λ-calculus goes to the tropics

Davide Barbarossa (joint with Paolo Pistone)

davide.barbarossa@unibo.it

https://lipn.univ-paris13.fr/~barbarossa/

Dipartimento di Informatica

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Journées d'hiver 2022-23 GdT Scalp

CIRM, 16/02/2023

Differential/resource/Taylor approximation

"Approximate" MN via $\sum_{n\in\mathbb{N}} \frac{1}{n!}(\mathtt{D}^n M \bullet N^n)0$.

	Analysis	Differential λ-calculus
Same shape of $\mathcal{T}(\cdot)$	$\sum_{n\in\mathbb{N}} \frac{1}{n!}(\mathtt{D}^n f \bullet x^n)0$	$\sum_{n\in\mathbb{N}} \frac{1}{n!}(\mathtt{D}^n M \bullet N^n)0$
Recover a complicated object from simple ones	monomials (simple = finite degree)	Resource terms, i.e. of shape $(\mathtt{D}^n M \bullet N^n)0$ (simple = *fixed number* of duplications allowed)
Interpretation of $\mathtt{D}(\cdot) \bullet (\cdot)$	The usual differential	A generalisation of it
Can be *used as* an approximation technique	Yes	Yes
Meaning of "to approximate"	Can control the *distance* between simple objects and the complicated one	

Metric semantics, i.e. control error amplification

Yes for the *linear* (and *affine*) STλC

The category pMET of pseudo-metric spaces and non-expansive functions is SMCC.

No for STλC

pMET is *not* CCC (However it contains several interesting sub-CCC)

Yes for bounded-duplication calculi

The maps $X \mapsto \mathcal{M}_{\leq \mathrm{n}}(X)$ lift to functors $!_n : \mathrm{pMET} \to \mathrm{pMET}$ forming a graded linear exponential comonad on pMET.

A program

$$\vdash M : !_n X \to Y$$

is interpreted as a n-Lipschitz function

$$[\![X]\!] \to [\![Y]\!].$$

Two quantitative approaches to duplication

If a program calls x exactly 3 times, say $(\mathtt{D}^3 M \bullet x^3)0$, then:

	Differential λ-calculus	Bounded-duplication calculi
$[\![(\mathtt{D}^3 M \bullet x^3)0]\!]$	a degree 3 function, say x^3	a 3-Lipschitz function, say $3x$

Question

Is there a way to relate this two approaches?

$$x^3 \xmapsto{\ \cdot := +\ } 3x$$

Tropicalisation

Tropical semiring

$\mathbb{L}$ is the semiring $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ with min as sum and $+$ as product.
Equivalently, it is a quantale with $\geq$ as order and $+$ as product.

The zero is $+\infty$, and sum is idempotent: $\min\{x, x\} = x$.

$$x^3 \xmapsto{\cdot := +} 3x$$

$$\sum_n a_n x^n \xmapsto{+ := \inf,\ \cdot := +} \inf_n \{a_n + nx\}$$

Tropical relational model: the category $\mathbb{L}\mathrm{Rel}$

Objects: sets; Morphisms $X \to Y$: maps $X \times Y \to \mathbb{L}$.
Composition of $X \xrightarrow{t} Y$ and $Y \xrightarrow{s} Z$:

$$(s \circ t)_{a,c} := \inf_{b \in Y} \{s_{b,c} + t_{a,b}\}.$$

In $\mathbb{L}\mathrm{Rel}^{\mathrm{op}}$ you find the usual undergraduate linear algebra formulas for the product matrix-matrix and matrix-vector. Here we use the transpose ones, so that:

$\mathbb{L}\mathrm{Rel}(X, Y) \simeq$ the set of linear maps from the $\mathbb{L}$-semimodule $\mathbb{L}^X$ to $\mathbb{L}^Y$.

Model of the *linear* STλC (or IMLL)

$\mathbb{L}\mathrm{Rel}$ is SMCC w.r.t. the tensor product $\otimes$ and internal hom $\multimap$ which act, on objects, as the Cartesian product.

Modeling duplications/erasures

The SMCC $\mathbb{L}\mathrm{Rel}$ is Lafont w.r.t. the comonad ! which acts, on objects, as the finite multisets operator. Thus:

Model of STλC

The coKleisli $\mathbb{L}\mathrm{Rel}_!$ is CCC.

Model of ST$\partial\lambda$C

$\mathbb{L}\mathrm{Rel}_!$ is CC∂C w.r.t. the differential $D : \mathbb{L}\mathrm{Rel}(!X, Y) \to \mathbb{L}\mathrm{Rel}(!(X + X), Y)$:

$$(Df)_{\mu \oplus \rho, b} := f_{\rho + \mu, b} \text{ if } \#\mu = 1 \text{ and } +\infty \text{ otherwise.}$$

Taylor

In $\mathbb{L}\mathrm{Rel}_!$ all morphisms coincide with their Taylor expansion and $[\![\mathcal{T}(M)]\!] = [\![M]\!]$.

An example with probabilities

$\mathbb{L}\mathrm{Rel}$ models a STλC with a coin toss $\oplus_p$ constructor, with bias p. Consider

$$\vdash M := (\mathtt{true} \oplus_p \mathtt{false}) \oplus_p ((\mathtt{true} \oplus_p \mathtt{false}) \oplus_p (\mathtt{false} \oplus_p \mathtt{true})) \;:\; \mathrm{Bool}$$

$$\mathbb{P}(M \twoheadrightarrow \mathtt{true}) = p^2 + p^2 q + q^3 \qquad \mathbb{P}(M \twoheadrightarrow \mathtt{false}) = pq + 2pq^2$$

An example with probabilities

$\mathbb{L}\mathrm{Rel}$ models a STλC with a coin toss $\oplus_p$ constructor, with bias p. Consider

$$\vdash M := (\mathtt{true} \oplus_p \mathtt{false}) \oplus_p ((\mathtt{true} \oplus_p \mathtt{false}) \oplus_p (\mathtt{false} \oplus_p \mathtt{true})) \ : \ \mathrm{Bool}$$

$$\mathbb{P}(M \twoheadrightarrow \mathtt{true}) = p^2 + p^2 q + q^3 \qquad \mathbb{P}(M \twoheadrightarrow \mathtt{false}) = pq + 2pq^2$$

Choosing $[\![\mathrm{Bool}]\!] := \{0, 1\}$, $[\![M]\!] : \mathbb{L}^{[\![\mathrm{Bool}]\!]} \to \mathbb{L}$ gives the tropicalisations: $[\![M]\!]_1 = \min\{2x, 2x + y, 3y\}$, $[\![M]\!]_0 = \min\{x + y, x + 2y\}$.

An example with probabilities

$\mathbb{L}\mathrm{Rel}$ models a STλC with a coin toss $\oplus_p$ constructor, with bias p. Consider

$$\vdash M := (\mathtt{true} \oplus_p \mathtt{false}) \oplus_p ((\mathtt{true} \oplus_p \mathtt{false}) \oplus_p (\mathtt{false} \oplus_p \mathtt{true})) : \mathrm{Bool}$$

$$\mathbb{P}(M \twoheadrightarrow \mathtt{true}) = p^2 + p^2 q + q^3 \qquad \mathbb{P}(M \twoheadrightarrow \mathtt{false}) = pq + 2pq^2$$

Choosing $[\![\mathrm{Bool}]\!] := \{0, 1\}$, $[\![M]\!] : \mathbb{L}^{[\![\mathrm{Bool}]\!]} \to \mathbb{L}$ gives the tropicalisations: $[\![M]\!]_1 = \min\{2x, 2x + y, 3y\}$, $[\![M]\!]_0 = \min\{x + y, x + 2y\}$.

Knowing that M tossed $\mathtt{true}$, what is the most likely choice for p s.t. the tossed occurrence is the rightmost one? I.e., when $\max\{p^2, p^2 q, q^3\} = q^3$?

Solution: tropicalise + change of variables! $[\![M]\!]_1 = 3y$ iff $y \leq \frac{2}{3}x$; $x := -\log p$, $y := -\log(1 - p)$; The problem is equivalent to $-\log(1 - p) \leq -\frac{2}{3}\log p$, i.e. $1 - p \geq p^{\frac{2}{3}}$. E.g., $p = \frac{1}{4}$.

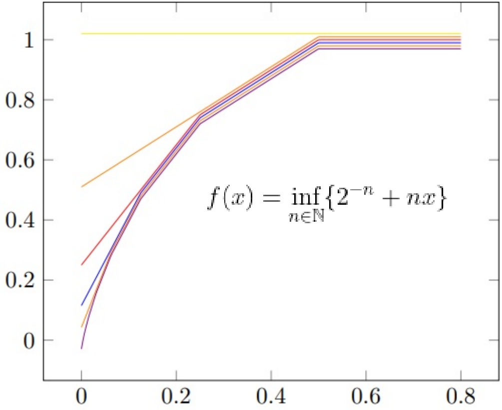
A selection of results

Expressing a coKleisli (linear) map $f : !X \to Y$ "in the base X", yields a *tropical Laurent series*, i.e. a (non-linear) $f : \mathbb{L}^X \to \mathbb{L}^Y$ of shape:

$$f(x)_b := \inf_{\mu \in \mathcal{M}_{\mathrm{fin}}(X)} \{a_{\mu,b} + \mu x\}.$$

Let f as above, with $X = Y = \{*\}$. $\forall \epsilon \in (0, +\infty)$, $\exists \mathcal{F}_\epsilon \subseteq_{\mathrm{fin}} \mathbb{N}$ s.t. $f(x)$ coincides on $[\epsilon, +\infty]$ with the tropical *polynomial* $\min_{n \in \mathcal{F}_\epsilon} \{a_n + nx\}$.

$[\![\Gamma \vdash_{ST\lambda C} M : X]\!]$ is a *locally* Lipschitz function, inf of Lipschitz ones.



Generalise: no more coordinates in a base!

Let $\mathbb{L}\mathrm{Mod} :=$ the category of $\mathbb{L}$-semimodules and $\mathbb{L}\mathrm{CCat} :=$ the one of complete Lawvere generalised metric spaces (*complete* in a stronger sense than Cauchy). E.g. the metric on the function space is the usual sup distance.

$\mathbb{L}\mathrm{Mod}$ and $\mathbb{L}\mathrm{CCat}$ are isomorphic categories.

Model of STλC

$\mathbb{L}\mathrm{Mod}$ is a SMCC and it admits a Lafont exponential !. Thus the coKleisli $\mathbb{L}\mathrm{Mod}_!$ is CCC.

For semimodules of shape $\mathbb{L}^X$, the SMCC structure and ! coincide with that of $\mathbb{L}\mathrm{Rel}$.

Model of ST$\partial\lambda$C

We can define a differential operator on $\mathbb{L}\mathrm{Mod}_!$ making it a CC∂C.


ANY
QUESTIONS
?