.

Our main regret, however, is that the final characterization is not a complete one: there is (*a priori*) filter models that are approximable and not positively stratifyable, or even models that are sensible but not appriximable! To illustrate this remark, we look at four filter models that are generated by the atoms $\alpha, \beta, \gamma, \delta$ and the following four sets of equations:[8]

$$\alpha = \omega{\to}\alpha \qquad \beta = \omega{\to}\alpha \qquad \gamma = (\gamma \wedge \delta){\to}\beta \qquad \delta = \omega{\to}\omega{\to}\alpha \tag{1}$$
$$\alpha = \omega{\to}\alpha \qquad \beta = \omega{\to}\alpha \qquad \gamma = (\gamma \wedge \delta){\to}\beta \qquad \delta = \alpha{\to}\alpha{\to}\alpha \tag{2}$$
$$\alpha = \omega{\to}\alpha \qquad \beta = \omega{\to}\alpha \qquad \gamma = (\gamma \wedge \delta){\to}\beta \qquad \delta = \omega{\to}\alpha{\to}\alpha \tag{3}$$
$$\alpha = \omega{\to}\alpha \qquad \beta = (\beta{\to}\alpha){\to}\alpha \qquad \gamma = (\gamma \wedge \delta){\to}\beta \qquad \delta = \omega{\to}\alpha{\to}\alpha \tag{4}$$

Notice that the notation $\omega{\to}\omega{\to}\alpha$ is simply syntactic sugar for $\omega{\to}\gamma'$ for $\gamma' = \omega{\to}\alpha$. Considering that we omit the full description of $\wedge$ and $\mathbf{ext}_D$ which are the free ones, each of these lines forms a partial DEFiM.

In the first model, $\delta \leq \gamma$ since $\delta = \omega{\to}\alpha$ and $\gamma = (\gamma \wedge \delta){\to}\alpha$ with $\omega \geq (\gamma \wedge \delta)$ (remember that ω is a top). Thus the equation $\gamma = \delta{\to}\alpha$ is now positive, and the generated model is positively stratified.

On the other hand, in the second model, $\delta \geq \gamma$; thus $\gamma = \gamma{\to}\beta$ is an unsafe equation breaking sensibility because $\gamma \in [\![\Omega]\!]$. The third one is more interesting; in this case, neither $\delta \geq \gamma$ not $\delta \leq \gamma$; it is conjectured that this model is sensible and approximable but no proof have be found yet.

The last example is even more surprising: it is also conjectured sensible for the same reason, but it can be shown non-approximable. This is an example that appears[9] in Kerth's thesis [16], he showed (more or less) that if we consider the λ-term $V := (\lambda xy.y(xx))\,(\lambda xy.y(xx))$, then β is in the interpretation of V, but $\tau_\beta(V)$ diverges. None of these two facts are difficult to obtain and we invite our reader to verify it as an exercise.

Conclusion

With this highly theoretical and exploratory article, we only aim at questioning the limits of our models by pointing on unusual behaviors of well known semantical objects.

Indeed, we have seen that approximability and sensibility are properties that are surprisingly hard to separate by traditional filter models. The possible causes are easy to see:

- Either it may rise from a new internal incompleteness of the considered class of model, which would join the incompleteness of [9].
- But it is more probably a logical weakness of the methods we know for proving the sensibility of a model.

In the second case, this would be an indication that the realizability methods are in fact limited when joining coinductive types and subtyping. It is, however, impossible to discern at which point level is the blockage.

All we know is that this must be somehow related to our knowledge on the *non-constructive* determination of a solution for linear but non-monotonous constraints in a highly non trivial functional space. In fact, it is easy to show that in our case, the solution is unic when it exists, which means that there is still a lot of symmetry that we where unable to use.

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