An overview on the Taylor expansion of programs

Davide Barbarossa

davide.barbarossa@unibo.it

https://lipn.univ-paris13.fr/~barbarossa/

Dipartimento di Informatica

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

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Differential λ-calculus

Good old undergraduate Taylor:

$$F(U) = F(0) + (DF \bullet U)(0) + \sum_{n \geq 2} \frac{1}{n!} \, (D^n F \bullet U^n)(0)$$

where $D(\_) \bullet (\_) \; : \; (F, U) \longmapsto \dfrac{d}{dx} F \cdot U$

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$$\textit{Best linear approximation of } F(\_) \;=\; (DF \bullet \_)(0) \;=\; \left.\frac{d}{dx}\right|_{x=0} F \cdot (\_)$$

$$=\; F \textit{ ``forced'' to use } (\_) \textit{ exactly once}$$

Differential λ-calculus [ER03]

Linearisation (in analysis and computer science) of $F = \left.\frac{d}{dx}\right|_{x=0} F \cdot (\_)$

Differential λ-terms

The $\mathbb{Q}$-module $\partial\Lambda$ given by the words:

$$F ::= 0 \mid x \mid \lambda x.F \mid FF \mid (\mathtt{D}F) \bullet F \mid pF + qF$$

quotiented under α-equivalence and other natural equations involving "$+$".

Reductions

The usual β-reduction plus: $\mathtt{D}(\lambda x.F) \bullet U \to_\partial \lambda x. \left(\frac{d}{dx} F \cdot U\right)$.

$$\frac{d}{dx}(PQ) \cdot U := \left(\frac{d}{dx} P \cdot U\right) Q + \left(\mathtt{D}P \bullet \left(\frac{d}{dx} Q \cdot U\right)\right) Q$$

A quick categorical parenthesis [BCS09, Man12]

Cartesian Closed Differential Category

It is a CCC enriched in commutative monoids (i.e. in each $\mathrm{Hom}(X, Y)$ there is 0 and we can do $f + g$) with $(f + g) \circ h = f \circ h + g \circ h$, $0 \circ f = 0$, cartesian closed structure compatible with $+$, plus a *differential map* $D : \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X \times X, Y)$ verifying some axioms.

Cartesian Closed Differential λ-Category

It is a Cartesian Closed Differential Category satisfying the D-Curry axiom: $D\,\mathrm{curry} f = \mathrm{curry}(Df \circ \langle \pi_X \times 0_Y, \pi_Y \times \mathrm{id}_Y \rangle)$, for all $f \in \mathrm{Hom}(X \times Y, Z)$.

Example: convenient vector spaces

Objects: particular kinds of locally convex topological vector spaces
Morphisms: smooth maps with pointwise addition
Differential $Df : X \times X \to Y$ of smooth $f : X \to Y$ given by:

$$Df(x, u) := \frac{d}{dt}\Big|_{t=0} f(x + tu).$$

The (quantitative) Taylor expansion of a λ-term

Morally:

$$FU = \sum_{n \in \mathbb{N}} \frac{1}{n!} \left(\mathtt{D}^n F \bullet U^n\right)(0)$$

The (quantitative) Taylor expansion of a λ-term

Morally:

$$\Theta(FU) := \sum_{n\in\mathbb{N}} \frac{1}{n!} \left(\mathtt{D}^n F \bullet U^n\right)(0)$$

The (quantitative) Taylor expansion of a λ-term

Morally:

$$\Theta(FU) := \sum_{n\in\mathbb{N}} \frac{1}{n!} \left(\mathtt{D}^n\Theta(F) \bullet \Theta(U)^n\right)(0)$$

The (quantitative) Taylor expansion of a λ-term [ER08]

Rigorously:

It is the map $\Theta : \Lambda \to \partial\Lambda$ given by:

$$
\begin{aligned}
\Theta(x) &:= x \\
\Theta(\lambda x.F) &:= \lambda x.\Theta(F) \\
\Theta(FU) &:= \sum_{n\in\mathbb{N}} \frac{1}{n!} \left(\mathtt{D}^n\Theta(F) \bullet \Theta(U)^n\right)(0)
\end{aligned}
$$

Qualitative Taylor expansion [ER08]

Resource λ-terms

It is the set Λ^r defined by:

$$t ::= x \mid \lambda x.t \mid t[t, \dots, t]$$

with a reduction simulating $\to_\partial$. We also have translation $(.)^\partial : \Lambda^r \to \partial\Lambda$ whose interesting case is $(s[u_1, \dots, u_n])^\partial := \left(\mathtt{D}^n s^\partial \bullet (u_1^\partial, \dots, u_n^\partial)\right)(0)$.

One has:

$$\Theta(F) = \sum_{t \in \mathcal{T}(F)} \frac{1}{m(t)}\, t^\partial$$

where $\mathcal{T} : \Lambda \to \mathscr{P}(\Lambda^r)$ is called the *qualitative Taylor expansion*, given by:

$$\mathcal{T}(x) := \{x\}$$

$$\mathcal{T}(\lambda x.F) := \{\lambda x.t \mid t \in \mathcal{T}(F)\}$$

$$\mathcal{T}(FU) := \{s[u_1, \dots, u_n] \mid s \in \mathcal{T}(F) \text{ and } u_i \in \mathcal{T}(U)\}.$$

Classic results via labelled reduction

Scott's Continuity → Berry's Stability

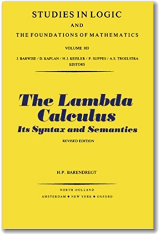
Khan & Plotkin's Sequentiality → Berry's Stability

Khan & Plotkin's Sequentiality → Perpendicular Lines Property → ∄ parallel-or

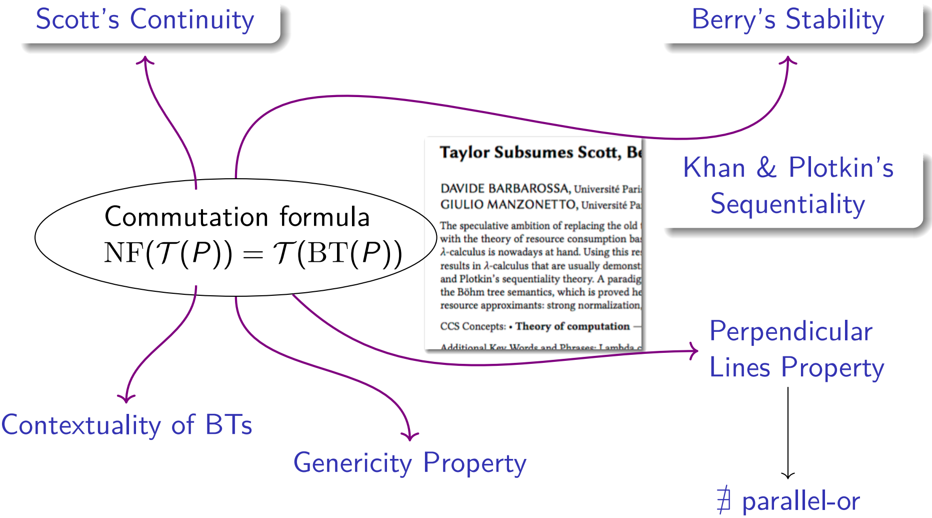
Scott's Continuity → Contextuality of BTs

Scott's Continuity → (topological argument) → Genericity Property

Chapter 14



Classic results via Resource Approximation [BM20]




That's all Folks!

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