Random subgroups of free groups

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Joint work with
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General context 1/2

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- A subgroup $H \leq F(A)$ is finitely generated if and only H is rational (Anisimov and Seifert, 1975)

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- Gromov, Arjantseva, Ol'shanskii, Kapovich, Miasnikov, Schupp, Shpilrain, Ollivier, Jitsukawa, ...

Which distribution on fg subgroups?

- Classical approach: a subgroup is generated by a random tuple of reduced words. A k-tuple (few-generators), or a s_n^d-tuple, where $s_n =$ cardinality of the sphere of radius n and $0 < d < 1$ (Gromov's density model)

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- This graph is efficiently computable (Touikan), opens the way to countless efficient (and elegant) decision or computation algorithms on fg subgroups. A natural finite discrete structure attached to a subgroup.
- The idea: use these graphs to define what a random subgroup is. There are finitely many possible Stallings graphs with n vertices: draw one uniformly at random.

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$$\begin{aligned} h_1 &= a^3 b^{-1} \\ h_2 &= a^3 c a^{-2} \\ h_3 &= a^2 c d^{-1} b^{-1} \\ h_4 &= a^2 d e^{-1} d^{-1} b^{-1} \end{aligned}$$

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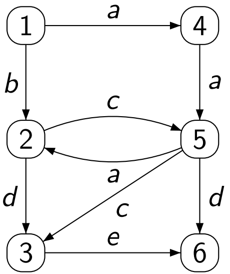

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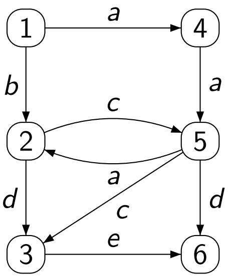

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rank $= E - V + 1$

conjugation, finite index

intersection of subgroups, malnormality

effective separability

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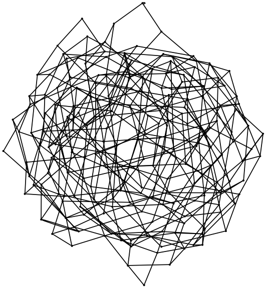
A picture with $n = 200$



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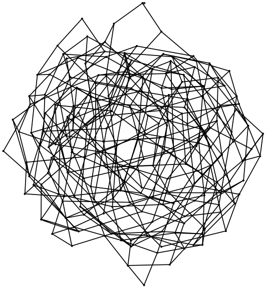

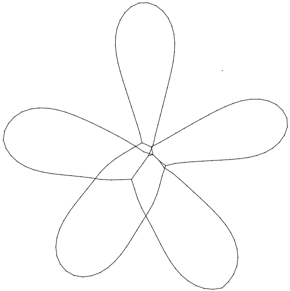

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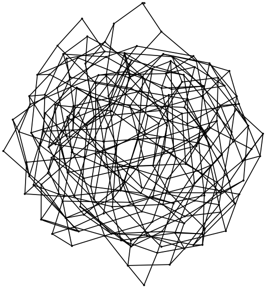

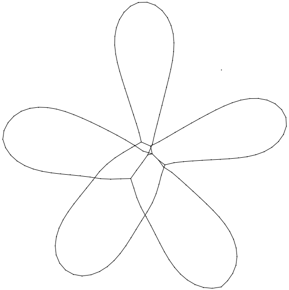

- Many more edges, many more cycles in the graph based distribution. Higher rank, lesser probability of malnormality, etc.

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- every vertex has valency at least 2, except maybe the base vertex.
- There are many! although estimating that number is non-trivial

General strategy to draw a random Stallings graph

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- to estimate the probability of non-admissibility – we show that it tends to 0 as n tends to infinity

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- Draw a size m of an orbit, decide whether it is a cycle or a sequence; and draw another random partial injection of size $n - m$

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- sets of structures $\mathcal{A}$: $\exp(A(z))$

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- Let $I(z) = \sum_n \frac{I_n}{n!} z^n$. We will be interested in an asymptotic equivalent of the coefficients of $I(z)$

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Use a theorem of Bender (with $F(z, y) = \log(1 + y)$)

Let $F(z, y)$ is a real function, analytic at $(0, 0)$. Let $J(z) = \sum_{n>0} j_n z^n$, $C(z) = \sum_{n>0} c_n z^n$ and $D(z) = \sum_{n>0} d_n z^n$ with $C(z) = F(z, J(z))$ and $D(z) = \frac{\partial F}{\partial y}(z, J(z))$. If $j_{n-1} = o(j_n)$ and there exists such that $\sum_{k=s}^{n-s} |j_k j_{n-k}| = \mathcal{O}(j_{n-s})$, then $c_n = \sum_{k=0}^{s-1} d_k j_{n-k} + \mathcal{O}(j_{n-s})$.

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- Then… dive into complex analysis!
- $\mathbb{P}(\mathrm{connected}_n) = 1 - \frac{2^r}{n^{r-1}} + o(\frac{1}{n^{r-1}})$
- Generically, every r-tuple of partial injections is connected

Number of sequences, admissibility 1/2

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- So: study the random variable sequence$_n$, which counts the number of sequences in a partial injection: use an analogous calculus for bivariate EGSs, to study $I(z,u) = \sum_{n,k} \frac{I_{n,k}}{n!} z^n u^k$, where $I_{n,k}$ is the number of partial injections of size n with k sequences

Number of sequences, admissibility 2/2

- $$I(z, u) = \frac{1}{1-z} \exp\left(\frac{zu}{1-z}\right)$$

Number of sequences, admissibility 2/2

- $$I(z,u) = \frac{1}{1-z}\exp(\frac{zu}{1-z})$$
- More complex analysis (and more complicated!) shows that $\mathbb{E}(\text{sequence}_n) = \sqrt{n} + o(\sqrt{n})$, with standard deviation $o(\sqrt{n})$

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- Requires a pre-computation phase, to compute the I_k $(k \leq n)$.
- We have $I(z) = \sum_n \frac{I_n}{n!} z^n = \frac{1}{1-z} \exp(\frac{z}{1-z})$ and $I'(z) = \sum_n \frac{I_{n+1}}{n!} z^n$, we find that

$$(1-z)^2 I'(z) = (2-z)I(z) \text{ and}$$

$$I_n = 2nI_{n-1} - (n-1)^2 I_{n-2} \text{ with } I_0 = 1 \text{ and } I_1 = 2.$$

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- With probablility tending to e^{-r}, H contains a conjugate of a letter.
- H is *minimal* if for every automorphism φ of $F(A)$, $\varphi(H)$ is not smaller than H (in terms of the number of vertices of its Stallings graph). This is a generic property

Thank you for your attention!