RANDOM PLANAR GRAPHS

MathStic day combinatorics and probability

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DEFINITION OF PLANAR GRAPHS

The left and center diagrams are labeled with red 'X's, indicating that they are not planar graphs. The right diagram is marked with a green checkmark, indicating that it is a planar graph.
Question: How many planar graphs with \( n \) vertices are there?

- Giménez, Noy (2009): The number \( g_n \) of planar graphs with vertices labelled from 1 to \( n \) satisfies
  \[
  g_n \sim g n^{-7/2} \rho_G^{-n} n!
  \]
  for constants \( g, \rho_G > 0 \).

- The asymptotic behaviour of the number \( \tilde{g}_n \) of unlabelled planar graphs is unknown.
NUMBER OF LABELLED PLANAR GRAPHS

• Denise, Vasconcellos, Welsh (1996): $g_n \leq n!(75.8)^{n+\omega(n)}$, $(g_n/n!)^{1/n}$ converges

• Bender, Gao, Wormald (2002): $g_n \geq n!(26.1)^{n+\omega(n)}$, $b_n \sim b n^{-7/2} \rho_B^{-n} n!$

• Osthus, Prömel, Taraz (2003): $g_n \leq n!(37.3)^{n+\omega(n)}$

• (Further estimates of growth constants... sorry for omitting those)

• Giménez, Noy (2009): $g_n \sim gn^{-7/2} \rho_G^{-n} n!$ by analytic integration


• Stufler (2019+): recover $g_n \sim gn^{-7/2} \rho_G^{-n} n!$ without integration, random walk approach, uses large deviation results by Denisov, Dieker, Shneer (2008)
Question: What are the properties of a uniform random planar graph $\mathcal{P}_n$ with $n$ labelled vertices?
Fastest known sampling algorithm was invented and implemented by Fusy (2008). It generates planar graphs...

- with size in $[n(1 - \epsilon), n(1 + \epsilon)]$ in expected time $O(n)$.
- with size $n$ in expected time $O(n^2)$.

Uniform planar graph with roughly 10k vertices
GIANT CONNECTED COMPONENT

• McDiarmid (2008): Giant connected component, remainder admits a finite Boltzmann-Poisson Random Graph as limit

• McDiarmid (2009): Universality: in general, random graphs from proper addable minor-closed classes of graphs have a remainder with a Boltzmann-Poisson Random Graph as limit.

• Stufler (2018): Universality: Small block-stable classes of graphs. (If such a class fails to be small, the random graph is connected with high probability. See for example Stufler (2020).)
MAXIMUM DEGREE

• McDiarmid and Reed (2008): The maximum degree $\Delta_n$ satisfies whp $c_1 \log n < \Delta_n < c_2 \log n$ for suitable constants $0 < c_1 < c_2$.

• Drmota, Giménez, Noy, Panagiotou, Steger (2012): whp $|\Delta_n - c \log n| = O(\log \log n)$ for a constant $c > 0$. 
**DEGREE DISTRIBUTION**

- McDiarmid, Steger, Welsh (2004): Number $d_k(n)$ of vertices of a degree $k$ is $\Theta(n)$

- Drmota, Giménez, Noy (2011): Degree of a random vertex has a limit distribution

- Panagiotou, Steger (2011): Recovered degree distribution via different methods

- Stufler (2019+): Degree of a random vertex converges to the degree of the root of a new Uniform Infinite Planar Graph (UIPG)
LOCAL DISTANCE

\( \mathcal{M} = \text{collection of vertex-rooted locally finite unlabelled graphs} \)

\( p_k : \mathcal{M} \to \mathcal{M} \) projection to \( k \)-neighbourhood of the root vertex

\[
d_{\text{loc}}(G, H) = \frac{1}{1 + \sup\{k \in \mathbb{N}_0 | p_k(G) = p_k(H)\}}
\]

\((\mathcal{M}, d_{\text{loc}})\) is a Polish space
LOCAL CONVERGENCE: UIPG

Annealed Version (Stufler 2019+):

The uniform $n$-vertex planar graph $\mathcal{P}_n$ rooted at a uniformly selected vertex $v_n$ admits a distributional limit $\hat{\mathcal{P}}$.

We call $\hat{\mathcal{P}}$ the Uniform Infinite Planar Graph (UIPG).
LOCAL CONVERGENCE: UIPG

Annealed Version (Stufler 2019+):

The uniform $n$-vertex planar graph $\mathcal{P}_n$ rooted at a uniformly selected vertex $v_n$ admits a distributional limit $\hat{\mathcal{P}}$.

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Quenched Version (Stufler 2019+):

The regular conditional law $\mathbb{L}((\mathcal{P}_n, v_n) | \mathcal{P}_n)$ satisfies

$$\mathbb{L}((\mathcal{P}_n, v_n) | \mathcal{P}_n) \overset{p}{\to} \mathbb{L}(\hat{\mathcal{P}}).$$
THE UIPG IS ALMOST SURELY RECURRENT

• (Benjamini and Schramm, 2001) Let $M < \infty$. If a random locally finite rooted graph $G$ is a distributional limit of rooted random unbiased finite planar graphs (not necessarily uniform) with degrees bounded by $M$, then with probability one $G$ is recurrent.

• (Gurel-Gurevich and Nachmias, 2013) Instead of a uniform bound on the degrees, it suffices to assume that degree of the root of $G$ has an exponential tail.

• Consequence: the UIPG is almost surely recurrent
NON-EXHAUSTIVE LIST OF MODELS WITH LOCAL LIMITS

• Kesten's tree: Simply generated trees (Kennedy, 1975)
• UIPT: Planar Triangulations (Angel, Schramm 2003)
• UIPQ: Planar Quadrangulations (Krikun 2005)
• UIPM: Planar Maps (Ménard, Nolin 2013)
• UI3PM: 3-connected Planar Maps (Addario-Berry 2014)
• IBPM: Boltzmann Maps (Björnberg, Stefánsson 2014, Stephenson 2018)
• PSHT: Triangulations with a high genus (Budzinsky, Louf 2020)
• UIPG, UI2PG, UI2PM: Planar Graphs (S. 2019+)
Planar graphs

Connected planar graphs

2-connected planar graphs (n vertices)

2-connected planar graphs (n edges)

Weighted blow-ups of 3-connected planar graphs/maps

4-type branching processes

Weighted planar maps

Weighted non-separable planar maps
LOCAL CONVERGENCE: UI2PM

Non-separable Maps (Stufler 2019+):

The uniform \( n \)-edge 2-connected (= non-separable) planar map \( \mathcal{N}_n \) rooted at a uniformly selected corner \( c_n \) admits a novel Uniform Infinite 2-connected Planar Map (UI2PM) \( \hat{\mathcal{N}} \) as quenched local limit

\[
\mathcal{L}((\mathcal{N}_n, c_n) \mid \mathcal{N}_n) \xrightarrow{p} \mathcal{L}(\hat{\mathcal{N}}).
\]
NEW NON-BIJECTIVE PROOF METHOD

"Two steps down, one step up"

Planar map $\mathcal{M}_n$

Non-separable planar map $\mathcal{N}_n$

Non-separable core $\mathcal{N}(\mathcal{M}_n)$

"2.9-connected"-core $\mathcal{R}(\mathcal{N}_n)$

"2.9-connected"-core $\mathcal{R}(\mathcal{N}(\mathcal{M}_n))$
NEW NON-BIJECTIVE PROOF METHOD

\[ \mathcal{M}_n \] consists of 2-con. core \( \mathcal{N}(\mathcal{M}_n) \) and components \( (\mathcal{M}_i(\mathcal{M}_n))_{1 \leq i \leq |\mathcal{N}(\mathcal{M}_n)|} \)

- For the purpose of proving local convergence, we may pretend that the components are i.i.d. copies of a Boltzmann map

- Waiting time paradox: the component containing a uniformly selected corner \( c_n \) follows a size-biased distribution
\( \mathcal{M}_n \)

\( N(\mathcal{M}_n) \)

\[ p_k(\mathcal{M}_n, c_n) = M \]
$\mathcal{N}(\mathcal{M}_n)$
NEW NON-BIJECTIVE PROOF METHOD

"Two steps down, one step up"

Non-separable planar map $N_n$

Planar map $M_n$

"2.9-connected"-core $R(N_n)$

Non-separable core $N(M_n)$

"2.9-connected"-core $R(N(M_n))$
NEW NON-BIJECTIVE PROOF METHOD

- $N(\mathcal{M}_n)$ consists of 2.9-con. core $\mathcal{R}(N(\mathcal{M}_n))$ and components that substitute its edges

- For the purpose of proving local convergence, we may pretend that the components are i.i.d. copies of a Boltzmann map

- Waiting time paradox: the component containing a uniformly selected corner $c_n$ follows a size-biased distribution
\( N(\mathcal{M}_n) \)

\[ \mathcal{R}(N(\mathcal{M}_n)) \]

\[ p_k(N(\mathcal{M}_n), c_n^\mathcal{N}) = M \]
Problem: $k$-neighbourhood of core structure $R(N(M_n))$ could have more edges than $k$-neighbourhood of $N(M_n)$.

Induction does not work for neighbourhoods.

Solution: Use communities instead of neighbourhoods.
NEW NON-BIJECTIVE PROOF METHOD

"Two steps down, one step up"

Non-separable planar map $\mathcal{N}_n$

"2.9-connected"-core $\mathcal{R}(\mathcal{N}_n)$

Planar map $\mathcal{M}_n$

Non-separable core $\mathcal{N}(\mathcal{M}_n)$

"2.9-connected"-core $\mathcal{R}(\mathcal{N}(\mathcal{M}_n))$
NEW NON-BIJECTIVE PROOF METHOD

- $\mathcal{R}(N(M_{3n}))$ and $\mathcal{R}(N_n)$ are distributed like mixtures $\mathcal{R}_X$ and $\mathcal{R}_Y$.

- There are $\mu, a, b > 0, h$ density of a $3/2$-stable law, such that uniformly for $\ell \in \mathbb{N}$
  $$\mathbb{P}(X_n = \ell) = \frac{1}{an^{2/3}} \left( h \left( \frac{\mu n - \ell}{an^{2/3}} \right) + o(1) \right)$$
  $$\mathbb{P}(Y_n = \ell) = \frac{1}{bn^{2/3}} \left( h \left( \frac{\mu n - \ell}{bn^{2/3}} \right) + o(1) \right)$$

- For any $\epsilon > 0$ there exists $M, c, C > 0$ such that $I_n := [\mu n - Mn^{2/3}, \mu n + Mn^{2/3}]$ satisfies for all large enough $n$
  $$\mathbb{P}(X_n \notin I_n), \mathbb{P}(Y_n \notin I_n) < \epsilon$$
  and uniformly for $\ell \in I_n$
  $$c < \frac{\mathbb{P}(X_n = \ell)}{\mathbb{P}(Y_n = \ell)} < C$$
NEW NON-BIJECTIVE PROOF METHOD

"Two steps down, one step up"

Planar map $\mathcal{M}_n$

Non-separable core $\mathcal{N}(\mathcal{M}_n)$

"2.9-connected"-core $\mathcal{R}(\mathcal{N}(\mathcal{M}_n))$

Non-separable planar map $\mathcal{N}_n$

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NEW NON-BIJECTIVE PROOF METHOD

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"2.9-connected"-core $\mathcal{R}(\mathcal{N}(\mathcal{M}_n))$

Non-separable planar map $\mathcal{N}_n$

"2.9-connected"-core $\mathcal{R}(\mathcal{N}_n)$
DIAMETER AND SCALING LIMITS

• (Chapuy, Fusy, Giménez, Noy) There exists a $c > 0$ such that the diameter $D(\mathcal{P}_n)$ satisfies for each small enough $\epsilon > 0$ and all $n > n_0(\epsilon)$
  $$\mathbb{P}(D(\mathcal{P}_n) \notin [n^{1/4-\epsilon}, n^{1/4+\epsilon}]) < \exp(-n^{c\epsilon}).$$

• **Open problem:** What happens when we rescale distances by $n^{-1/4}$?
SCALING LIMITS

500k vertex simply generated tree in the universality class of the Brownian continuum random tree. Colours correspond to the height of the vertex.

Simulation: GRANT (Generate RANdom Trees), available here: http://github.com/BenediktStufler/grant
Uniform labelled tree with 1M vertices. Colours correspond to closeness centrality of the vertex.
Thm. (Aldous, 1991) The uniform labelled tree $T_n$ with its graph distance $d_{T_n}$ and the uniform measure $\mu_n$ on its vertices satisfies

\[
\left( T_n, \frac{1}{2\sqrt{n}} d_{T_n}, \mu_{T_n} \right) \rightarrow (T, d_T, \mu_T)
\]

for a limiting random measured metric space $(T, d_T, \mu_T)$. 

SCALING LIMITS
SCALING LIMITS

Thm. (Chassaing and Schaeffer, 2004) The height $H(Q_n)$ of a uniform random quadrangulation with $n$ faces admits the width $r$ of Aldous’ one-dimensional ISE as scaling limit:

$$(8n/9)^{-1/4}H(Q_n) \to r$$
SCALING LIMITS

Thm. (Chassaing and Schaeffer, 2004) The height $H(Q_n)$ of a uniform random quadrangulation with $n$ faces admits the width $r$ of Aldous' one-dimensional ISE as scaling limit:

$$(8n/9)^{-1/4}H(Q_n) \to r$$

Miermont (2013), Le Gall (2013): GHP scaling limit called the Brownian map $(M, d_M, \mu_M)$:

$$(Q_n, (8n/9)^{-1/4}d_{Q_n}, \mu_{Q_n}) \to (M, d_M, \mu_M)$$
Uniform simple triangulation of the sphere with 1M faces

UNIFORM SPANNING TREE
Question: What are the properties of a uniform random spanning tree of a uniform random planar graph $\mathcal{P}_n$ with $n$ labelled vertices?
Uniform spanning tree of a uniform planar map with 1M edges
UNIFORM SPANNING TREE

Histogram for the height of the UST of a uniform random planar map with $n = 10000$ edges
UNIFORM SPANNING TREE

Histogram for the height of the UST of a uniform random planar map with $n = 100000$ edges
UNIFORM SPANNING TREE

Histogram for the height of the UST of a uniform random planar map with $n = 500000$ edges
Histogram for the height of the UST of a uniform random planar map with $n = 1000000$ edges.
**UNIFORM SPANNING TREE**

\( h(n) \): average height of simulations of UST of uniform planar map with \( n \) edges.

\[ \alpha(n) = \log\left(\frac{h(10n)}{h(n)}\right)/\log n \]

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<th>( 10^4 )</th>
<th>( 10^5 )</th>
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<td>0.459914</td>
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Non-rigorous Knizhnik-Polyakov-Zamolodchikov (KPZ) formula predicts:

\[ \alpha = \frac{5 - \sqrt{10}}{4} = 0.4594305... \]

Many thanks to Nathanaël Berestycki for explaining this to me.

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Thanks for your attention.