Approximating functional programs: Taylor subsumes Scott, Berry, Kahn and Plotkin

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Possible behaviours of a program $F = F\{x\}$

Normalizing		Meaningless

$z \leftarrow 1$
$z \leftarrow \frac{z + x/z}{2}$
write i-th digit of z
		while(True){
		DoNothing
		}

$F(2) = 1,5$		$F(2)$ produces no information!

Possible behaviours of a program $F = F\{x\}$

Normalizing	Solvable (Babylonians)	Meaningless
$z \leftarrow 1$ $z \leftarrow \frac{z + x/z}{2}$ write i-th digit of z	$z \leftarrow 1; i \leftarrow 0$ while$(\mathrm{True})\{$ $i++$ $z \leftarrow \frac{z + x/z}{2}$ write i-th digit of z $\}$	while$(\mathrm{True})\{$ DoNothing $\}$
$F(2) = 1,5$	$F(2) \twoheadrightarrow 1,41$ $\twoheadrightarrow 1,414$ $\twoheadrightarrow 1,4142$ $\cdots \twoheadrightarrow_{\infty} \sqrt{2}$	$F(2)$ produces no information!

Böhm Trees

λ-calculus (Church)

The set Λ of programs is given by $M ::= x \mid \lambda x.M \mid MM$.
Computation step: $(\lambda x.M)N \to M\{N/x\}$.

Böhm trees (Barendregt)

The map $\mathrm{BT} : \Lambda \to \mathcal{B}$ associates each λ-term F with its *Böhm tree*:

$$\mathrm{BT}(F) := \mathrm{BT}(\mathrm{hnf}(F)), \qquad \mathrm{BT}(F) := \bot \text{ if } F \text{ is unsolvable,}$$

$$\mathrm{BT}(\lambda\vec{x}.y\, Q_1 \dots Q_k) := \quad \lambda\vec{x}.y$$

$$\mathrm{BT}(Q_1) \quad \cdots \quad \mathrm{BT}(Q_k)$$

The equivalence $=_{\mathrm{BT}}$ is a λ-theory. So $\mathcal{B}_\Lambda \simeq \Lambda/_{=_{\mathrm{BT}}}$ is a semantics for Λ.
The set of all normal forms is dense in $\mathcal{B}_\Lambda$ (in analogy with $\mathbb{Q}$ dense in $\mathbb{R}$).

Finite approximants

- The set $\mathcal{A}$ of finite approximants is defined as:

$$P \ ::= \ \bot \mid \lambda\vec{x}.y\, P \dots P$$

  with the intuition that $\bot$ means *no information*.
- Fix $\leq$ the prorder on $\mathcal{A}$ generated by taking $\bot \leq P$ for all P.
- The set $\mathcal{A}(F)$ of the finite approximants of $F \in \Lambda$ is:

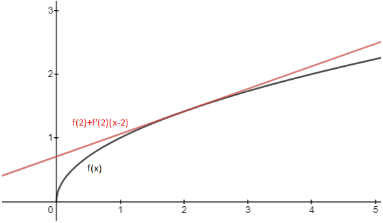
$$\mathcal{A}(F) := \{P \in \mathcal{A} \text{ s.t } \exists N \in \Lambda \text{ s.t } F \twoheadrightarrow_\beta N \geq P\}$$

Approximation Theorem

$$\mathrm{BT}(F) = \sup_{P \in \mathcal{A}(F)} P$$

in analogy to the fact that $\sqrt{2}$ is the limit of $\mathrm{BabylonianProgram}(2)$.

Derivatives!



	Analysis	λ-calculus
Application	$F(x)$	$F\ x$
Taylor expansion $\Theta(\cdot)$	$\sum_n \frac{1}{n!} F^{(n)}(0)x^n$	$\sum_n \frac{1}{n!} (\mathrm{D}^n \Theta(F) \bullet x^n)0$

Differential λ-calculus

Programs live in the module $\mathbb{Q}^+\langle\Lambda^r\rangle_\infty$ and are subject to the equation:

$$\mathtt{D}(\lambda x.M) \bullet N = \lambda x.\left(\frac{d}{dx} M \cdot N\right)$$

where $\frac{d}{dx}(PQ) \cdot N := \left(\frac{d}{dx} P \cdot N\right) Q + \left(\mathtt{D}P \bullet \left(\frac{d}{dx} Q \cdot N\right)\right) Q$
is the linear substitution of N in M for x.

Ehrhard and Régnier:

Θ defines a function $\Lambda \to \mathbb{Q}^+\langle\Lambda^r\rangle_\infty$ (called the *full Taylor expansion*):

$$\Theta(\cdot) = \sum_{t \in \mathcal{T}(\cdot)} \frac{1}{\mathrm{m}(t)} t$$

where $\mathrm{m}(t) \in \mathbb{N}$ is difficult and $\mathcal{T}(\cdot) : \Lambda \to \mathcal{P}(\Lambda^r)$ is easy (i.e. inductive).
Furthermore,

$$\mathrm{NF}(\Theta(\cdot)) = \Theta(\mathrm{BT}(\cdot)).$$

The Resource Calculus

Define the set Λ^r of Resource terms:

$$t ::= x \mid \lambda x.t \mid t\,[t, \ldots, t]$$

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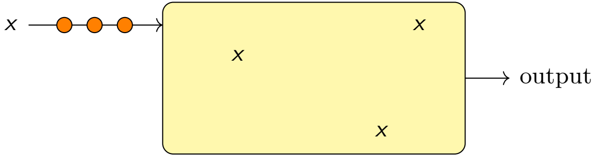
$$(\lambda x.t)[s_1, s_2, s_3] \to \; ?$$



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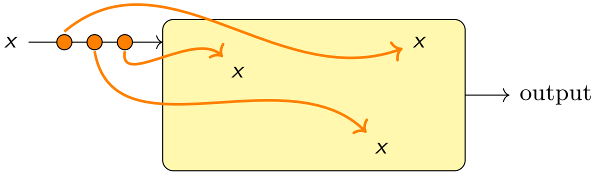
$$(\lambda x.t)[s_1, s_2, s_3] \to t\{s_1/x^{(1)}, s_2/x^{(2)}, s_3/x^{(3)}\}$$



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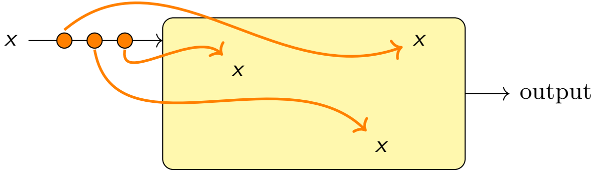
Define the set Λ^r of Resource terms:

$$t ::= \; x \;\mid \lambda x.t \;\mid t\,[t, \ldots, t]$$

We need formal (*idempotent*) sum $\mathbb{T} = \; t_1 + \cdots + t_n$ of resource terms.
Reduction:

$$(\lambda x.t)[s_1, s_2, s_3] \to \sum_{\sigma \in \mathfrak{S}_3} t\{s_{\sigma(1)}/x^{(1)}, s_{\sigma(2)}/x^{(2)}, s_{\sigma(3)}/x^{(3)}\}$$



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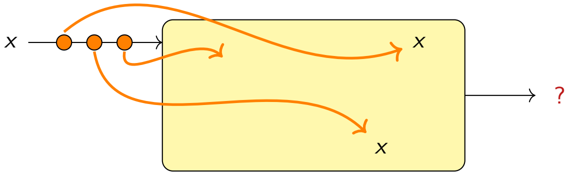

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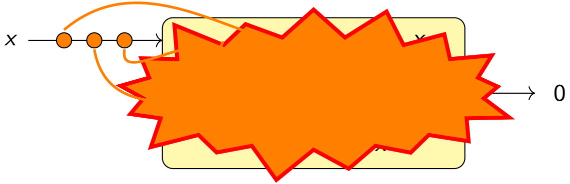
Reduction:

$$(\lambda x.t)[s_1, s_2, s_3] \to 0$$



Resource terms live a tough life

They may experience non-determinism:

$$\Delta[x,y] := (\lambda x.x[x])[y,y'] \to y[y'] + y'[y]$$

But also starvation:

$$\Delta[\Delta,\Delta] \to (\lambda x.x[x])[\Delta] \to 0$$

As well as surfeit:

$$(\lambda x \lambda y.x)[I][I] \to (\lambda y.I)[I] \to 0$$

Summing up: $(\lambda x.t)[s_1,\dots,s_n] \not\twoheadrightarrow 0 \quad \Rightarrow \quad t$ uses each s_i exactly once!

Main Properties:

- Linearity: Cannot erase non-empty bags (unless annihilating). □
- Strong Normalization: Trivial, as there is no duplication. □
- Confluence: Locally confluent + strongly normalizing. □

Qualitative Taylor Expansion

The (support of the full) Taylor expansion is the map $\mathcal{T}(\cdot) : \Lambda \to \mathcal{P}(\Lambda^r)$:

$$\begin{array}{lcl} \mathcal{T}(x) & = & \{x\} \\ \mathcal{T}(\lambda x.M) & = & \{\lambda x.t \in \Lambda^r \text{ s.t. } t \in \mathcal{T}(M)\} \\ \mathcal{T}(MN) & = & \{t[s_1, \ldots, s_k] \in \Lambda^r \text{ s.t. } k \in \mathbb{N}, t \in \mathcal{T}(M), s_i \in \mathcal{T}(N)\}. \end{array}$$

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\end{array}
$$

Examples:

$$
\begin{array}{lcl}
\mathcal{T}(\lambda x.x) & = & \{\lambda x.x\} \\
\mathcal{T}(\lambda x.xx) & = & \{\lambda x.x[x^n] \mid n \in \mathbb{N}\} \\
\mathcal{T}(\Omega) & = & \{(\lambda x.x[x^{n_0}])[\lambda x.x[x^{n_1}], \ldots, \lambda x.x[x^{n_k}]] \mid k, n_0, \ldots, n_k \in \mathbb{N}\} \\
\mathcal{T}(\Delta_f) & = & \{\lambda x.f[x^n][x^k] \mid n, k \in \mathbb{N}\} \\
\mathcal{T}(Y) & = & \{\lambda f.t[s_1, \ldots, s_k] \mid k \in \mathbb{N},\ t, s_1, \ldots, s_k \in \mathcal{T}(\Delta_f)\}
\end{array}
$$

where $Y = \lambda f.\Delta_f \Delta_f$ and $\Delta_f = \lambda x.f(xx)$.

Approximating through resources

Computing the normal form:

$$\mathrm{NF}(\mathcal{T}(M)) = \bigcup_{t\in\mathcal{T}(M)} \mathrm{nf}(t)$$

Examples

$\mathrm{NF}(\mathcal{T}(Y)) = \{\lambda f.f1, \lambda f.f[f1], \lambda f.f[f1, f1], \lambda f.f[f1, f[f1], f[f[f1]]], \dots\}$.

$\mathrm{NF}(\mathcal{T}(\Omega)) = \emptyset$. This is the case for all unsolvables.

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Taylor Expansion of Böhm Trees

$$\mathcal{T}(\bot) := \emptyset$$

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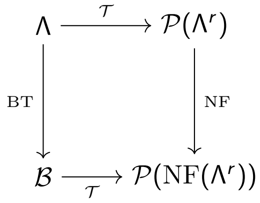

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The diagramm commutes!

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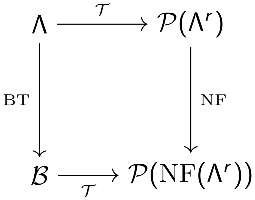

A common structure

1. Source language $\mapsto$ Resource version

   (gain confluence and strong normalization)

2. via a Taylor Expansion, providing:
   - static analysis (coherence/cliques),
   - dynamic analysis (normalization)

3. and (when possible) a:

Commutation Theorem

$$\mathrm{NF}(\mathcal{T}(P)) = \mathcal{T}(\mathrm{BT}(P))$$

Corollary

$$\mathrm{BT}(M) = \mathrm{BT}(N) \Leftrightarrow \mathrm{NF}(\mathcal{T}(M)) = \mathrm{NF}(\mathcal{T}(N))$$


On the Taylor Expansion of Probabilistic λ-terms
(Long Version)
Ugo Dal Lago Thomas Leventis
Abstract

REVISITING CALL-BY-VALUE BÖHM TREES IN LIGHT OF THEIR TAYLOR EXPANSION
EMMA KERINEC, GIULIO MANZONETTO, AND MICHELE PAGANI

A semantical and operational account of call-by-value solvability
Alberto Carraro[1,2] and Giulio Guerrieri[2]

NORMALIZING THE TAYLOR EXPANSION OF NON-DETERMINISTIC λ-TERMS, VIA PARALLEL REDUCTION OF RESOURCE VECTORS
LIONEL VAUX

A common structure

1. Source language $\mapsto$ Resource version

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2. via a Taylor Expansion, providing:

3. anc

Commut

> *"Understanding the relation between the term and its full Taylor expansion might be the starting point of a renewing of the theory of approximations".*
> Ehrhard and Régnier

$$\mathrm{NF}(\mathcal{T}(P)) = \mathcal{T}(\mathrm{BT}(P))$$

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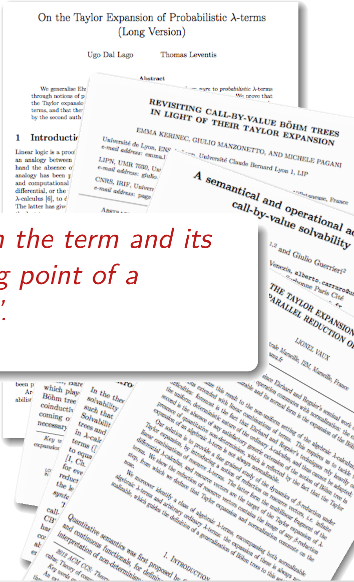


Classic results via labelled reduction

Scott's Continuity → Berry's Stability

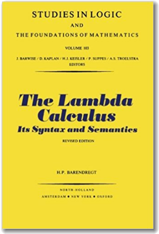
Khan & Plotkin's Sequentiality → Berry's Stability

Khan & Plotkin's Sequentiality → Perpendicular Lines Property → ∄ parallel or

Scott's Continuity → (topological argument) → Contextuality of BTs

Scott's Continuity → (topological argument) → Genericity Property

Chapter 14



Classic results via Resource Approximation

Scott's Continuity

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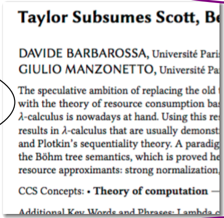
$\mathrm{NF}(\mathcal{T}(P)) = \mathcal{T}(\mathrm{BT}(P))$


Taylor Subsumes Scott, Be

DAVIDE BARBAROSSA, Université Paris
GIULIO MANZONETTO, Université Pa

The speculative ambition of replacing the old t
with the theory of resource consumption bas
λ-calculus is nowadays at hand. Using this res
results in λ-calculus that are usually demonst
and Plotkin's sequentiality theory. A paradig
the Böhm tree semantics, which is proved he
resource approximants: strong normalization,

CCS Concepts: • **Theory of computation** —

Additional Key Words and Phrases: Lambda c

Khan & Plotkin's Sequentiality

Perpendicular Lines Property

$\nexists$ parallel or

Contextuality of BTs

Genericity Property

Equality mod BT is a λ-theory

Contextuality of Böhm trees

Let $C(\!|\cdot|\!)$ be a context.

$$\mathrm{BT}(M) = \mathrm{BT}(N) \quad \Rightarrow \quad \mathrm{BT}(C(\!|M|\!)) = \mathrm{BT}(C(\!|N|\!))$$

Equality mod NFT is a λ-theory

Monotonicity of contexts w.r.t. $\leq_{\mathrm{NFT}}$

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Proof. Induction on C. The interesting case is $C = C_1\, C_2$.

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$t \in \mathrm{NF}(\mathcal{T}(C(\!|M|\!))) \quad \Rightarrow \quad \exists t' \in \mathcal{T}((C_1(\!|M|\!))(C_2(\!|M|\!)))$ such that :

$$\begin{array}{ccc} t' = s_1[u_1, \ldots, u_k] & \twoheadrightarrow & t + \mathbb{T} \\ \downarrow\!\!\downarrow & \nearrow\!\!\!\nearrow & \\ \mathrm{nf}(s_1)[\mathrm{nf}(u_1), \ldots, \mathrm{nf}(u_k)] & & \end{array}$$

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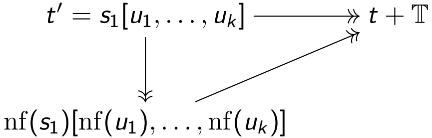

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and $\mathrm{nf}(u_1), \ldots, \mathrm{nf}(u_k) \subseteq \mathrm{NF}(\mathcal{T}(C_2(\!|M|\!))) \subseteq \mathrm{NF}(\mathcal{T}(C_2(\!|N|\!)))$.

Easily conclude that $t \in \mathrm{NF}(\mathcal{T}(C(\!|N|\!)))$. $\square$

Unsolvables are computationally meaningless

Genericity Property

Let U unsolvable. If $C(\!|U|\!)$ has a β-nf, then $C(\!|U|\!) =_\beta C(\!|M|\!)\ \forall M \in \Lambda$.

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Proof. $C(\!| U |\!)$ normalizable $\Rightarrow$ $\exists t \in \mathrm{NF}(\mathcal{T}(C(\!| U |\!)))$ such that:

$$\text{"}\mathrm{nf}_\beta(C(\!| U |\!)) = t\text{" and all its bags are singletons.}$$

So $\exists t' \in \mathcal{T}(C(\!| U |\!))$ such that:

$$t' = c(\!| s_1, \dots, s_k |\!) \twoheadrightarrow t + \mathbb{T}$$

for some $c \in \mathcal{T}(C(\!| \cdot |\!))$ and $s_1, \dots, s_k \in \mathcal{T}(U)$.

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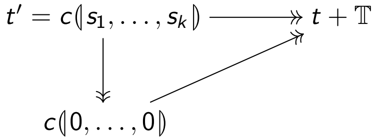

for some $c \in \mathcal{T}(C(\!|\cdot|\!))$ and $s_1, \ldots, s_k \in \mathcal{T}(U)$. $\quad$ (U unsolvable $\Rightarrow \mathrm{nf}(s_i) = 0$)

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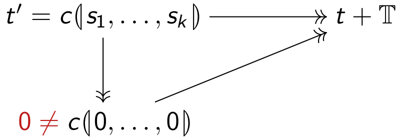


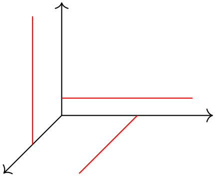
for some $c \in \mathcal{T}(C(\!| \cdot |\!))$ and $s_1, \dots, s_k \in \mathcal{T}(U)$. No hole can occur in c!

$$\text{Therefore}: t' = c(\!| s_1, \dots, s_k |\!) = c \in \mathcal{T}(C(\!| M |\!)) \text{ and hence } t \in \mathrm{NF}(\mathcal{T}(C(\!| M |\!))).$$

Since and bags of t are singletons, "$t = \mathrm{nf}_\beta(C(\!| M |\!))$". □

Perpendicular Lines Property

PLP: *If a context* $C(\!|\cdot,\ldots,\cdot|\!) : \Lambda^n \to \Lambda$ *is constant on n perpendicular lines, then it must be constant everywhere.*



Perpendicular Lines Property

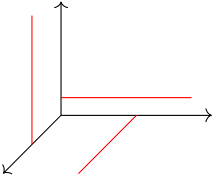
PLP: *If a context $C(\!|\cdot,\ldots,\cdot|\!) : \Lambda^n \to \Lambda$ is constant on n perpendicular lines, then it must be constant everywhere.*

True in $\Lambda/_{=_{\mathrm{BT}}}$, Barendregt's Book 1982
Proof: via Sequentiality.

?? in $\Lambda^o/_{=_{\mathrm{BT}}}$

False in $\Lambda^o/_{=_\beta}$, Barendregt & Statman 1999
Counterexample: via Plotkin's terms.

True in $\Lambda/_{=_\beta}$, De Vrijer & Endrullis 2008
Proof: via Reduction under Substitution.



PLP	β	BT
open	✓	✓
closed	X	?

Idea of the proof

Perpendicular Lines Property

$$\forall Z \left\{ \begin{array}{lcl} C(\!| Z, M_{12}, \ldots\ldots, M_{1n} |\!) & =_{\mathrm{BT}} & N_1 \\ C(\!| M_{21}, Z, \ldots\ldots, M_{2n} |\!) & =_{\mathrm{BT}} & N_2 \\ \ddots & \vdots & \vdots \\ C(\!| M_{n1}, \ldots, M_{n(n-1)}, Z |\!) & =_{\mathrm{BT}} & N_n \end{array} \right. \quad \Rightarrow \quad \forall \vec{Z}\,,\ C(\!| \vec{Z} |\!) =_{\mathrm{BT}} N_1.$$

Idea of the proof

Perpendicular Lines Property

$$\forall Z \begin{cases} C(\!| Z, M_{12}, \ldots\ldots, M_{1n} |\!) & =_{\mathrm{BT}} & N_1 \\ C(\!| M_{21}, Z, \ldots\ldots, M_{2n} |\!) & =_{\mathrm{BT}} & N_2 \\ \qquad\qquad \ddots & \vdots & \vdots \\ C(\!| M_{n1}, \ldots, M_{n(n-1)}, Z |\!) & =_{\mathrm{BT}} & N_n \end{cases} \quad \Rightarrow \quad \forall \vec{Z}\,,\ C(\!| \vec{Z} |\!) =_{\mathrm{BT}} N_1.$$

How can a context $C(\!|\cdot|\!)$ be constant in $\Lambda/_{=_{\mathrm{BT}}}$?

1. $C(\!|\cdot|\!)$ does not contain the hole at all (the trivial case);
2. the hole is erased during its reduction;
3. the hole is "hidden" behind an unsolvable;
4. the hole is never erased but "pushed into infinity".

Idea of the proof

Perpendicular Lines Property

$$\forall Z \begin{cases} C(\!| Z, M_{12}, \ldots\ldots, M_{1n} |\!) & =_{\mathrm{NFT}} & N_1 \\ C(\!| M_{21}, Z, \ldots\ldots, M_{2n} |\!) & =_{\mathrm{NFT}} & N_2 \\ \ddots & \vdots & \vdots \\ C(\!| M_{n1}, \ldots, M_{n(n-1)}, Z |\!) & =_{\mathrm{NFT}} & N_n \end{cases} \quad \Rightarrow \quad \forall \vec{Z}\,,\ C(\!| \vec{Z} |\!) =_{\mathrm{NFT}} N_1.$$

How can a $c \in \mathcal{T}(C(\!| \cdot |\!))$ s.t. $\mathrm{nf}(c) \neq 0$ be constant in Λ^r ?

1. c does not contain the hole at all (the trivial case);
2. the hole is erased during its reduction ;
3. the hole is "hidden" behind an unsolvable;
4. the hole is never erased but "pushed into infinity".

Idea of the proof

Perpendicular Lines Property

$$\forall Z \begin{cases} C(\!|Z, M_{12}, \ldots\ldots, M_{1n}|\!) & =_{\mathrm{NFT}} & N_1 \\ C(\!|M_{21}, Z, \ldots\ldots, M_{2n}|\!) & =_{\mathrm{NFT}} & N_2 \\ \ddots & \vdots & \vdots \\ C(\!|M_{n1}, \ldots, M_{n(n-1)}, Z|\!) & =_{\mathrm{NFT}} & N_n \end{cases} \Rightarrow \quad \forall \vec{Z}\,,\ C(\!|\vec{Z}|\!) =_{\mathrm{NFT}} N_1.$$

How can a $c \in \mathcal{T}(C(\!|\cdot|\!))$ s.t. $\mathrm{nf}(c) \neq 0$ be constant in Λ^r ?

1. c does not contain the hole at all (the trivial case);
2. ~~the hole is erased during its reduction~~ (linearity);
3. the hole is “hidden” behind an unsolvable;
4. the hole is never erased but “pushed into infinity”.

Idea of the proof

Perpendicular Lines Property

$$\forall Z \begin{cases} C\langle\!| Z, M_{12}, \ldots\ldots, M_{1n} |\!\rangle & =_{\mathrm{NFT}} \quad N_1 \\ C\langle\!| M_{21}, Z, \ldots\ldots, M_{2n} |\!\rangle & =_{\mathrm{NFT}} \quad N_2 \\ \qquad\qquad \ddots & \quad\ \vdots \qquad\ \ \vdots \\ C\langle\!| M_{n1}, \ldots, M_{n(n-1)}, Z |\!\rangle & =_{\mathrm{NFT}} \quad N_n \end{cases} \quad \Rightarrow \quad \forall \vec{Z}\,,\ C\langle\!| \vec{Z} |\!\rangle =_{\mathrm{NFT}} N_1.$$

How can a $c \in \mathcal{T}(C\langle\!| \cdot |\!\rangle)$ s.t. $\mathrm{nf}(c) \neq 0$ be constant in Λ^r ?

1. c does not contain the hole at all (the trivial case);
2. ~~the hole is erased during its reduction~~ (linearity);
3. ~~the hole is "hidden" behind an unsolvable~~ (strong normalization);
4. the hole is never erased but "pushed into infinity".

Idea of the proof

Perpendicular Lines Property

$$\forall Z \begin{cases} C(\!|Z, M_{12}, \ldots\ldots, M_{1n}|\!) & =_{\mathrm{NFT}} & N_1 \\ C(\!|M_{21}, Z, \ldots\ldots, M_{2n}|\!) & =_{\mathrm{NFT}} & N_2 \\ \ddots & \vdots & \vdots \\ C(\!|M_{n1}, \ldots, M_{n(n-1)}, Z|\!) & =_{\mathrm{NFT}} & N_n \end{cases} \Rightarrow \quad \forall \vec{Z}\,,\ C(\!|\vec{Z}|\!) =_{\mathrm{NFT}} N_1.$$

How can a $c \in \mathcal{T}(C(\!|\cdot|\!))$ s.t. $\mathrm{nf}(c) \neq 0$ be constant in Λ^r ?

1. c does not contain the hole at all (the trivial case!);
2. ~~the hole is erased during its reduction~~ (linearity);
3. ~~the hole is "hidden" behind an unsolvable~~ (strong normalization);
4. ~~the hole is never erased but "pushed into infinity"~~ (finiteness).

Idea of the proof

Perpendicular Lines Property

$$\forall Z \begin{cases} C(\!|Z, M_{12}, \ldots\ldots, M_{1n}|\!) & =_{\mathrm{NFT}} & N_1 \\ C(\!|M_{21}, Z, \ldots\ldots, M_{2n}|\!) & =_{\mathrm{NFT}} & N_2 \\ \quad\ddots & \vdots & \vdots \\ C(\!|M_{n1}, \ldots, M_{n(n-1)}, Z|\!) & =_{\mathrm{NFT}} & N_n \end{cases} \quad \Rightarrow \quad \forall \vec{Z}\,,\ C(\!|\vec{Z}|\!) =_{\mathrm{NFT}} N_1.$$

Claim.

If $c \in \mathcal{T}(C(\!|\cdot|\!))$ then:

$$\mathrm{nf}(c) \neq 0 \quad \Rightarrow \quad c \text{ contains no hole.}$$

PLP	β	BT
open	✓	✓
closed	X	?

By induction on the size of c.

Idea of the proof

Perpendicular Lines Property

$$\forall Z \begin{cases} C(\!| Z, M_{12}, \ldots\ldots, M_{1n} |\!) & =_{\mathrm{NFT}} & N_1 \\ C(\!| M_{21}, Z, \ldots\ldots, M_{2n} |\!) & =_{\mathrm{NFT}} & N_2 \\ \quad\ddots & \vdots & \vdots \\ C(\!| M_{n1}, \ldots, M_{n(n-1)}, Z |\!) & =_{\mathrm{NFT}} & N_n \end{cases} \Rightarrow \quad \forall \vec{Z}\,,\ C(\!| \vec{Z} |\!) =_{\mathrm{NFT}} N_1.$$

Our proof does not need open terms!

PLP holds in $\Lambda^o/_{=_{\mathrm{BT}}}$ ✓

PLP	β	BT
open	✓	✓
closed	X	✓

The End!