Efficient Solutions for the $\lambda$-coloring Problem on Classes of Graphs

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\[ d(u, v) = \text{distance between } u \text{ and } v. \]

\[ \text{diameter} = \max\{d(u, v) \mid u, v \in V(G)\} \]

\[ \text{Ex: } d(u, v) = 3; \text{ diameter of the graph is 3}. \]
coloring of a graph $G = (V, E)$

$f: V \rightarrow \mathbb{N}^*$, such that

if $uv \in E$, then $f(u) \neq f(v)$
\( \lambda \)-coloring of a graph \( G = (V, E) \)

\[ f : V \rightarrow \mathbb{N} \]

such that

- if \( uv \in E \), then \( |f(u) - f(v)| \geq 2 \),

- if \( \text{dist}(u, v) = 2 \), then \( f(u) \neq f(v) \)
Motivation
Motivation
Motivation

\[ \chi(G) = 4 \]
Motivation

\[ \lambda(G) = 9 \]
L(2,1)-coloring Problem

*Instance:* $G = (V, E), k \in \mathbb{N}$

*Question:* Is there an $\lambda$-coloring $f$ of $G$ with $f : V \rightarrow \{0, 1, ..., k\}$?

The minimum span is denoted $\lambda$
Examples

$\lambda = 5$

$\lambda(\text{Kn}) = 2n - 2$

$\lambda \geq \Delta + 1$
<table>
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<th>Comp.</th>
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<td>trees</td>
<td>$P$</td>
<td>diameter 2</td>
<td>$NP-c$</td>
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<tr>
<td>p-quasi trees</td>
<td>$P$</td>
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<td>bipartite chain</td>
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<td>$P$</td>
<td>P4-tidy</td>
<td>$P$</td>
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<td></td>
<td></td>
<td>graphs (q, q-4), q fixed</td>
<td>$P$</td>
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</tbody>
</table>
$\lambda = \Delta + 1 \text{ or } \Delta + 2$
[Griggs e Yeh 92] conjectured $\lambda$-col. of trees was NP-complete.

[Chang e Kuo 96] showed an $O(n^{4.5})$ algorithm.

[Hasunuma et al. 09] gave a linear time algorithm.

It is still open a structural characterization of trees.
$pv(G) = minimum\ number\ of\ disjoint\ paths$

$pv(G) = 3$
$\lambda(G)$ and $pv(G)$

[Griggs e Yeh 92]

$\lambda(G \land K_1) = n \iff G^c$ is hamiltonian

[Georges et al. 94]

$pv(G^c) \geq 2 \iff \lambda = n + pv(G^c) - 2$

$pv(G^c) = 1 \iff \lambda \leq n - 1$
A graph \( G \) is \((q, q-4)\) if each set of \( q \) vertices induces at most \( q - 4 \) P4's. [Babel e Olariu]

**Ex.:**
- \( q = 4 \) a.k.a. **cografos** (G is cograph \( \iff \) P4-free)
- \( q = 5 \) a.k.a. **P4-sparse**
- \( q = 7 \) superclass of **P4-lite** (P4-tidy and perfect)
Teo (Jamison e Olariu): If $G$ is $(q, q-4)$, then:

(i) union of two $(q, q-4)$ graphs or;

(ii) join of two $(q, q-4)$ graphs or;

(iii) spider where the head is a $(q, q-4)$ graph or;

(iv) it has a separable p-component $H = (H_1, H_2)$,

$$|H| \leq q,$$

$$G[V \setminus H_2] = G[V \setminus H] \uparrow G[H_1],$$

$$G[V \setminus H_1] = G[V \setminus H] \cup G[H_2].$$
If \( G = (V, E) \) is a spider, then

\[ V = S \cup K \cup R. \]

- \( S \) is a stable set.
- \( K \) is a clique,
- \( |S| = |K| \)

\[ G[R \cup K] = G[R] \uplus G[K] \]

- bijective function \( f : S \rightarrow K \)

(a) edges: thin spider

(b) no edges: thick spider
Thin spider and Thick spider

(a)

(b)
Separable p-component
$\lambda$-coloring of union and join

$\lambda$-coloring of $G \cup H$

$\lambda = \max\{ \lambda(G), \lambda(H) \}$

$\lambda$-coloring $G \uplus H$

$\lambda = \lambda'(G) + \lambda'(H) + 2$
If $G$ is thin spider with $|K| > 3$.

then $\lambda = \max\{ |R| - 1, \lambda(G[R]) \} + 2 |K|$

$|K| - 1 \in \{0, \ldots, 2|K| - 2\}$
If $G$ is thick spider with $|K| \geq 3$

$$
\lambda = \begin{cases} 
\lambda(G[R]) + 2 |K| & \text{if } \lambda(G[R]) \geq |R| + \left\lceil \frac{|K|}{2} \right\rceil - 2 \\
n + \left\lceil \frac{|K|}{2} \right\rceil - 2 & \text{otherwise}
\end{cases}
$$
If $G$ has a separable p-component $H = (H_1, H_2)$, then

$$pv(G) = \min \left\{ \max_{\psi \in CH} \{ pv(G \setminus H) - |B_1(\psi)|, \left\lceil \frac{|B_3(\psi)|}{2} \right\rceil, 1 \} + |B_2(\psi)| \right\}$$
FPT (fixed parameter tractable) in \( q(G) \)

\[ q(G) = \text{smallest } q \text{ for which } G \text{ is } (q, q-4) \text{ graph} \]

Algorithm FPT in \( q(G) \)

Linear algorithms for \((q, q-4) \text{ graphs}\) with \(q\) fixed

Ex: \(O(2^q n)\) or \(O(q^q n)\)
\[ \lambda \text{-coloring of separable p-component} \]

**Theorem** If \( G \) is \((q, q-4)\) graph, \( q \) fixed, with a separable p-component then \( \lambda \) can be obtained in linear time.

**Proof.**

\( G^c \) is \((q, q-4)\) graph and \( H^c \) is a separable p-component.

If \( G \) has less than \( 2q \) vertices, one can obtain \( \lambda \) in \( O(2q^{4q}) \).

Otherwise, \( pv(G^c) \) can be obtained in \( O(n) \), as \( |CH| \leq q^q \).
\begin{proof}

If \( d(u,v) \geq 3 \), then \( u, v \in H_1 \cup H_2 \).

\end{proof}
**λ-coloring of separable p-component**

**Teo** If $G$ is $(q, q-4)$ graph, $q$ fixed, with a separable p-component then $\lambda$ can be obtained in linear time.

**proof.**

Let $G'$ be obtained from $G$ merging vertices in the same class.

If $pv(G^c) > 1$, then $\lambda(G') = n' + pv(G^c) - 2$ (Georges et al.)

If $pv(G^c) = 1$, then $\lambda(G') = n' - 1$ (use hamiltonian path.)
\textbf{λ-coloring of separable p-component}

**Theorem** If G is \((q, q-4)\) graph, \(q\) fixed, with a separable p-component then \(\lambda\) can be obtained in linear time.

**Proof.**

Assign the \textbf{same color} to the \textbf{merged vertices}.

For each \(O(q^q)\) possible \(G'\) one can obtain \(\lambda'\).

\(\lambda\) will be the \textbf{minimum} among all these \(\lambda'\).

**Complexity:** \(O(n \cdot 2q^{5q})\)
Example
Example
Example
Example
Example
Example

$G \setminus H$

$H_1$

$H_2$
Example

\[(G \setminus H)^C\]
Example

\[(G\setminus H)^C\]
Example

\[(G \setminus H)^C\]
Example
Example

\[ G \setminus H \]

\[ H_1 \]

\[ H_2 \]

13

13
Example

G\H

H₁

H₂

5

5

5
Example
Interval graph: $G = \Omega(l)$
Comparability graph: \( \exists \) transitive orientation of the edges of the graph.

Cocomparability graph: \( G^c \) is a comparability graph.
Permutation graph: $G = \Omega(S)$
Split graph: $G = (V, E), V = S \cup K$.

- $K$ is a clique
- $S$ is a stable set
Split permutation graph: \( G \) is \textit{split} and \textit{permutation}.

[Brandstäd, Bang Le and Spinrad-99]
There are $\theta \left( \frac{4^n}{\sqrt{n}} \right)$ split permutation graphs.

Split permutation $\subset$ clique Helly

Extended triangle

$G$ is clique-Helly $\iff$ every extended triangle of $G$ has an universal vertex
Our work:

For a split permutation graph $G$,

$$\lambda(G) = \max\{ \lambda(G_R), \lambda(G_L) \}$$

$G_L = G \setminus S_R$

$G_R = G \setminus S_L$

$\lambda(G)$ can be computed in linear time.

$O(n^2)$ algorithm that obtain an $\lambda$-coloring with this span
split permutation graphs
∃ Chain ordering \( a_1 < a_2 < \ldots < a_{L+M} \) such that: \( N(a_1) \subseteq N(a_2) \subseteq \ldots \subseteq N(a_{L+M}) \)
\[ \forall c \in S_M \Rightarrow K_L \subseteq N(c) \text{ or } K_R \subseteq N(c). \]
For a split permutation graph $G$, $\lambda(G) \geq \max\{\lambda(G_R), \lambda(G_L)\}$.

For a split permutation graph $G_L$, $\lambda(G_L) = n_L + pv(G_L^c) - 2$.

$diameter$ of $G_L$ is 2

$\lambda(G_L) \geq \lambda(G_L)$. 

$G_L$ subgraph of $G$
\[ \lambda(G) = \max\{\lambda(G_R), \lambda(G_L)\} \]
split permutation graphs

\[ G^c \]

\[ S_L, a+12, a+10, a+8 \]

\[ S_M, a+6, a+4, a+3, a+1 \]

\[ K_L, a, a+5, a+7, K_R \]

\[ K_M, a+9, a+2 \]
Interval model has to be modified:

(a) Split permutation graphs

(b) Split permutation graphs

(c) Split permutation graphs
A linear-time algorithm to find $\lambda(G_R)$ and $\lambda(G_L)$.

For split permutation graphs $\lambda(G) = \max\{\lambda(G_R), \lambda(G_L)\}$.

This proof also gives a $O(n^2)$ algorithm to find an optimum $\lambda$-coloring of graphs on this class.
Griggs and Yeh Conjecture:

\[ \lambda \leq \Delta^2 \]

only proved for a few classes of graphs,

and it is still open for bipartite graphs.

\[ \lambda \leq \Delta^2 + \Delta - 2 \]  
[ Gonçalves 06]
<table>
<thead>
<tr>
<th>Class</th>
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<tr>
<td>diameter 2</td>
<td>$\Delta^2 [\text{Griggs and Yeh}]$</td>
</tr>
<tr>
<td>regular grids</td>
<td>$\Delta + 2 [\text{Calamoneri et al.}]$</td>
</tr>
<tr>
<td>cocomparability</td>
<td>$4\Delta - 1 [\text{Calamoneri et al.}]$</td>
</tr>
<tr>
<td>cograph</td>
<td>$n + \text{pv}(G^c) - 2 [\text{Chang e Kuo}], 2\Delta [\text{C. and P.}]$</td>
</tr>
<tr>
<td>planar</td>
<td>$2\Delta + 25 [\text{van den Heuvel and McGuinness}]$</td>
</tr>
<tr>
<td>bipartite permutation</td>
<td>$\text{wb}(G)+1 [\text{Araki}]$</td>
</tr>
<tr>
<td>weakly chordal</td>
<td>$\Delta^2 [\text{C. and P.}]$</td>
</tr>
<tr>
<td>split</td>
<td>$0.385\Delta^{1.5} + 2\Delta + \Delta^{0.5} - 2 [\text{C. and P.}]$</td>
</tr>
<tr>
<td>interval</td>
<td>$2\Delta [\text{Calamoneri et al.}]$</td>
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