Distribution of parameters in certain fragments of the linear and planar $\lambda$-calculus

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What is the \( \lambda \)-calculus?

- A \textit{universal} formal system for expressing computation.
- Its terms are formed using the following grammar:
  - A variable is a valid term.
  - If \( x \) a variable and \( t \) is a valid term, then so is \( (\lambda x.t) \).
  - If \( s \) and \( t \) are valid terms, then so is \( (s \, t) \).
- The \( \lambda \) calculus also provides us with tools to transform terms, including the operation of \( \beta \)-reduction:

\[
((\lambda x.t) \, s) \xrightarrow{\beta} t[x := s]
\]

Some examples of terms:

\[
(\lambda x.(xx))(\lambda x.(xx))
\]
\[
\lambda x.\lambda y.(x \, (x \, y))
\]
\[
\lambda x.(z \, (\lambda y.y))
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• General terms are quite complicated. Growth is super-exponential, generating functions are not analytic. \(^1\) Asymptotic number of general terms still (?) unresolved!

• We focus on *linear* terms: bound variables must appear exactly once: \(\lambda x. (x \ x), \ \lambda x. \lambda y. (a \ (y \ x))\).

• We also consider *planar* terms: bound variables must appear in the order they are introduced: \(\lambda x. \lambda y. (y \ x), \ \lambda x. \lambda y. (a \ (x \ y))\).

\(^1\)For the notion of term size given recursively by:

\(|\text{var}| = 1, |(s \ t)| = |s| + |t| + 1, |\lambda x. t| = |t| + 1.|
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The $\lambda$-calculus and maps

- Maps: graphs embedded in an oriented surface without boundary.
- Closed linear terms are combinatorially intriguing: they correspond to rooted connected trivalent maps! [1, 2] Closed planar terms correspond to planar such maps. Open terms allow for univalent vertices too.
The $\lambda$-calculus and maps

- Maps: graphs embedded in an oriented surface without boundary.
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  Closed planar terms correspond to planar such maps. Open terms allow for univalent vertices too.
An example of a term and its corresponding map

$$\lambda x.\lambda y. y((\lambda z. z)\ x)$$

Where $\lambda$ annotates abstractions and $@$ applications.
Purpose of this work

- How do “typical” (random, of large size) linear and planar terms behave?
- How many free variables do they have? How often is a typical term an abstraction?
- Using tools from analytic combinatorics to obtain parameter distributions.
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In this talk

We’ll sketch the following results:

**Linear $\lambda$-Terms**
(Differentially Algebraic, Divergent)

- Limit distribution of free variables
- Limit distribution of identity-subterms in closed terms.
- Limit distribution of closed subterms in closed terms.
- Probability that term is an abstraction.

**Planar $\lambda$-Terms**
(Algebraic, Analytic)

- Limit distribution of free variables for regular and bridgless terms.
- Probability that regular or bridgless open term is an abstraction.
Free variables in closed linear terms

- Free variables are those not bound by an abstraction. For example: $\lambda x.(a \ x)$

**Proposition**

The limit distribution of free variables in linear $\lambda$-terms of size $n$ is Gaussian with mean and variance $\mu = \sigma^2 \sim \sqrt[3]{n}$.

Starting point (follows from definition of combinatorial maps):

$$L(z^2, u) = uz^2 + z^4 + z^5 \frac{\partial}{\partial z} \left( \ln \left( \exp(z^2/2) \odot \exp(z^3/3 + uz) \right) \right)$$

where $L$ counts open linear $\lambda$-terms with $u$ tagging free variables.
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Free variables in closed linear terms

Proof Sketch

Saddle-point analysis of Hadamard product yields:

\[
[z^n] \exp \left( \frac{z^3}{3} + uz \right) = \left( \frac{1}{6} \frac{\sqrt{2} \sqrt{3} n^{-\frac{1}{2}}}{\sqrt{\pi}} - \frac{1}{36} \frac{\sqrt{2} \sqrt{3} u^2 n^{-\frac{5}{6}}}{\sqrt{\pi}} + O \left( n^{-\frac{7}{6}} \right) \right) e^{un^{1/3} + n/3 n - n/3}
\]

\[
[z^n] \exp \left( \frac{z^2}{2} \right) \sim \frac{1}{2} \frac{e^{1/2+n/2}}{(\sqrt{1+n})^{1+n} \sqrt{\pi}} - \frac{1}{2} \frac{e^{1/2+n/2}}{(-\sqrt{1+n})^{1+n} \sqrt{\pi}}
\]

While an application of Bender’s theorem [3, Theorem 1] gives

\[
2[z^n] \frac{d}{dz} \ln(A(z^{1/2}, u)) = n \left( [z^n] h(z, u) - \frac{1}{2} [z^{n-2}] h(z, u) \right) + O \left( [z^{n-4}] h(z, u) \right)
\]

for \( A(x, u) = \exp(z^2/2) \odot \exp(z^3/3 + uz) \)
• Identity terms: terms which are $\alpha$-equivalent to $\lambda x. x$. For example: $\lambda x. (x \ (\lambda y. y))$.

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• They appear as loops in the corresponding map.

\[(\lambda x.x)(\lambda y.y(\lambda z.z(\lambda w.w)))\]
The limit distribution of identity-subterms in closed linear \( \lambda \)-terms is Poisson of parameter \( \lambda = 1 \).

Proof Sketch: Use moment pumping on

\[
G = (u - 1)z^2 + zG^2 + \frac{\partial}{\partial u} G
\]

where \( G \) counts closed linear terms with \( u \) tagging identity-subterms.
Distribution of identity-subterms in closed linear terms

Justification for

$$G = (u - 1)z^2 + zG^2 + \frac{\partial}{\partial u}G$$

Terms are either identity-terms, applications, or
For the pumping, note that the $k$-th derivative of the eq. may be written as

$$\frac{\partial^k}{\partial u^k} G - S - 2z G \frac{\partial^k}{\partial u^k} G = \frac{\partial^{k+1}}{\partial u^{k+1}} G$$

with $S$, depending on the parity of $k$, being as follows

$$\sum_{l=1}^{\lfloor k/2 \rfloor} 2z \binom{k}{l} \frac{\partial^l}{\partial u^l} G \frac{\partial^{k-l}}{\partial u^{k-l}} G$$, for odd $k$

$$\sum_{l=1}^{\lfloor k/2 \rfloor - 1} 2z \binom{k}{l} \frac{\partial^l}{\partial u^l} G \frac{\partial^{k-l}}{\partial u^{k-l}} G + z \binom{k}{\lfloor k/2 \rfloor} \left( \frac{\partial^{\lfloor k/2 \rfloor}}{\partial u^{\lfloor k/2 \rfloor}} G \right)^2$$, for even $k$
Distribution of closed subterms in closed linear terms

- Closed subterms: subterms having no free variables.
- Corresponding to non-root-containing connected components resulting from the deletion of some bridge in the respective map.
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\[ \lambda x.x(\lambda y.y(\lambda z.z)) \]
**Proposition**

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**Proof Sketch:** Use moment pumping on

\[
\frac{\partial W}{\partial v} = \frac{-(zv^2W^2 + z^2 - W)W}{zv^2(v - 1)W^2 + (1 - v)W + vz^2}
\]

where \( W \) counts closed linear terms with \( v \) tagging identity-subterms.
Probability that a closed linear term is an abstraction

**Proposition**

Asymptotically almost surely a random closed linear λ-term is an abstraction.

**Proof Sketch:**

It can be shown that $[z^n]L_c \sim k \cdot 6^n \cdot n!$ for some constant $k$. Compare the coefficients of $L_c$ and $2z^4 \frac{\partial}{\partial z} L_c$ in

$$L_c = z^2 + zL_c^2 + 2z^4 \frac{\partial}{\partial z} L_c.$$

where $L_c$ enumerates closed linear λ-terms.
Distribution of identity-subterms in closed linear terms

Justification for

\[ G = L_c = z^2 + zL_c^2 + 2z^4 \frac{\partial}{\partial z} L_c. \]

Terms are either identity-terms, applications, or
Distribution of free variables in planar and bridgeless planar terms

**Proposition**

The limit distribution of free variables in planar $\lambda$-terms of size $n$ is Gaussian with mean $\mu = \frac{n}{8}$ and variance $\sigma^2 = \frac{9n}{32}$.

**Proposition**

The limit distribution of free variables in bridgeless planar $\lambda$-terms of size $n$ is Gaussian with mean $\mu = \frac{n}{5}$ and variance $\sigma^2 = \frac{9n}{25}$.
Both results follow similar steps.

Our starting points are the following two equations

\[
P(z, u) = uz + zQ(z, u)^2 + \frac{z(P(z, u) - P(z, 0))}{u}
\]

\[
Q(z, u) = uz + zQ(z, u)^2 + \frac{z(Q(z, u) - u[u^1]Q(z, u))}{u}
\]

with \( P \) and \( Q \) counting planar and bridgeless planar terms respectively and \( u \) tagging free variables.

Sketch: use elimination and the quadratic method to obtain closed form solutions. Proceed by applying, [4, Proposition IX.17].
Distribution of free variables in planar and bridgeless planar terms

\[ Q(z, u) = \frac{1}{2} z^{-1} - \frac{1}{2} u^{-1} \]

\[ + \frac{1}{2} \frac{1}{u z} \left( \frac{1}{3} u^2 \frac{3}{3} \sqrt{-1458 z^6 + 6 \sqrt{19683} z^8 - 4374 z^5 + 324 z^2} - 8 z^{-1} z^2 - 270 z^3 + 1 \right) \]

\[ + 36 u^2 z^3 \frac{1}{\frac{3}{3} \sqrt{-1458 z^6 + 6 \sqrt{19683} z^8 - 4374 z^5 + 324 z^2} - 8 z^{-1} z^2 - 270 z^3 + 1} \]

\[ + \frac{1}{3} \frac{1}{u^2} \frac{1}{\frac{3}{3} \sqrt{-1458 z^6 + 6 \sqrt{19683} z^8 - 4374 z^5 + 324 z^2} - 8 z^{-1} z^2 - 270 z^3 + 1} \]

\[ + \frac{1}{3} \left( u^2 - 4 u^3 z^2 - 2 u z + z^2 \right)^{1/2} . \]
Distribution of free variables in planar and bridgeless planar terms

While $P(z) = A(z, u) + B(z, u) \cdot C(z, u)^{-1/2}$ with

$$A(z, u) = \frac{1}{2z} - \frac{1}{2u}, \quad B(z, u) = \frac{1}{2uz}$$

$$C(z, u) = -4u^3z^2$$

$$+ \frac{1}{48} \frac{u^3 \sqrt[3]{1492992z^{12} + 8640z^6 + 96}\sqrt[3]{80621568z^{18} - 559872z^{12} + 1296z^6 - 1z^3 - 1}}{z^2}$$

$$+ 72 \frac{u \sqrt[3]{1492992z^{12} + 8640z^6 + 96}\sqrt[3]{80621568z^{18} - 559872z^{12} + 1296z^6 - 1z^3 - 1}}{uz^4}$$

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$$- \frac{1}{48} \frac{u}{z^2} + u^2 + z^2$$
Probability that an open planar or bridgeless planar term is an abstraction

Proposition

Asymptotically, the probability that an random open planar (bridgeless planar) term is an abstraction is $\rho_P = \frac{\sqrt{2}}{4}$ ($\rho_{PB} = \frac{2}{5}$).

Proof Sketch: Estimate

$$\frac{[z^n] z(P(z, 1) - P(z, 0))}{[z^n] P(z, 1)} \quad \text{and} \quad \frac{[z^n] z(Q(z, 1) - ([u^1]Q(z, u))|_{u=1})}{[z^n] Q(z, 1)}$$

Both $P(z, 0)$ and $[u^1]Q(z, u)|_{u=1}$ are analytic at the respective singularities $\rho_P$ and $\rho_{PB}$ of $P$ and $Q$. Use the singular expansions of $P, Q$ at the corresponding singularities to obtain the desired result.
Conclusions

• Clear distinctions between the divergent/differentially-algebraic case of linear terms and the algebraic one of planar terms.

• Need for: more tools to handle divergent combinatorial classes, algebraicity results for closed planar terms.

• Future directions: study of $\beta$-reduction, typing of linear terms.

Thank you!
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