Combinatoire des jeux, des mots et numération

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http://www.discmath.ulg.ac.be/

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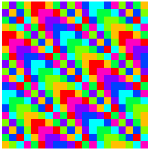
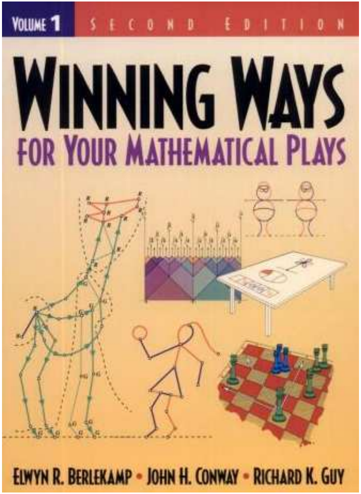
UNIVERSITÉ de Liège



Plan de l'exposé

1. Concepts classiques :
   - définition d'un jeu combinatoire,
   - quelques exemples,
   - graphe et noyau d'un jeu,
   - position perdante/gagnante,
   - Nim-somme,
   - fonction de Sprague-Grundy et somme de jeux
2. Jeu de Wythoff et liens avec la combinatoire des mots
3. Travaux récents et questions

E. R. Berlekamp, J. H. Conway, R. K. Guy, Winning Ways for Your Mathematical Plays, vol. 1–4, A K Peters, Ltd (2001).



Qu'est qu'un jeu combinatoire

- There are just **two players**.
- There are several, usually finitely many, positions, and often a particular starting position.
- There are **clearly defined rules** that specify the moves that either player can make from a given position (options).
- The two players play **alternatively**.
- Both players know what is going on (**complete information**).
- There are **no chance moves**.
- In the **normal play convention** a player unable to move loses.
- The rules are such that play will always come to an end because some player will be unable to move (**ending condition**).

WE CONSIDER THE EASIEST FRAMEWORK:

Impartial (vs. partizan) and acyclic (vs. cyclic) games: the allowable moves depend only on the position and not on which of the two players is currently moving.

QUELQUES EXEMPLES

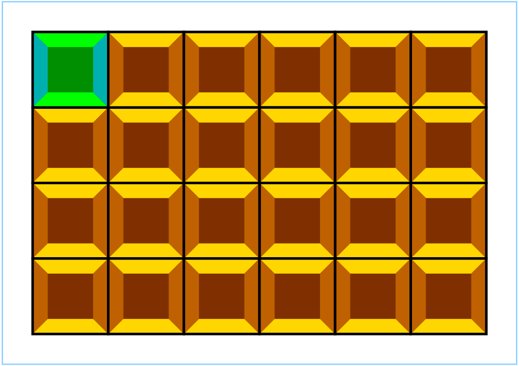
- CHOMP !
  → graphe de jeu, position perdante/gagnante

  D. Gale, A Curious Nim-Type Game, *American Math. Monthly* **81** (8), 1974, 876–879.

  F. Schuh, The game of divisions, *Nieuw Tijdschrift voor Wiskunde* **39** (1952), 299–304.
- NIM
  → Nim-somme, fonction de Sprague–Grundy

CHOMP

Chomp ou le jeu de la tablette de chocolat empoisonnée
(F. Schuh 1952, D. Gale 1974)



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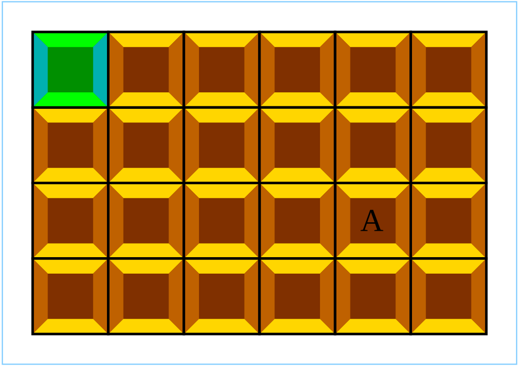

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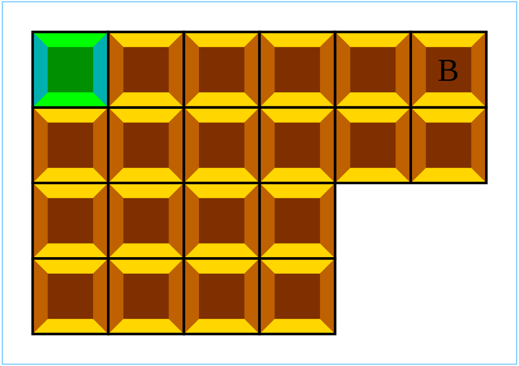

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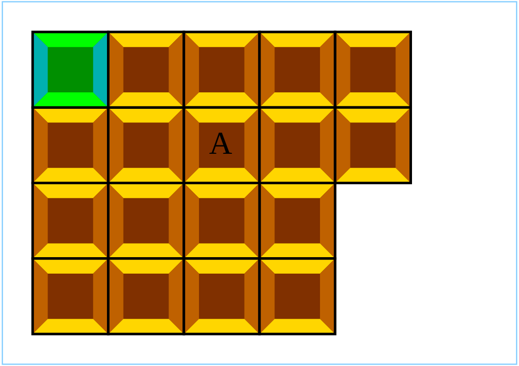

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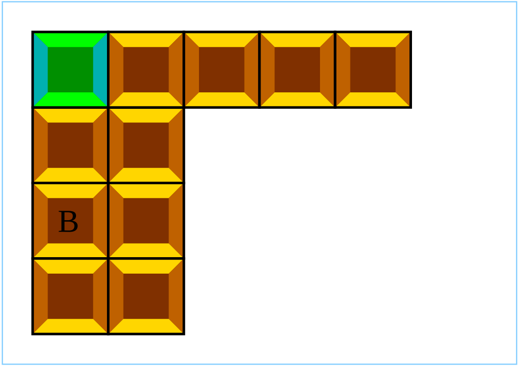

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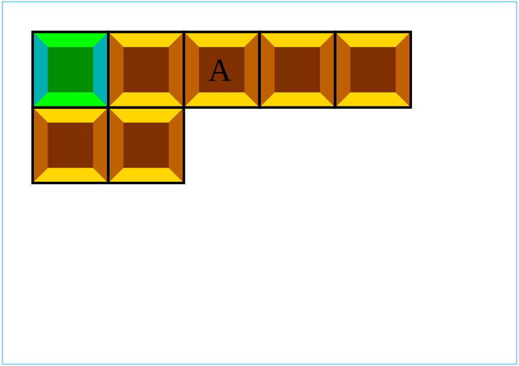

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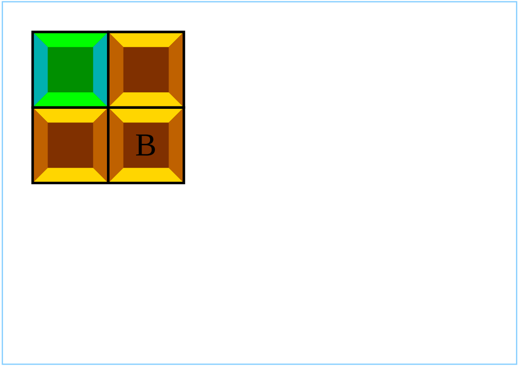

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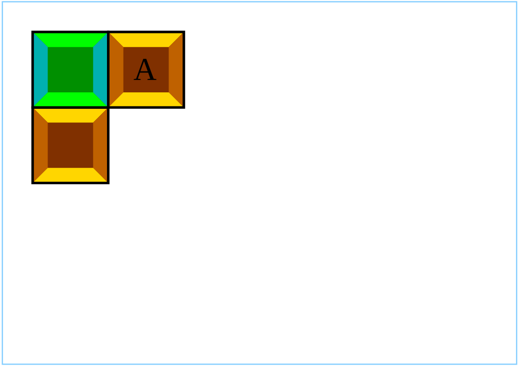

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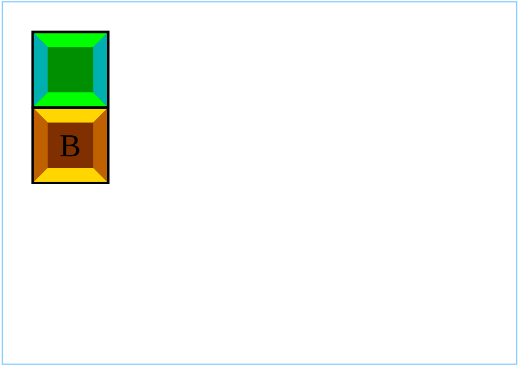

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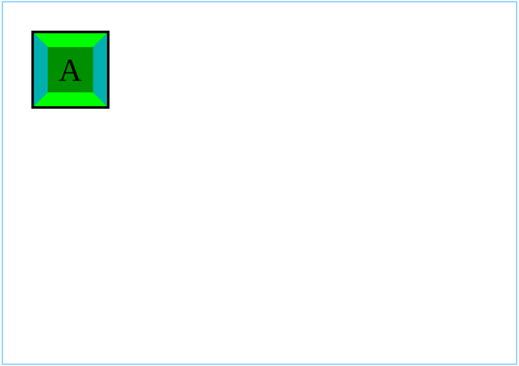
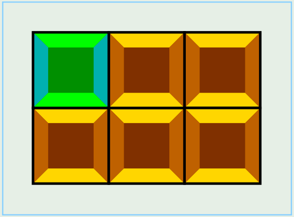


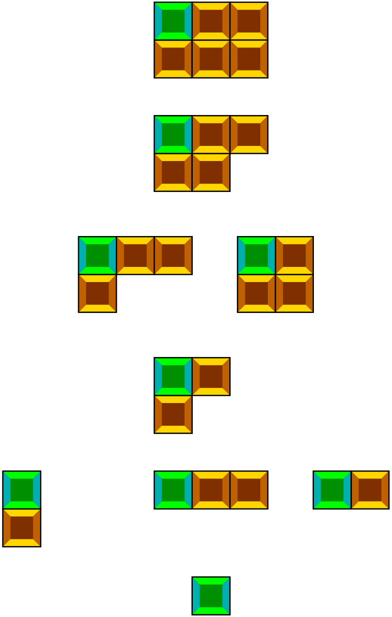
CHOMP

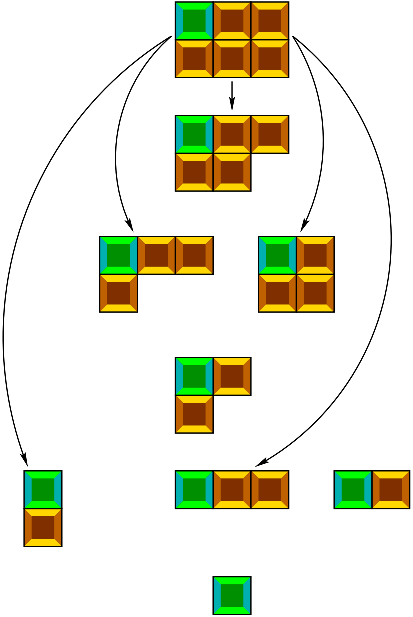
QUESTIONS

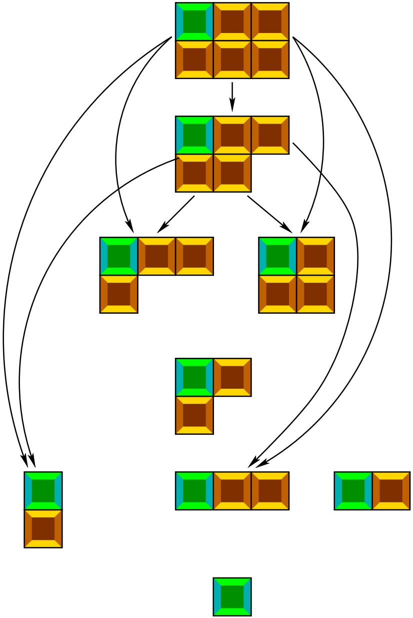
- Qui peut gagner la partie à partir d'une tablette $m \times n$?
- Quel coup doit-on jouer pour gagner ?
- Complexité sous-jacente ?

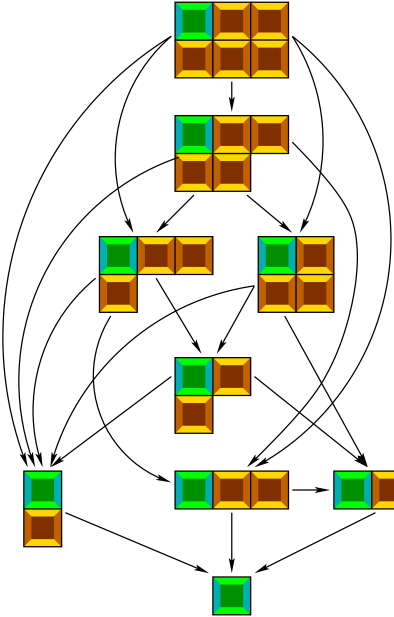
ANALYSE DU CAS 2×3

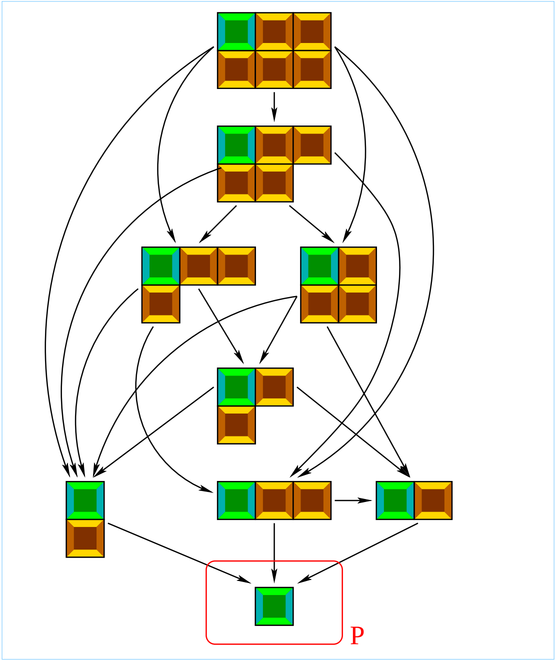








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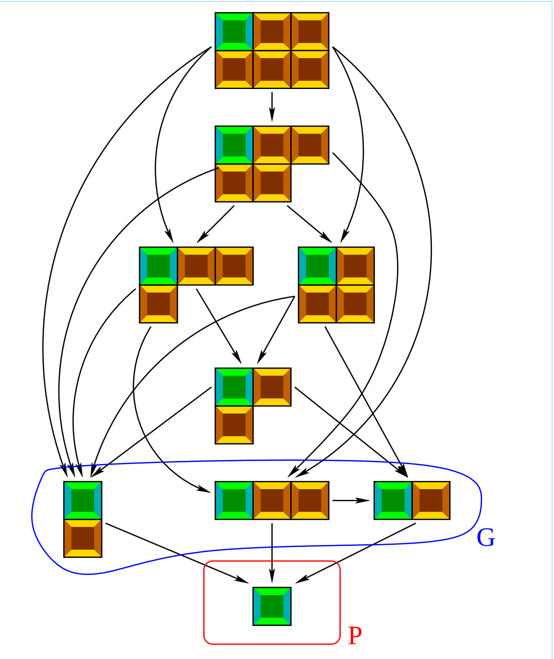
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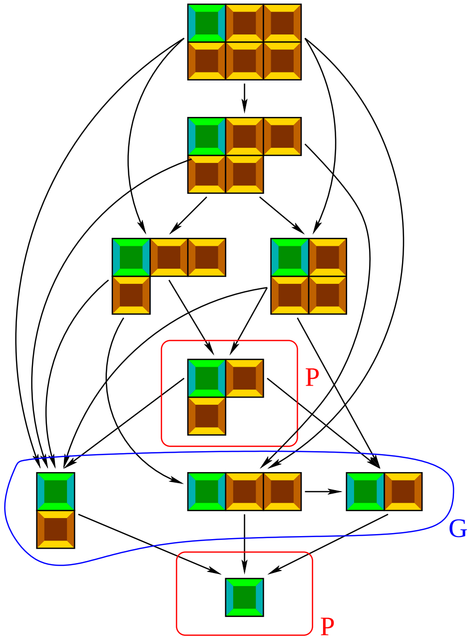
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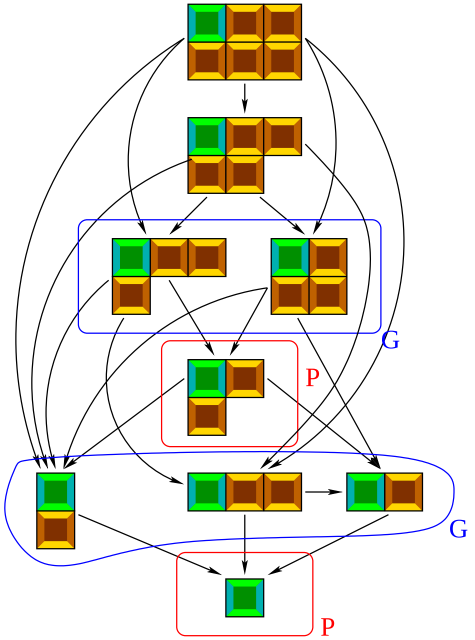
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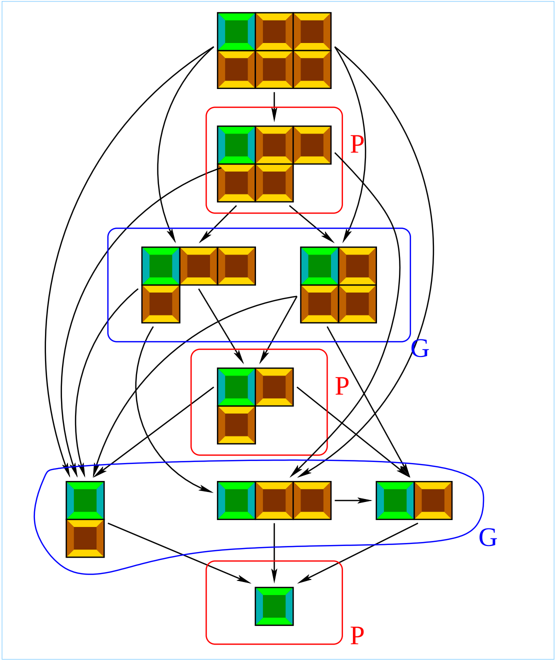
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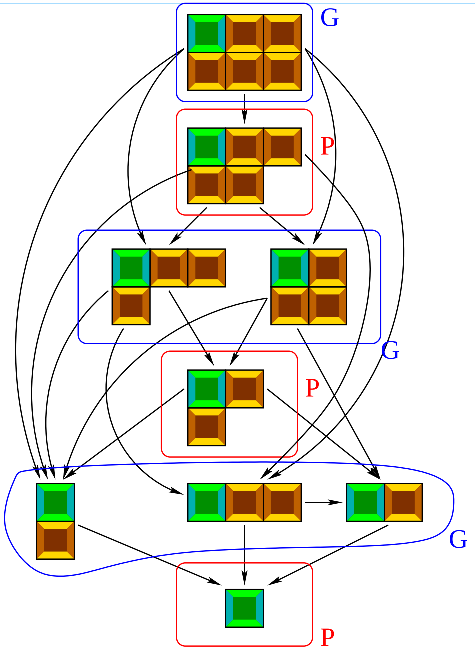
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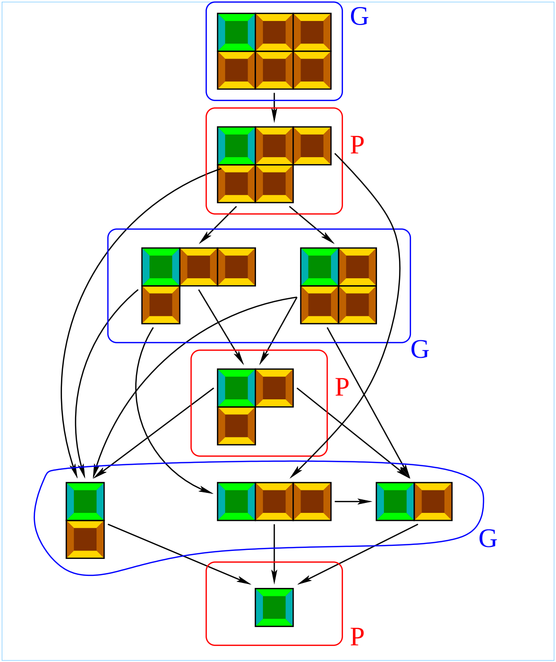
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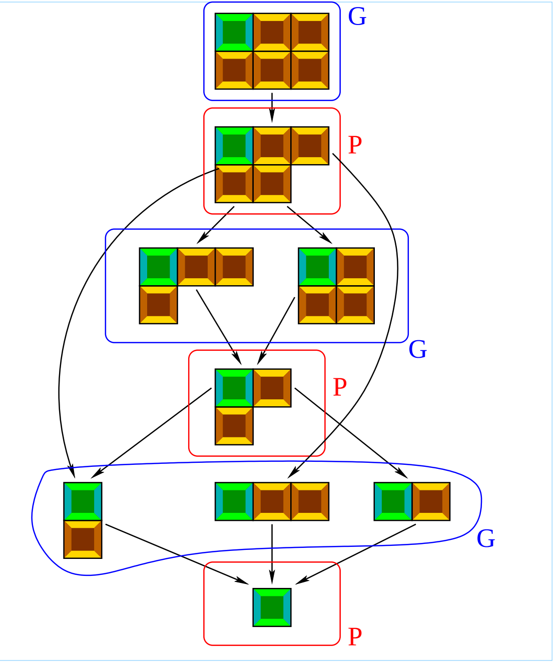
G
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G
P
G
P

- P = {positions perdantes},
  quoi que le joueur fasse, l'autre joueur *peut* gagner.
- G = {positions gagnantes},
  le joueur peut gagner, *quoi que fasse* son adversaire.
- Stratégie gagnante : choisir une bonne option depuis une position gagnante pour assurer *in fine* le gain.

REMARQUE

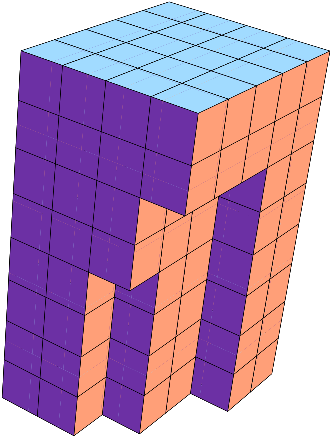
Pour une tablette $m \times n$, toute position est gagnante ou perdante.

THÉORÈME (EXISTENTIEL)

A partir d'une tablette $m \times n$, il existe toujours une stratégie gagnante pour le joueur qui débute.

Remarque

- Stratégie (facile) connue pour $2 \times m$
- Stratégie (facile) connue pour $m \times m$
- Généralisations...



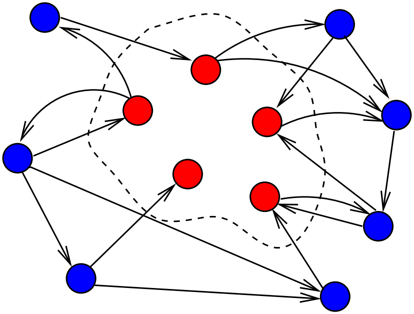
TRANSLATION IN GRAPH THEORETICAL TERMS (C. BERGE)

A kernel is a stable and absorbing subset of vertices. If G is an acyclic (simply connected) digraph then G has a unique kernel.

L'ensemble des positions perdantes

=

noyau du graphe de jeu



Stratégie : toujours "jouer vers le noyau"

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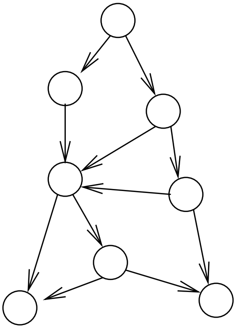
Consider the *game graph* where the vertices represent positions of the game.



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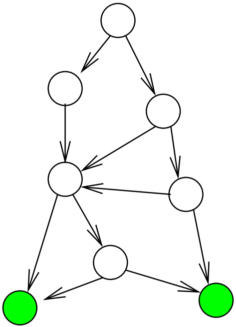

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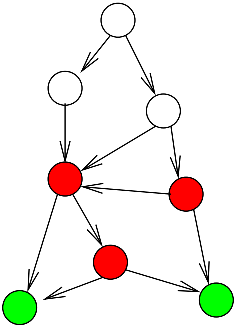

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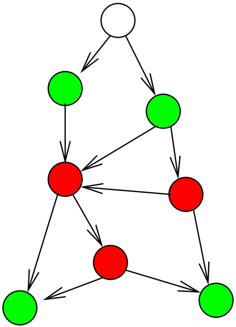

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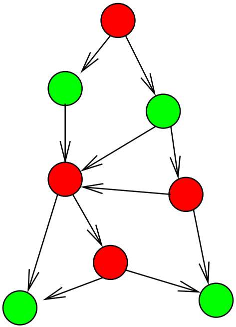
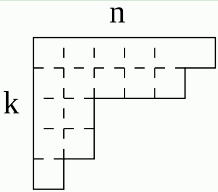


Tout va pour le mieux dans le meilleur des mondes...
Sauf que le graphe croît trop vite !

LE CAS $k \times n$ POUR CHOMP !



$(x_1, \ldots, x_k)$

avec $x_1 \geq x_2 \geq \cdots \geq x_k \geq 0$ et $1 \leq x_1 \leq n$

nombre de positions : $\sum_{x_1=1}^{n} \sum_{x_2=0}^{x_1} \cdots \sum_{x_k=0}^{x_{k-1}} 1 \sim n^k / k!$

nombre de coups : $\sum_{x_1=1}^{n} \sum_{x_2=0}^{x_1} \cdots \sum_{x_k=0}^{x_{k-1}} (x_1 + x_2 + \cdots + x_k - 1) \sim n^{k+1} / 2\,k!$

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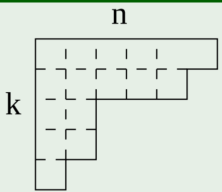

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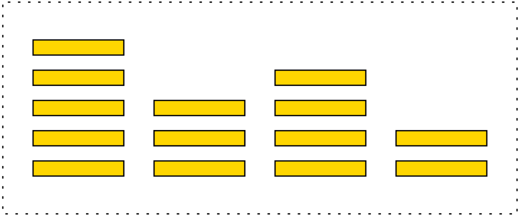
- Two players
- k piles of tokens, $n_1, \ldots, n_k > 0$
- Players play alternatively and
  remove any number of tokens from *one* pile
- The one who takes the last token wins the game.



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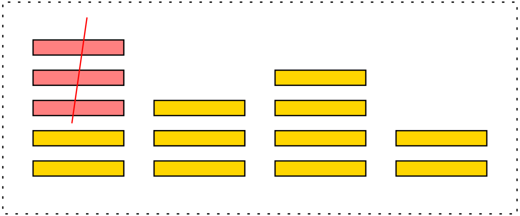

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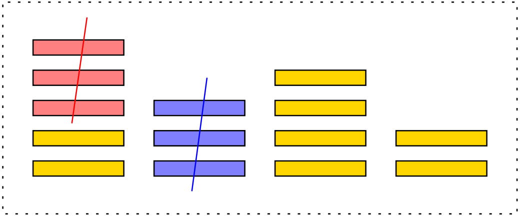

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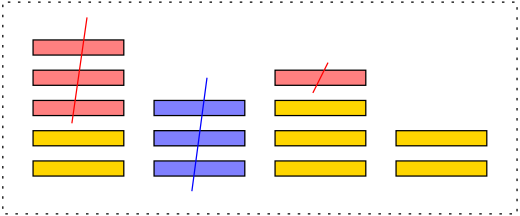


Strategy for the game of Nim, Bouton 1905

A player wins iff the addition in base 2 without carry is zero,

$$\bigoplus_{i=1}^{k} n_i = 0$$

Winner is known since the beginning of the game !

5	101
3	11
4	100
2	10
	000

- Les positions perdantes sont à “Nim-somme” nulle : tout coup joué depuis une position à “Nim-somme” nulle amène dans une position à “Nim-somme” nulle → non nulle.

5	101		2	10		2	10
3	11		3	11		3	11
4	100	$\longrightarrow$	4	100	$\longrightarrow$	4	11
2	10		2	10		2	10
	000			111			000

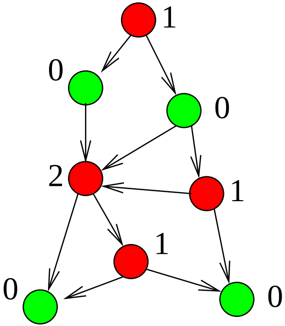
- Pour toute position à “Nim-somme” non nulle, il existe un coup vers une position de “Nim-somme” nulle (stratégie).

Fonction de Sprague-Grundy

Définition

La *fonction de Sprague–Grundy* d'un graphe orienté G (sans cycle) est définie par

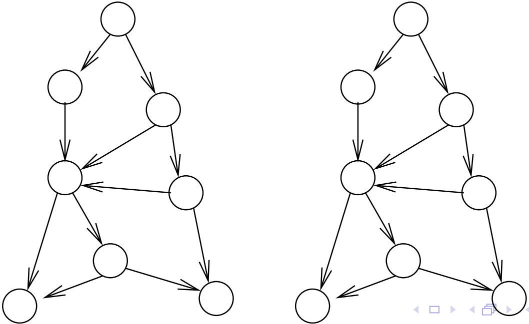
$$\forall v \in V, \quad g(v) = \mathrm{mex}\{g(y) \mid y \in \mathrm{Succ}(v)\}.$$



Somme de jeux

On peut définir la *somme de jeux* $J_1, \ldots, J_n$

- Si le jeu J_i a pour graphe $G_i = (V_i, E_i)$, le jeu $J_1 + \cdots + J_n$ a un graphe ayant $V_1 \times \cdots \times V_n$ comme ensemble de sommets.
- Un déplacement dans $J_1 + \cdots + J_n$ revient à jouer sur une seule des composantes.



Somme de jeux

Théorème

Si g_i est la fonction de S.-G. de J_i,
alors $J_1 + \cdots + J_n$ a pour fonction de S.-G.

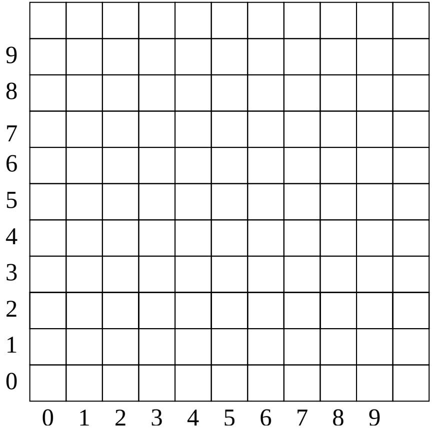
$$g(x_1, \ldots, x_n) = g_1(x_1) \oplus \cdots \oplus g_n(x_n).$$

e.g., Thomas S. Ferguson, p.22

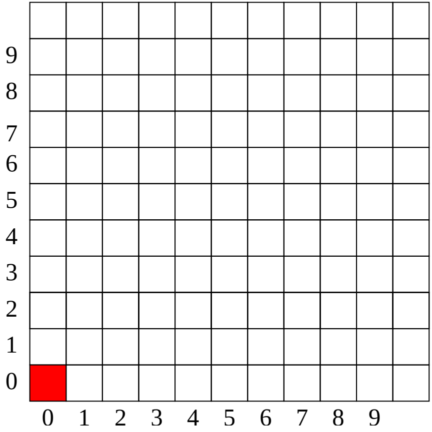
Exemple

Le jeu de Nim sur un seul tas $\rightarrow$ sur k tas.

Le jeu de Wythoff



Le jeu de Wythoff



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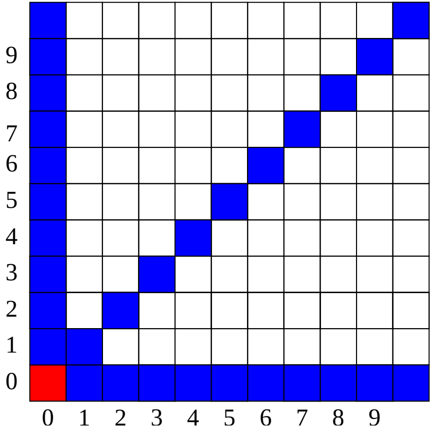

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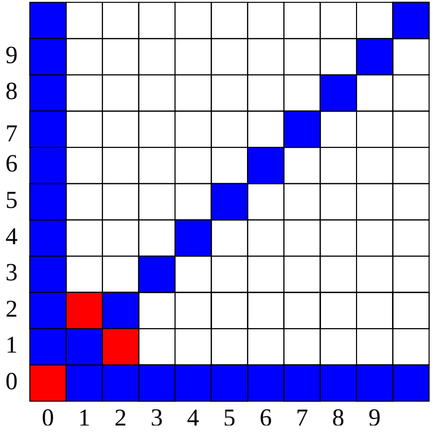

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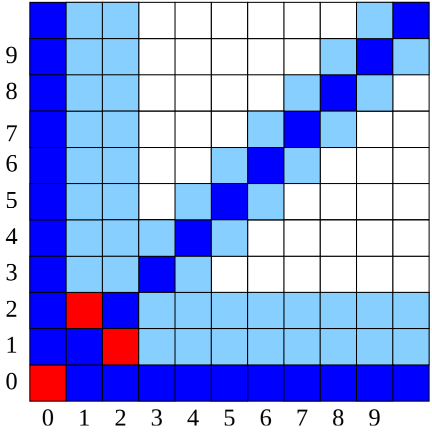

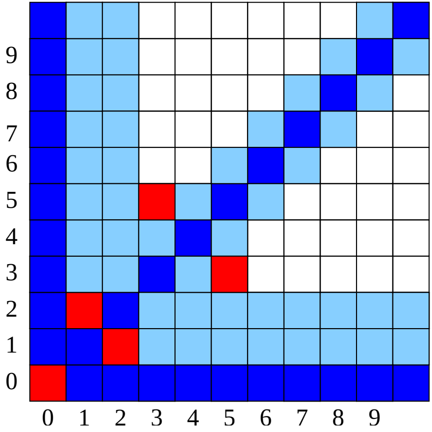
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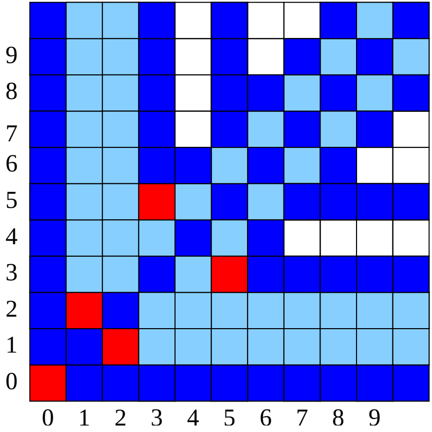
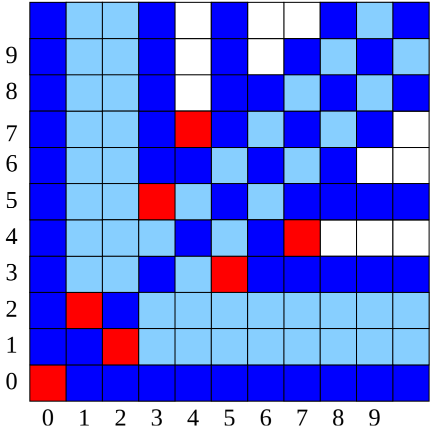
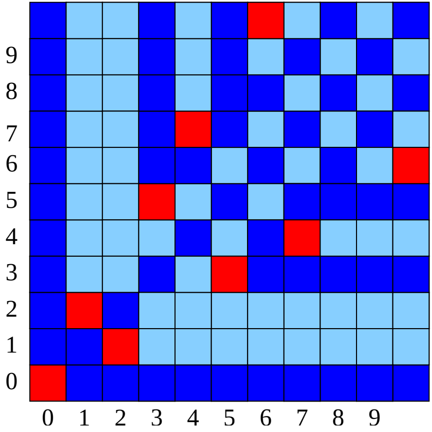
Le jeu de Wythoff



Le jeu de Wythoff



Le jeu de Wythoff



QUESTION

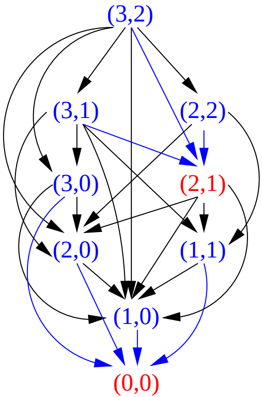
Peut-on décider “rapidement” si on se trouve sur une position gagnante ou perdante ?

$$(20365015276, 32951286898) \text{ est une position perdante}$$

$$(2180961, 2181194) \text{ est une position gagnante}$$

Le jeu de Wythoff

On peut faire la même analyse avec le (*graphe du jeu*) :



On a le même problème : praticable uniquement pour de “petites” valeurs !

Le jeu de Wythoff

Les premières positions perdantes sont

$$(1,2),\ (3,5),\ (4,7),\ (6,10),\ (8,13),\ (9,15),\ldots$$

Théorème (Wythoff 1907)

Pour tout entier $n \geq 1$,

$$(A_n, B_n) = (\lfloor n\,\varphi \rfloor, \lfloor n\,\varphi^2 \rfloor) = (\lfloor n\,\varphi \rfloor, \lfloor n\,\varphi \rfloor + n)$$

où $\varphi = (1+\sqrt{5})/2$ est le nombre d'or.

Théorème (S. Beatty 1927)

Si $\alpha, \beta > 1$ sont irrationnels et vérifient $1/\alpha + 1/\beta = 1$, alors $\{\lfloor n\,\alpha \rfloor \mid n \geq 1\}$ et $\{\lfloor n\,\beta \rfloor \mid n \geq 1\}$ partitionnent $\mathbb{N}_{\geq 1}$.

Autres caractérisations de l'ensemble des P-positions

- récursive

$$\forall n \geq 0, \quad \begin{cases} A_n = Mex\{a_i, b_i \mid i < n\} \\ B_n = a_n + n \end{cases}$$

- morphique

$$f(a) = ab, \; f(b) = a$$

$$\lim_{n\to\infty} f^n(a) = abaababaabaababaababa\cdots$$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	$\cdots$
$\mathcal{F}$	a	b	a	a	b	a	b	a	a	b	a	a	b	a	

- syntaxique

écriture dans le système de Zeckendorf

E. Zeckendorf, Représentation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de Lucas, *Bull. Soc. Roy. Sci. Liège* **41** (1972), 179–182

LE JEU DE WYTHOFF


suite de Fibonacci $F_{i+2} = F_{i+1} + F_i$, $F_0 = 1$, $F_1 = 2$

$\ldots, 610, 377, 233, 144, 89, 55, 34, 21, 13, 8, 5, 3, 2, 1$

1	1	8	10000	15	100010
2	10	9	10001	16	100100
3	100	10	10010	17	100101
4	101	11	10100	18	101000
5	1000	12	10101	19	101001
6	1001	13	100000	20	101010
7	1010	14	100001	21	1000000

$$(1,2),\ (3,5),\ (4,7),\ (6,10),\ (8,13),\ (9,15),\ldots$$

$$\ldots, 610, 377, 233, 144, 89, 55, 34, 21, 13, 8, 5, 3, 2, 1$$

1	2	1	10
3	5	100	1000
4	7	101	1010
6	10	1001	10010
8	13	10000	100000
9	15	10001	100010
11	18	10100	101000
12	20	10101	101010

première composante : nombre pair de zéros
seconde composante : décalé d'un cran vers la gauche

QUESTION

Peut-on décider “rapidement” si on se trouve sur une position gagnante ou perdante ?

$(20365015276, 32951286898)$ est une position perdante

$(2180961, 2181194)$ est une position gagnante

$$20365015276 = F_{49} + F_{17} + F_6$$

10000000000000000000000000000000100000000001000000

$$32951286898 = F_{50} + F_{18} + F_7$$

100000000000000000000000000000001000000000010000000

$$2180961 = F_{30} + F_{16} + F_8 + F_5$$

1000000000000010000000100100000

Lien mots/numération

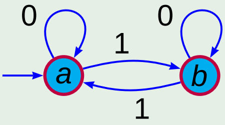
Théorème de Cobham (1973)

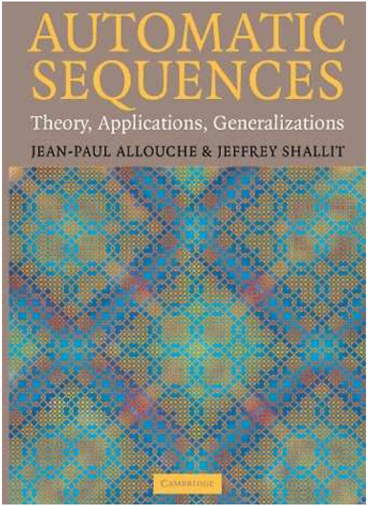
Une suite est un codage lettre-à-lettre d'une suite engendrée par morphisme uniforme de longueur k SSI elle est k-automatique.

Généralisation aux numérations *abstraits* et aux mots morphiques [A. Maes, M.R. 2002]

Le classique Thue–Morse

$f(a) = ab,\ f(b) = ba$, $abbabaabbaababbabaababba\cdots$




AUTOMATIC
SEQUENCES
Theory, Applications, Generalizations
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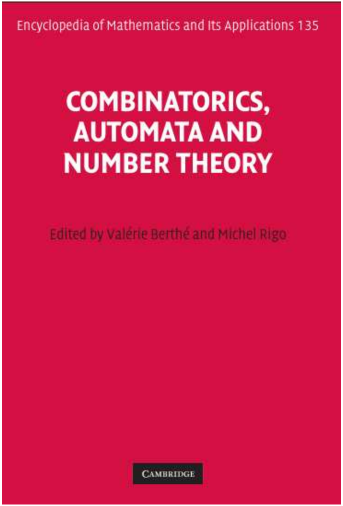

Encyclopedia of Mathematics and Its Applications 135
COMBINATORICS,
AUTOMATA AND
NUMBER THEORY
Edited by Valérie Berthé and Michel Rigo
CAMBRIDGE

Travaux récents...

- A. Fraenkel, Heap Games, Numeration Systems and Sequences, Annals of Combin. **2** (1998), 197-–210.
- A. Fraenkel, How to beat your Wythoff games' opponent on three fronts, Amer. Math. Monthly **89** (1982), 353—361.
- E. Duchêne, M. R., A morphic approach to combinatorial games : the Tribonacci case, Theor. Inform. Appl. **42** (2008), 375–393.
- E. Duchêne, M. R., Invariant games, Theoret. Comp. Sci. **411** (2010), 3169–3180, `orbi.ulg.ac.be/handle/2268/35496`
- E. Duchêne, A. Fraenkel, R. Nowakowski, M. R., Extensions and restrictions of wythoff's game preserving Wythoff's sequence as set of P positions, J. Combinat. Theory Ser. A **117** (2010), 545–567, `orbi.ulg.ac.be/handle/2268/17591`

$f(a) = ab, \; f(b) = ac, \; f(c) = a$

$\mathbf{t} = abacabaabacababacabaabacabacabaabacabab \cdots .$

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{A_n}$	0	1	3	5	7	8	10	12	14	16	18	20	21
$\mathbf{B_n}$	0	2	6	9	13	15	19	22	26	30	33	37	39
$\mathbf{C_n}$	0	4	11	17	24	28	35	41	48	55	61	68	72

This sequence was studied in

- L. Carlitz, R. Scoville, V.E. Hoggatt Jr., Fibonacci representations of higher order, *Fibonacci Quart.* **10** (1972), 43–69.
- E. Barcucci, L. Bélanger, S. Brlek, On Tribonacci sequences, *Fibonacci Quart.* **42** (2004), 314–319.

E. DUCHÊNE, M.R.

I. Any positive number of tokens from up to two piles can be removed.

II. Let $\alpha, \beta, \gamma > 0$ such that $2\max\{\alpha, \beta, \gamma\} \leq \alpha + \beta + \gamma$. Then one can remove α (resp. β, γ) from the first (resp. second, third) pile.

III. Let $\beta > 2\alpha > 0$. From (a, b, c) one can remove the same number α of tokens from any two piles and β tokens from the unchosen one. But the configuration $a' < c' < b'$ is not allowed.

Different sets of moves / more piles

$\downarrow$

Different sets of $\mathcal{P}$-positions to characterize...

PRESERVING WYTHOFF'S P-POSITIONS

OUR GOAL / DUAL QUESTION

Consider invariant extensions or restrictions of Wythoff's game that keep the set of $\mathcal{P}$-positions of Wythoff's game unchanged.

Characterize the different sets of moves...

$\downarrow$

Same set of $\mathcal{P}$-positions as Wythoff's game

DEFINITION, E. DUCHÊNE, M. R., TCS 411 (2010)

A removal game G is invariant, if for all positions $p = (p_1, \dots, p_\ell)$ and $q = (q_1, \dots, q_\ell)$ and any move $x = (x_1, \dots, x_\ell)$ such that $x \preceq p$ and $x \preceq q$ then, the move $p \to p - x$ is allowed if and only if the move $q \to q - x$ is allowed.

PRESERVING WYTHOFF'S P-POSITIONS

- Nim or Wythoff game are invariant games
- Raleigh game, the Rat and the Mouse game, Tribonacci game, Cubic Pisot games,... are NOT invariant

NON-INVARIANT GAME

Remove an odd number of tokens from a position (a, b) if a or b is a prime number, and an even number of tokens otherwise.

Very recently, Nhan Bao Ho (La Trobe Univ., Melbourne), *Two variants of Wythoff's game preserving its $\mathcal{P}$-positions*:

- A restriction of Wythoff's game in which if the two entrees are not equal then removing tokens from the smaller pile is not allowed.
- An extension of Wythoff's game obtained by adjoining a move allowing players to remove k tokens from the smaller pile and ℓ tokens from the other pile provided $\ell < k$.

PRESERVING WYTHOFF'S P-POSITIONS

OUR GOAL / DUAL QUESTION

Consider invariant extensions or restrictions of Wythoff's game that keep the set of $\mathcal{P}$-positions of Wythoff's game unchanged.

- We characterize all moves that can be adjoined while preserving the original set of $\mathcal{P}$-positions.
- Testing if a move belong to such an extended set of rules can be done in polynomial time.

Preserving Wythoff's P-positions

Can we get a "morphic characterization" of the Wythoff's matrix ?

$$
(P_{i,j})_{i,j\geq 0} = \begin{array}{|cccccccccccc}
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \\
0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \\
\vdots & & & & & & & & & & & \ddots
\end{array}
$$

Let's try something...

$$\varphi : a \mapsto \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \quad b \mapsto \begin{array}{|c|}\hline i \\ \hline e \\ \hline\end{array} \quad c \mapsto \begin{array}{|c|c|}\hline i & j \\ \hline\end{array} \quad d \mapsto \begin{array}{|c|}\hline i \\ \hline\end{array} \quad e \mapsto \begin{array}{|c|c|}\hline f & b \\ \hline\end{array}$$

$$f \mapsto \begin{array}{|c|c|}\hline g & b \\ \hline h & d \\ \hline\end{array} \quad g \mapsto \begin{array}{|c|c|}\hline f & b \\ \hline h & d \\ \hline\end{array} \quad h \mapsto \begin{array}{|c|c|}\hline i & m \\ \hline\end{array} \quad i \mapsto \begin{array}{|c|c|}\hline i & m \\ \hline h & d \\ \hline\end{array}$$

$$j \mapsto \begin{array}{|c|}\hline k \\ \hline c \\ \hline\end{array} \quad k \mapsto \begin{array}{|c|c|}\hline l & m \\ \hline c & d \\ \hline\end{array} \quad l \mapsto \begin{array}{|c|c|}\hline k & m \\ \hline c & d \\ \hline\end{array} \quad m \mapsto \begin{array}{|c|}\hline i \\ \hline h \\ \hline\end{array}$$

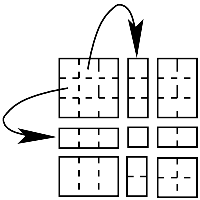
and the coding

$$\mu : e, g, j, l \mapsto 1, \quad a, b, c, d, f, h, i, k, m \mapsto 0$$

O. Salon, Suites automatiques à multi-indices, *Séminaire de théorie des nombres*, Bordeaux, 1986–1987, exposé 4.

Shape-Symmetric morphism [A. Maes '99]

If P is the infinite bidimensional picture that is the fixpoint of φ, then for all $i, j \in \mathbb{N}$, if $\varphi(P_{i,j})$ is a block of size $k \times \ell$ then $\varphi(P_{j,i})$ is of size $\ell \times k$



$a \mapsto$

a	*b*
c	*d*

$\mapsto$

a	*b*	*i*
c	*d*	**e**
i	**j**	*i*

$\mapsto$

a	*b*	*i*	*i*	*m*
c	*d*	**e**	*h*	*d*
i	**j**	*i*	*f*	*b*
i	*m*	*k*	*i*	*m*
h	*d*	*c*	*h*	*d*

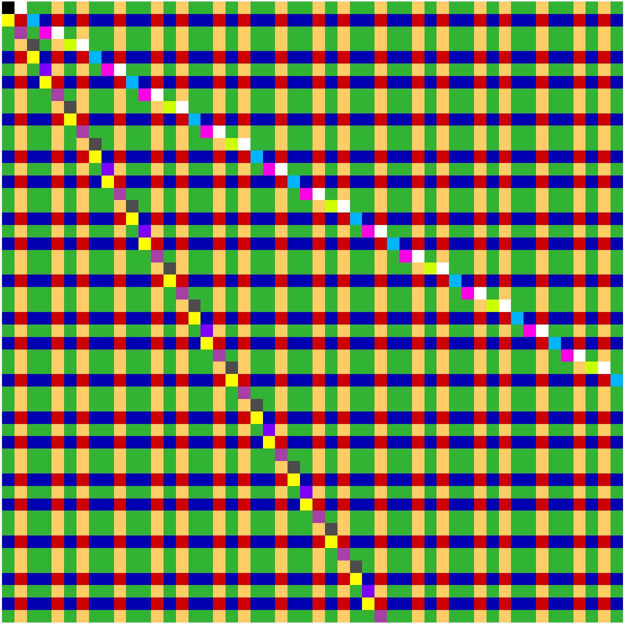
sizes : 1, 2, 3, 5

$\cdots \mapsto$

a	b	i	i	m	i	m	i
c	d	**e**	h	d	h	d	h
i	**j**	i	f	b	i	m	i
i	m	k	i	m	**g**	b	i
h	d	c	h	d	h	d	**e**
i	m	i	**l**	m	i	m	i
h	d	h	c	d	h	d	h
i	m	i	i	**j**	i	m	i

$\mapsto \cdots$

size : 8,...



MORPHISMS $\longrightarrow$ AUTOMATA

We can do the same as for the unidimensional case :
Automaton with input alphabet

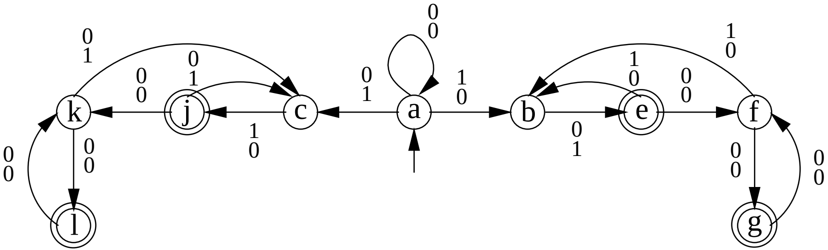
$$\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$\varphi(r) = \begin{array}{|c|c|} \hline s & t \\ \hline u & v \\ \hline \end{array}, \quad \begin{array}{|c|c|} \hline s & t \\ \hline \end{array}, \quad \begin{array}{|c|} \hline s \\ \hline u \\ \hline \end{array} \quad \text{or} \quad \begin{array}{|c|} \hline s \\ \hline \end{array}$$

we have transitions like

$$r \xrightarrow{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} s, \quad r \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} t, \quad r \xrightarrow{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} u, \quad r \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} v.$$

We get (after trimming useless part with four states)



This automaton accepts the words

$$\begin{pmatrix} 0w_1 \cdots w_\ell \\ w_1 \cdots w_\ell 0 \end{pmatrix} \text{ and } \begin{pmatrix} w_1 \cdots w_\ell 0 \\ 0w_1 \cdots w_\ell \end{pmatrix}$$

where $w_1 \cdots w_\ell$ is a valid F-representation ending with an even number of zeroes.

EXTENSION PRESERVING SET OF $\mathcal{P}$-POSITIONS

To decide whether or not a move can be adjoined to Wythoff's game without changing the set K of $\mathcal{P}$- positions, it suffices to check that it does not change the stability property K.

Remark : absorbing property holds true whatever the adjoined move is.

CONSEQUENCE

A move (i,j) can be added IFF it prevents to move from a $\mathcal{P}$-position to another $\mathcal{P}$-position.

In other words, a necessary and sufficient condition for a move $(i,j)_{i<j}$ to be adjoined is that it does not belong to

$$\{(A_n - A_m, B_n - B_m) : n > m \geq 0\} \cup \{(A_n - B_m, B_n - A_m) : n > m \geq 0\}$$

Thanks to the previous characterizations of A_n, B_m,

PROPOSITION

A move $(i,j)_{i<j}$ can be adjoined to without changing the set of $\mathcal{P}$-positions IFF

$$(i,j) \neq (\lfloor n\tau \rfloor - \lfloor m\tau \rfloor, \lfloor n\tau^2 \rfloor - \lfloor m\tau^2 \rfloor) \ \forall n > m \geq 0$$

and

$$(i,j) \neq (\lfloor n\tau \rfloor - \lfloor m\tau^2 \rfloor, \lfloor n\tau^2 \rfloor - \lfloor m\tau \rfloor) \ \forall n > m \geq 0$$

For all $i, j \geq 0$, $W_{i,j} = 0$ IFF Wythoff's game with the adjoined move (i, j) has Wythoff's sequence as set of $\mathcal{P}$-positions,

$$(W_{i,j})_{i,j\geq 0} = \begin{array}{|ccccccccccccc}
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & \\
0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \\
0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & \\
0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & 0 & \mathbf{1} & 0 & \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \\
0 & 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \\
0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \\
0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & \mathbf{1} & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \\
\vdots & & & & & & & & & & & \ddots
\end{array}$$

Corollary

Let $I \subseteq \mathbb{N}$. Wythoff's game with adjoined moves

$$\{(x_i, y_i) : i \in I, x_i, y_i \in \mathbb{N}\}$$

has the same sequence (A_n, B_n) as set of $\mathcal{P}$-positions

IFF

$W_{x_i, y_i} \neq 1$ for all $i \in I$.

ARE WE DONE ? **Complexity issue**

We investigate tractable extensions of Wythoff's game, we also need to test these conditions in polynomial time. And the winner can consummate a win in at most an exponential number of moves.

MANY "EFFORTS" LEAD TO THIS

For any pair (i, j) of positive integers, we have $W_{i,j} = 1$ if and only if one the three following properties is satisfied :

- $(\rho_F(i-1), \rho_F(j-1)) = (u0, u01)$ for any valid F-representation u in $\{0,1\}^*$.
- $(\rho_F(i-2), \rho_F(j-2)) = (u0, u01)$ for any valid F-representation u in $\{0,1\}^*$.
- $(\rho_F(j - A_i - 2), \rho_F(j - A_i - 2 + i)) = (u1, u'0)$ for any two valid F-representations u and u' in $\{0,1\}^*$.

Concluding result

Theorem

There is no redundant move in Wythoff's game. In particular, if any move is removed, then the set of $\mathcal{P}$-positions changes.

An open problem

- Sprague-Grundy function Mex(Opt(p)) for Nim is 2-regular [Allouche–Shallit, p.448].
- A two-dimensional array $(a(m,n))_{m,n\geq 0}$ is *k-regular* if there exist a finite number of two-dimensional arrays $(a_i(m,n))_{m,n\geq 0}$ such that each sub-array of the form $(a(k^e m + r, k^e n + s)_{m,n\geq 0}$ with $e \geq 0$, $0 \leq r, s < k^e$ is a $\mathbb{Z}$-linear combination of the a_i.

	0	1	2	3	4	5	6	7	8	9	$\cdots$
0	0	1	2	3	4	5	6	7	8	9	$\cdots$
1	1	0	3	2	5	4	7	6	9	8	
2	2	3	0	1	6	7	4	5	10	11	
3	3	2	1	0	7	6	5	4	11	10	
4	4	5	6	7	0	1	2	3	12	13	
5	5	4	7	6	1	0	3	2	13	12	
6	6	7	4	5	2	3	0	1	14	15	
7	7	6	5	4	3	2	1	0	15	14	
8	8	9	10	11	12	13	14	15	0	1	
9	9	8	11	10	13	12	15	14	1	0	
$\vdots$	$\vdots$										

AN OPEN PROBLEM

so what for Wythoff's game ?

	0	1	2	3	4	5	6	7	8	9	...
0	0	1	2	3	4	5	6	7	8	9	...
1	1	2	0	4	5	3	7	8	6	10	
2	2	0	1	5	3	4	8	6	7	11	
3	3	4	5	6	2	0	1	9	10	12	
4	4	5	3	2	7	6	9	0	1	8	
5	5	3	4	0	6	8	10	1	2	7	
6	6	7	8	1	9	10	3	4	5	13	
7	7	8	6	9	0	1	4	5	3	14	
8	8	6	7	10	1	1	5	3	4	15	
9	9	10	11	12	8	7	13	14	15	16	
⋮											⋱

A. S. Fraenkel, the Sprague-Grundy function for Wytoff's game, **TCS**'90