Partitions d’entiers et groupes de Coxeter

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Plan

1. Nekrasov-Okounkov type formulas

2. Generalizations through Littlewood decomposition

3. Coxeter groups and automata theory
1 Nekrasov-Okounkov type formulas

2 Generalizations through Littlewood decomposition

3 Coxeter groups and automata theory
A partition $\lambda$ of $n$ is a non-increasing sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. We represent a partition by its Ferrers diagram.
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![Ferrers diagram](image.png)

**Figure:** The Ferrers diagram of $\lambda=(5,4,3,3,1)$
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Figure: The Ferrers diagram of $\lambda=(5,4,3,3,1)$ and its hook lengths
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\[
\begin{array}{ccccccc}
+ & + & + & & & & \\
+ & + & - & & & & \\
+ & - & - & - & & & \\
- & - & - & - & - & & \\
\end{array}
\]

**Figure:** The Ferrers diagram of $\lambda=(5,4,3,3,1)$ and the sign $\varepsilon_h$ of its boxes.

Set $\varepsilon_h = \begin{cases} +1 & \text{if } h \text{ is strictly above the diagonal} \\ -1 & \text{else} \end{cases}$
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$\mathcal{H}_t(\lambda)$ the multi-set of hook lengths which are multiple of $t$
Let $t \geq 2$ be an integer. A partition is a \textit{t-core} if its hook lengths set does not contain $t$. It is equivalent to the fact that the hook lengths set does not contain any integral multiple of $t$, \textit{i.e.} $\mathcal{H}_t(\lambda) = \emptyset$. 
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\begin{figure}[h]
\centering
\begin{tabular}{cccc}
1 \\
2 \\
4 & 1 \\
7 & 4 & 2 & 1 \\
\end{tabular}
\caption{A 3-core}
\end{figure}

\textbf{Nakayama} (1940): introduction and conjectures in representation theory
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\begin{center}
\begin{tikzpicture}
\matrix (m) [draw, anchor=south] {
1 & 2 & 4 & 1 & 7 & 4 & 2 & 1 \};
\end{tikzpicture}
\end{center}

\textbf{Figure:} A 3-core

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\textbf{Garvan-Kim-Stanton} (1990): generating function, proof of Ramanujan’s congruences
\textbf{Ono} (1994): positivity of the number of t-cores
\textbf{Han} (2009): expansion of $\eta$ function in terms of hooks
Self-conjugate and doubled distinct partitions

**Self-conjugate partitions**

$SC$

$SC(t)$: subset of self-conjugate $t$-cores.
Self-conjugate and doubled distinct partitions

*Self-conjugate partitions*

$SC$

\[
\begin{array}{cccccc}
1 \\
2 \\
4 & 1 \\
7 & 4 & 2 & 1 \\
\end{array}
\]

$SC_{(t)}$: subset of self-conjugate $t$-cores.

In terms of Frobenius coordinates, we have:

\[
\begin{pmatrix}
a_1 & \ldots & a_k \\
a_1 & \ldots & a_k \\
\end{pmatrix}
\]
Self-conjugate and doubled distinct partitions

**Self-conjugate partitions**  
$\text{SC}$

<table>
<thead>
<tr>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
</tr>
<tr>
<td>4 1</td>
</tr>
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</tr>
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$\text{SC}(t)$: subset of self-conjugate $t$-cores.

**Doubled distinct partitions**  
$\text{DD}$

$\left(\begin{array}{cccc}
a_1 & \ldots & a_k \\
a_1 & \ldots & a_k \\
\end{array}\right)$

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**Self-conjugate and doubled distinct partitions**

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\[ SC \]

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4 \ 1 \\
7 \ 4 \ 2 \ 1
\end{array}
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\[ SC(t): \text{subset of self-conjugate } t\text{-cores.} \]

**Doubled distinct partitions**  
\[ DD \]

\[
\begin{array}{c}
1 \ 2 \\
4 \ 1 \\
7 \ 4 \ 2 \ 1
\end{array}
\]

\[
\begin{array}{c}
\text{In terms of Frobenius coordinates, we have:}
\end{array}
\]

\[
\left( a_1 \ \ldots \ a_k \right)
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\[
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Self-conjugate and doubled distinct partitions

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& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
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**Doubled distinct partitions**

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\[
\begin{array}{c}
8 6 2 \\
1 \\
4 1 \\
7 4 2 1
\end{array}
\]

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\end{pmatrix}
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**Doubled distinct partitions**

\[ DD \]

\[
\begin{array}{cccc}
 & & & 2 \\
 & & 6 & \\
 & 8 & &
\end{array}
\]

\[ DD(t) \]: subset of doubled distinct \( t \)-cores.
### Self-conjugate partitions $SC$

<table>
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<th>1</th>
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<tbody>
<tr>
<td>4</td>
<td>1</td>
</tr>
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<td>4</td>
</tr>
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</table>

$SC(t)$: subset of self-conjugate $t$-cores.

In terms of Frobenius coordinates, we have:

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\begin{pmatrix}
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  a_1 & \ldots & a_k
\end{pmatrix}
\]

### Doubled distinct partitions $DD$

<table>
<thead>
<tr>
<th>8</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8</td>
</tr>
</tbody>
</table>

$DD(t)$: subset of doubled distinct $t$-cores.

\[
\begin{pmatrix}
  b_1 + 1 & \ldots & b_k + 1 \\
  b_1 & \ldots & b_k
\end{pmatrix}
\]
We define Dedekind eta function by \( \eta(x) = x^{1/24} \prod_{i \geq 1} (1 - x^i). \)

\( \eta \) is a weight 1/2 modular form.
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Lehmer’s conjecture (1947)

Coefficients of the expansion of \( \eta^{24} \) are nonzero.
Theorem (Nekrasov-Okounkov, 2006; Westbury, 2006; Han, 2009; P., 2015)

For all complex number z, we have:

$$\prod_{k \geq 1} (1 - x^k)^{z-1} = \sum_{\lambda \in \mathcal{P}} x^{\lambda} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right)$$
Nekrasov-Okounkov type formulas

Theorem (Nekrasov-Okounkov, 2006; Westbury, 2006 ; Han, 2009 ; P., 2015)

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\]

\[
\prod_{k \geq 1} (1 - x^k)^{2z^2+z} = \sum_{\lambda \in \mathcal{D}D} \delta_\lambda x^{\lambda/2} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{2z + 2}{h \varepsilon_h}\right)
\]
Theorem (Nekrasov-Okounkov, 2006; Westbury, 2006; Han, 2009; P., 2015)

For all complex number $z$, we have:

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\prod_{k \geq 1} (1 - x^k)^{z-1} = \sum_{\lambda \in P} x^{||\lambda||} \prod_{h \in H(\lambda)} \left(1 - \frac{z}{h^2}\right)
\]

\[
\prod_{k \geq 1} (1 - x^k)^{2z^2+z} = \sum_{\lambda \in DD} \delta_\lambda x^{||\lambda||/2} \prod_{h \in H(\lambda)} \left(1 - \frac{2z + 2}{h \varepsilon_h}\right)
\]

\[
\left(\prod_{k \geq 1} \frac{(1 - x^{2k})^{z+1}}{1 - x^k}\right)^{2z-1} = \sum_{\lambda \in SC} \delta_\lambda x^{||\lambda||} \prod_{h \in H(\lambda)} \left(1 - \frac{2z}{h \varepsilon_h}\right)
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1 Nekrasov-Okounkov type formulas

2 Generalizations through Littlewood decomposition

3 Coxeter groups and automata theory
Generalization of the type $\tilde{C}$ Nekrasov-Okounkov formula

**Theorem (P., 2015)**

Let $t$ be a positive integer. For any complex numbers $y$ and $z$ we have

$$
\sum_{\lambda \in DD} \delta_\lambda x^{|\lambda|/2} \prod_{h \in \mathcal{H}_t(\lambda)} \left( y - \frac{tyz}{h \in h} \right)
$$

$$
= \begin{cases} 
\prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t'-1} \left( 1 - x^{kt}y^{2k} \right)^{(z-1)(zt+t-3)/2} & \text{if } t = 2t' + 1 \\
\prod_{k \geq 1} \frac{(1 - x^k)(1 - x^{kt})^{t'-1}}{1 - x^{kt'}} \left( \frac{1 - y^{2kx^{kt}}}{1 - y^{kx^{kt'}}} \right)^{z-1} & \text{if } t = 2t' 
\end{cases}
$$
The $t$-core of a partition

Let $t$ be a fixed integer. The $t$-core of a partition $\lambda$ is the partition obtained by successively deleting in $\lambda$ all the ribbons of length $t$, until we can not remove any ribbon.
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Exemple : $\lambda = (5, 5, 4, 4, 2)$

\[
\begin{array}{ccccccc}
\hline
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
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\begin{center}
\begin{tikzpicture}
\draw (0,0) rectangle (2,2);
\end{tikzpicture}
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has for 3-core
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The $t$-core of a self-conjugate partition is self-conjugate
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Exemple: $\lambda = (5, 5, 4, 4, 2)$ has for 3-core

Facts: The $t$-core of a partition is a $t$-core. The $t$-core of a self-conjugate partition is self-conjugate. The $t$-core of a doubled-distinct partition is doubled-distinct.
Theorem (Littlewood, 1951, probably)

The Littlewood decomposition maps a partition \( \lambda \) to \((\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})\) such that:

(i) \( \tilde{\lambda} \) is the \( t \)-core of \( \lambda \) and \( \lambda^0, \lambda^1, \ldots, \lambda^{t-1} \) are partitions;
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(iii) $\{h/t, h \in \mathcal{H}_t(\lambda)\} = \mathcal{H}(\lambda^0) \cup \mathcal{H}(\lambda^1) \cup \cdots \cup \mathcal{H}(\lambda^{t-1})$. 

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$t = 3$

\[ \text{Diagram of a partition with } t = 3 \]
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(ii) \( |\lambda| = |\tilde{\lambda}| + t(|\lambda^0| + |\lambda^1| + \cdots + |\lambda^{t-1}|) \)
(iii) \( \{ h/t, h \in \mathcal{H}_t(\lambda) \} = \mathcal{H}(\lambda^0) \cup \mathcal{H}(\lambda^1) \cup \cdots \cup \mathcal{H}(\lambda^{t-1}) \).

\( t = 3 \)

\[
\begin{array}{cccccc}
\ & \ & \ & 0 & 0 & \\
\ & \ & 0 & 1 & 1 & \\
\ & 0 & 0 & 0 & \ & \\
0 & 0 & 0 & \ & \ & \\
0 & 1 & 1 & 0 & 0 & \\
0 & 0 & \ & 0 & 0 & \\
0 & 1 & \ & 1 & 1 & \\
0 & \ & 0 & \ & \ & \\
0 & 0 & \ & \ & \ & \\
\end{array}
\]

\( w = \cdots 00110001.101110011 \cdots \)
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\[ t = 3 \]

\[
\begin{array}{c|ccc}
0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{array}
\]

$w = \cdots 00110001.101110011\cdots$

$w_0 = \cdots 1 \ 0 \ 1 \ 1 \ 0 \ \cdots$
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(i) $\tilde{\lambda}$ is the $t$-core of $\lambda$ and $\lambda^0, \lambda^1, \ldots, \lambda^{t-1}$ are partitions;
(ii) $|\lambda| = |\tilde{\lambda}| + t(|\lambda^0| + |\lambda^1| + \cdots + |\lambda^{t-1}|)$
(iii) $\{h/t, h \in \mathcal{H}_t(\lambda)\} = \mathcal{H}(\lambda^0) \cup \mathcal{H}(\lambda^1) \cup \cdots \cup \mathcal{H}(\lambda^{t-1})$.

$t = 3$

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
\end{array}
\]

$w = \cdots 00110001.101110011 \cdots$

$w_0 = \cdots 1 \ 0 \ 1 \ 1 \ 0 \ \cdots$

$w_1 = \cdots 0 \ 1 \ 0 \ 0 \ 1 \ 1 \ \cdots$

$w_2 = \cdots 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ \cdots$

Mathias Pétréolle (ICJ, Lyon)
The Littlewood decomposition maps a partition \( \lambda \) to \((\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})\) such that:

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\[ t = 3 \]
\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
\end{array}
\]

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\[ w_1 = \cdots 00101110 \cdots \]
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$t = 3$

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$w = \cdots 00110001.101110011 \cdots$ \hspace{1cm} $\lambda^0 = \begin{array}{c} 1 \\ 4 \\ 2 \\ 1 \end{array}$

$w_0 = \cdots 101110011 \cdots$ \hspace{1cm} $\lambda^1 = \begin{array}{c} 1 \\ 2 \end{array}$

$w_1 = \cdots 01001111 \cdots$ \hspace{1cm} $\lambda^2 = \begin{array}{c} 2 \\ 1 \end{array}$
When $\lambda \in DD$, its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ satisfies:

(i) $\tilde{\lambda}$ and $\lambda^0$ are doubled distinct partitions
Properties of the Littlewood decomposition

When \( \lambda \in DD \), its Littlewood decomposition \((\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})\) satisfies:

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(ii) \( \lambda^i \) and \( \lambda^{t-i} \) are conjugate for \( i \in \{1, \ldots, t-1\} \)
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\[
\begin{array}{cccc}
\tilde{\lambda} & \lambda^0 & \lambda^1 & \lambda^2 \\
12 & 6 & 3 & \emptyset & 1 & 4 & 2 & 1 & 1 & 2 & 2 & 1 \\
\end{array}
\]
When $\lambda \in DD$, its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ satisfies:

(iii) $\delta_{\lambda} = \delta_{\tilde{\lambda}} \delta_{\lambda^0}$ if $t$ is odd and $\delta_{\lambda} = \delta_{\tilde{\lambda}} \delta_{\lambda^0} \delta_{\lambda^{t'}}$ if $t = 2t'$ is even.
New properties of the Littlewood decomposition

When $\lambda \in DD$, its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ satisfies:

(iii) $\delta_{\lambda} = \delta_{\tilde{\lambda}} \delta_{\lambda^0}$ if $t$ is odd and $\delta_{\lambda} = \delta_{\tilde{\lambda}} \delta_{\lambda^0} \delta_{\lambda'}$ if $t$ is even.

(iv) Let $v$ be a box in $\lambda^0$ and $V$ its canonically associated box in $\lambda$. $v$ is strictly above the principal diagonal in $\lambda^0$ iff it is also the same for $V$ in $\lambda$. 

![Diagram of Littlewood decomposition]
New properties of the Littlewood decomposition

When $\lambda \in DD$, its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$ satisfies:

(v) Let $v = (j, k)$ be a box in $\lambda^i$, with $1 \leq i \leq t'$ and $v^* = (k, j)$ a box in $\lambda^{2t'+1-i} = \lambda^{i*}$. We denote by $V$ and $V^*$ the boxes of $\lambda$ associated with them. If $V$ is strictly above (resp. below) the principal diagonal of $\lambda$, then $V^*$ is strictly above (resp. below) this diagonal.
Proof of our generalization

- Fix $t = 2t' + 1$ an integer, $\lambda \in DD$ and its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$. 
Proof of our generalization

- Fix \( t = 2t' + 1 \) an integer, \( \lambda \in DD \) and its Littlewood decomposition \((\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})\).

- Write:

\[
\delta_{\lambda} x^{\frac{1}{2} |\lambda|} \prod_{h \in \mathcal{H}_t(\lambda)} \left( y - \frac{yt(2z + 2)}{\varepsilon_h h} \right)
\]
Proof of our generalization

- Fix $t = 2t' + 1$ an integer, $\lambda \in DD$ and its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$.

- Write:

$$
\delta_\lambda x^{\mid \lambda \mid / 2} \prod_{h \in \mathcal{H}_t(\lambda)} \left( y - \frac{yt(2z + 2)}{\varepsilon_h h} \right) = \delta_{\tilde{\lambda}} x^{\mid \tilde{\lambda} \mid / 2} 

\times \delta_{\lambda^0} x^{t\mid \lambda^0 \mid / 2} \prod_{h \in \mathcal{H}(\lambda^0)} \left( y - \frac{y(2z + 2)}{\varepsilon_h h} \right) 

\times \prod_{i=1}^{t'} x^{t\mid \lambda^i \mid} \prod_{h \in \mathcal{H}(\lambda^i)} \left( y^2 - \left( \frac{y(2z + 2)}{h} \right)^2 \right)
$$
Proof of our generalization

- Fix $t = 2t' + 1$ an integer, $\lambda \in DD$ and its Littlewood decomposition $(\tilde{\lambda}, \lambda^0, \lambda^1, \ldots, \lambda^{t-1})$.

- Write:

$$
\delta_\lambda x^{\lvert \lambda \rvert / 2} \prod_{h \in H_t(\lambda)} \left( y - \frac{yt(2z + 2)}{\varepsilon_h h} \right) = \delta_{\tilde{\lambda}} x^{\lvert \tilde{\lambda} \rvert / 2} \times \delta_{\lambda^0} x^{t\lvert \lambda^0 \rvert / 2} \prod_{h \in H(\lambda^0)} \left( y - \frac{y(2z + 2)}{\varepsilon_h h} \right) \times \prod_{i=1}^{t'} x^{t\lvert \lambda^i \rvert} \prod_{h \in H(\lambda^i)} \left( y^2 - \left( \frac{y(2z + 2)}{h} \right)^2 \right)
$$

- And sum over all doubled distinct partitions.
And for self-conjugate partitions?

Same types of properties apply for self-conjugate partitions.
And for self-conjugate partitions?

Same types of properties apply for self-conjugate partitions.

**Theorem (P., 2015)**

Let $t$ be a positive integer. For all complex numbers $y$ and $z$, we have:

$$
\sum_{\lambda \in SC} \delta_{\lambda} x^{\lambda}| \prod_{h \in H_t(\lambda)} \left( y - \frac{yzt}{h \in h} \right)
= \begin{cases} 
\prod_{k \geq 1} \frac{1 - x^k}{1 - x^{2k}} \left(1 - x^{2kt}\right)^{t'} \left(1 - y^{2k} x^{2kt}\right)(z^2-1)t' & \text{if } t = 2t' \\
\prod_{k \geq 1} \frac{1 - x^k}{1 - x^{2k}} \frac{(1 - x^{2kt})^{t'+1}}{1 - x^{kt}} \left(1 - y^{2k} x^{2kt}\right)(tz^2+z-t-1)/2 \left(1 - y^k x^{kt}\right)^{z-1} & \text{if } t = 2t' + 1
\end{cases}
$$
Corollary (P., 2015)

When $t = y = 1$, we recover the Nekrasov-Okounkov formula in types $\tilde{C}$ and $\tilde{C}^\sim$. 
Some consequences

Corollary (P., 2015)

When $t = y = 1$, we recover the Nekrasov-Okounkov formula in types $\tilde{C}$ and $\tilde{C}^\vee$.

Corollary (P., 2015)

We have:

$$\sum_{\lambda \in DD} \delta_\lambda x^{\frac{1}{2}|\lambda|} \prod_{h \in H_t(\lambda)} \frac{bt}{h \varepsilon_h} = \exp(-tb^2x^t/2) \prod_{k \geq 1} (1 - x^k)(1 - x^{kt})^{t' - 1}$$
New hook formulas

**Corollary**

*If $t$ is odd,*

$$
\sum_{\lambda \in DD, \, |\lambda|=2tn} \delta_{\lambda} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} = \frac{(-1)^n}{n!2^n t^n} 
$$

*If $t$ is even,*

$$
\sum_{\lambda \in SC, \, |\lambda|=2tn} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h} = \frac{1}{n!2^n t^n} 
$$
New hook formulas

Corollary

if $t$ is odd,

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\sum_{\lambda \in DD, \ |\lambda|=2tn} \delta_{\lambda} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h \varepsilon_h} = \frac{(-1)^n}{n!2^n t^n}
$$

$$
\sum_{\lambda \in SC, \ |\lambda|=tn} \delta_{\lambda} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{t}{h \varepsilon_h} = \left[x^{tn}\right] \exp(-x^t - tx^{2t}/2)
$$

if $t$ is even,

$$
\sum_{\lambda \in SC, \ |\lambda|=2tn} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h \varepsilon_h} = \frac{1}{n!2^n t^n}
$$

$$
\sum_{\lambda \in DD, \ |\lambda|=tn} \delta_{\lambda} \prod_{h \in \mathcal{H}_t(\lambda)} \frac{t}{h \varepsilon_h} = \left[x^{tn/2}\right] \exp(-x^{t/2} - tx^t/2)
$$
And after?

- Links with representation theory?
And after?

- Links with representation theory?
- Prove general theorem using the Littlewood decomposition
And after?

- Links with representation theory?
- Prove general theorem using the Littlewood decomposition
1. Nekrasov-Okounkov type formulas

2. Generalizations through Littlewood decomposition

3. Coxeter groups and automata theory
Coxeter groups

A Coxeter group is given by a matrix \((m_{s,t})_{s,t \in S}\)

\[
\begin{align*}
\begin{cases}
  s^2 = 1 \\
  sts \cdots = tst \cdots & \text{braid relations} \\
  m_{s,t} m_{s,t} & \text{if } m_{s,t} = 2, \text{ commutation relations}
\end{cases}
\end{align*}
\]
A Coxeter group is given by a matrix \((m_{s,t})_{s,t \in S}\)

\[ s^2 = 1 \]

Relations

\[
\begin{aligned}
&s t s \cdots = t s t \cdots & \text{braid relations} \\
&m_{s,t} m_{s,t} & \text{if } m_{s,t} = 2, \text{ commutation relations}
\end{aligned}
\]

Length of an element \(w := \ell(w) = \) minimal integer \(\ell\) such that

\(w = s_1 s_2 \ldots s_\ell\) with \(s_i \in S\)

A such word is a reduced decomposition of \(w \in W\)
A Coxeter group is given by a matrix $\begin{pmatrix} m_{s,t} \end{pmatrix}$ for all $s,t \in S$

Relations

\[
\begin{align*}
& s^2 = 1 \\
& \left\{ 
\begin{array}{ll}
sts \cdots = tst \cdots & \text{braid relations} \\
m_{s,t} m_{s,t} & \text{if } m_{s,t} = 2, \text{ commutation relations}
\end{array}
\right.
\]

Length of an element $w := \ell(w) = \text{minimal integer } \ell$ such that $w = s_1 s_2 \ldots s_\ell$ with $s_i \in S$

A such word is a reduced decomposition of $w \in W$

**Theorem (Matsumoto, 1964)**

Let $w$ be an element of $W$. Any two of its reduced decompositions are linked by a series of braid relations.
Cyclically fully commutative elements

**Definition**

An element \( w \) is **fully commutative** if, given two reduced decompositions of \( w \), there is a sequence of **commutation relations** which can be applied to transform one into the other.
Cyclically fully commutative elements

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An element $w$ is **fully commutative** if, given two reduced decompositions of $w$, there is a sequence of **commutation relations** which can be applied to transform one into the other.

$s_5 s_3 s_4 s_2 s_1 s_3 s_2 s_6 s_5$ FC,

$s_3 s_2 s_1 s_2 s_4 s_3 s_5$ not FC

$s_1 \ s_2 \ s_3 \ s_4 \ s_5 \ s_6$ $A_6$
Cyclically fully commutative elements

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An element $w$ is **fully commutative** if, given two reduced decompositions of $w$, there is a sequence of **commutation relations** which can be applied to transform one into the other.

$$s_5 s_3 s_4 s_2 s_1 s_3 s_2 s_6 s_5 \text{ FC,}$$
$$s_3 s_2 s_1 s_2 s_4 s_3 s_5 \text{ not FC}$$

**Definition**

An element $w$ is **cyclically fully commutative** if any cyclic shift of any of its reduced decompositions is FC.
Cyclically fully commutative elements

**Definition**

An element $w$ is **fully commutative** if, given two reduced decompositions of $w$, there is a sequence of **commutation relations** which can be applied to transform one into the other.

\[ s_5 s_3 s_4 s_2 s_1 s_3 s_2 s_6 s_5 \text{ FC, } \]
\[ s_3 s_2 s_1 s_2 s_4 s_3 s_5 \text{ not FC } \]

**Definition**

An element $w$ is **cyclically fully commutative** if any cyclic shift of any of its reduced decompositions is FC.

In $A_6$, \[ s_5 s_3 s_4 s_2 s_1 s_3 s_2 s_6 s_5 \text{ not CFC } \]
\[ s_2 s_1 s_3 s_2 s_1 \text{ not CFC } \]
\[ s_2 s_1 s_3 \text{ CFC } \]
Theorem (Stembridge, 1995)

A reduced expression correspond to a fully commutative element if and only if it does not contain up to commutation a subword $stst \ldots$ of length $m_{st}$. 
The set of reduced expressions of cyclically fully commutative elements is recognizable by an explicit finite state automaton.
Automaton

Theorem (P., 2015)

The set of reduced expressions of cyclically fully commutative elements is recognizable by an explicit finite state automaton.

Ideas of the proof:

- Construct the automaton state by state, checking Stembridge’s property along the reading of the word
The set of reduced expressions of cyclically fully commutative elements is recognizable by an explicit finite state automaton.

Ideas of the proof:

- Construct the automaton state by state, checking Stembridge’s property along the reading of the word.
- Encode also in the states the information about chains of type $stst \ldots$ at the beginning of the word.
Theorem (P., 2015)

The set of reduced expressions of cyclically fully commutative elements is recognizable by an explicit finite state automaton.

Ideas of the proof:

- Construct the automaton state by state, checking Stembridge’s property along the reading of the word
- Encode also in the states the information about chains of type $stst \ldots$ at the beginning of the word
- Define final states according to these informations
Theorem (P., 2015)

The generating function of CFC elements refined by length is rational.
Rationality

Theorem (P., 2015)

The generating function of CFC elements refined by length is rational.

Ideas of the proof:

- select one normal form among all reduced expressions of an element through another finite state automaton
Theorem (P., 2015)

The generating function of CFC elements refined by length is rational.

Ideas of the proof:

- select one normal form among all reduced expressions of an element through another finite state automaton
- conclude as generating series of languages recognizable by finite state automata are rational
Rationality

Theorem (P., 2015)

The generating function of CFC elements refined by length is rational.

Ideas of the proof:
- select one normal form among all reduced expressions of an element through another finite state automaton
- conclude as generating series of languages recognizable by finite state automata are rational

Corollary

Let $W$ be a Coxeter group. The generating function of CFC element is algorithmically computable.
Thank you for your attention!