Triangulated ternary disc packings that maximize the density

Daria Pchelina
supervised by
Thomas Fernique
September 29, 2020
What is a packing?

Discs:

Packing $P$:
(in $\mathbb{R}^2$)

\[
\delta(P) = \limsup_{n \to \infty} \frac{\text{area}(\{-n, n\} \cap P)}{\text{area}(\{-n, n\} \cap \mathbb{R}^2)}
\]
What is a packing?

Discs:

Packing $P$: (in $\mathbb{R}^2$)

Density:

$$
\delta(P) = \limsup_{n \to \infty} \frac{\text{area}([-n, n]^2 \cap P)}{\text{area}([-n, n]^2)}
$$
What is a packing?

Discs:

Packing $P$:
(in $\mathbb{R}^2$)

Density:

$$\delta(P) = \limsup_{n \to \infty} \frac{\text{area}([-n, n]^2 \cap P)}{\text{area}([-n, n]^2)}$$
What is a packing?

Discs:

Packing $P$: (in $R^2$)

Density:

$$\delta(P) = \limsup_{n \to \infty} \frac{\text{area}([-n, n]^2 \cap P)}{\text{area}([-n, n]^2)}$$
What is a packing?

Discs:

Packing $P$: (in $\mathbb{R}^2$)

Density:

$$\delta(P) = \limsup_{n \to \infty} \frac{\text{area}(\left[ -n, n \right]^2 \cap P)}{\text{area}(\left[ -n, n \right]^2)}$$
What is a packing?

Discs:

Packing $P$: (in $\mathbb{R}^2$)

Density:

$$\delta(P) = \limsup_{n \to \infty} \frac{\text{area}([-n, n]^2 \cap P)}{\text{area}([-n, n]^2)}$$
What is a packing?

Discs:

Packing $P$:
(in $R^2$)

Density:

$$
\delta(P) = \limsup_{n \to \infty} \frac{\text{area}([-n, n]^2 \cap P)}{\text{area}([-n, n]^2)}
$$
What is a packing?

Discs:

Packing $P$: (in $\mathbb{R}^2$)

Density:

$$\delta(P) = \limsup_{n \to \infty} \frac{\text{area}([-n, n]^2 \cap P)}{\text{area}([-n, n]^2)}$$

Which packings maximize the density?
What is a packing?

Discs:

Packing $P$: (in $R^2$)

Density:

$$\delta(P) = \limsup_{n \to \infty} \frac{\text{area}([-n, n]^2 \cap P)}{\text{area}([-n, n]^2)}$$

Which packings maximize the density?
Why do we study packings?

- To pack fruits

![Fruit Packings](image-url)
Why do we study packings?

- To pack fruits
- and vegetables
Why do we study packings?

- To pack fruits
- and vegetables
- To make compact materials

2D hexagonal $\circ$-packing: $\delta = \frac{\pi}{2\sqrt{3}}$

Lagrange, 1772

Hexagonal packing maximize the density among $\circ$ lattice packings.

Thue, 1910 (Toth, 1940)

Hexagonal packing maximize the density.
2D hexagonal packing: \( \delta = \frac{\pi}{2\sqrt{3}} \)

**Lagrange, 1772**

Hexagonal packing maximize the density among \( \bullet \) lattice packings.

**Thue, 1910 (Toth, 1940)**

Hexagonal packing maximize the density.

3D hexagonal packing: \( \delta = \frac{\pi}{3\sqrt{2}} \)

**Gauss, 1831**

Hexagonal packing maximize the density among lattice \( \bullet \) packings.

**Hales, Ferguson, 1998–2014**  
(Conjectured by Kepler, 1611)

Hexagonal packing maximize the density.
Two discs of radii 1 and $r$:

**Lower bound** on the density: $\frac{\pi}{2\sqrt{3}}$ (hexagonal packing with only 1 disc used)
Two discs of radii 1 and $r$:

**Lower bound** on the density: $\frac{\pi}{2\sqrt{3}}$ (hexagonal packing with only 1 disc used)

**Upper bound** on the density:

Florian, 1960

The density of a packing never exceeds the density in the following triangle:
A packing is called **triangulated** if each “hole” is bounded by three tangent discs.

Kennedy, 2006

There are 9 values of $r$ allowing triangulated packings.
A packing is called **triangulated** if each “hole” is bounded by three tangent discs.

Kennedy, 2006

There are 9 values of $r$ allowing triangulated packings.

Heppes 2000, 2003
Kennedy 2004
Bedaride, Fernique, 2019:

All these 9 packings maximize the density.
**Conjecture (Connelly, 2018)**

If a finite set of discs allows a saturated triangulated packing then the density is maximized on a saturated triangulated packing.

**True** for ○ and ⬤.
**Conjecture** (Connelly, 2018)

If a finite set of discs allows a saturated triangulated packing then the density is maximized on a saturated triangulated packing.

True for \( \bigcirc \) and \( \bigcirc \cdot \).

What happens with \( \bigcirc \cdot \cdot \)?
3 discs

- 164 \((r, s)\) with triangulated packings: (Fernique, Hashemi, Sizova 2019)
- 15 non saturated packings
- Case 53 is proved (Fernique 2019)
- 14 more cases (the internship)
- 164 \((r, s)\) with triangulated packings: (Fernique, Hashemi, Sizova 2019)
- 15 non saturated
- Case 53 is proved (Fernique 2019)
- 14 more cases (the internship)
- The others?
A Delaunay triangulation of a packing: no points inside a circumscribed circle

\[ \delta^* = \delta_{\Delta^*} = \frac{\pi}{2\sqrt{3}} \]

\[ \forall \, \Delta, \; \delta_{\Delta} \leq \delta_{\Delta^*} = \delta^* \]
A Delaunay triangulation of a packing: no points inside a circumscribed circle

\[ \delta^* = \delta_{\Delta^*} = \frac{\pi}{2\sqrt{3}} \]

\[ \forall \Delta, \delta_\Delta \leq \delta_{\Delta^*} = \delta^* \]
A Delaunay triangulation of a packing: no points inside a circumscribed circle

\[ \delta^* = \delta_{\Delta^*} = \frac{\pi}{2\sqrt{3}} \]

\[ \forall \ \Delta, \ \delta_{\Delta} \leq \delta_{\Delta^*} = \delta^* \]

- The largest angle of any \( \triangle \) is between \( \frac{\pi}{3} \) and \( \frac{2\pi}{3} \)

\[ R = \frac{|AC|}{2 \sin B} \geq \frac{1}{\sin B} \]
A Delaunay triangulation of a packing: no points inside a circumscribed circle

$$\delta^* = \delta_{\Delta^*} = \frac{\pi}{2\sqrt{3}}$$

$$\forall \Delta, \delta_\Delta \leq \delta_{\Delta^*} = \delta^*$$

- The largest angle of any $\Delta$ is between $\frac{\pi}{3}$ and $\frac{2\pi}{3}$

- The density of a triangle $\Delta$: $\delta_\Delta = \frac{\pi/2}{\text{area}(\Delta)}$

$$R = \frac{|AC|}{2 \sin B} \geq \frac{1}{\sin B}$$
Idea of the proof for

A Delaunay triangulation of a packing: no points inside a circumscribed circle

\[ \delta^* = \delta_{\Delta^*} = \frac{\pi}{2\sqrt{3}} \]

\[ \forall \Delta, \; \delta_{\Delta} \leq \delta_{\Delta^*} = \delta^* \]

- The largest angle of any \( \Delta \) is between \( \frac{\pi}{3} \) and \( \frac{2\pi}{3} \)
- The density of a triangle \( \Delta \): \( \delta_{\Delta} = \frac{\pi/2}{\text{area(\( \Delta \))}} \)
- The area of a triangle \( ABC \) with the largest angle \( \hat{B} \) is \( \frac{1}{2} |AB| \cdot |BC| \cdot \sin \hat{B} \) which is at least \( \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \)

\[ R = \frac{|AC|}{2 \sin \hat{B}} \geq \frac{1}{\sin \hat{B}} \]
Idea of the proof for a Delaunay triangulation of a packing: no points inside a circumscribed circle

\[ \delta^* = \delta_{\Delta^*} = \frac{\pi}{2\sqrt{3}} \]

\[ \forall \Delta, \; \delta_\Delta \leq \delta_{\Delta^*} = \delta^* \]

- The largest angle of any \( \Delta \) is between \( \frac{\pi}{3} \) and \( \frac{2\pi}{3} \)

\[ R = \frac{|AC|}{2 \sin B} \geq \frac{1}{\sin B} \]

- The density of a triangle \( \Delta \): \( \delta_\Delta = \frac{\pi/2}{\text{area}(\Delta)} \)

- The area of a triangle \( ABC \) with the largest angle \( \hat{B} \) is \( \frac{1}{2} |AB| \cdot |BC| \cdot \sin \hat{B} \)
  which is at least \( \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \)

- Thus the density of \( ABC \) is less or equal to \( \frac{\pi/2}{\sqrt{3}} \)
Idea of the proof for Delaunay triangulation → weighted by the disc radii

Triangles have different densities:

\[ \delta(\text{blue}) \neq \delta(\text{red}) \]

What to do?

Redistribution of the densities:

Some triangles "share their density" with neighbors
Idea of the proof for Delaunay triangulation $\rightarrow$ weighted by the disc radii

Triangles have different densities:

$$\delta(\text{red}) \neq \delta(\text{blue})$$

What to do?

Redistribution of the densities:
Idea of the proof for Delaunay triangulation → weighted by the disc radii

Triangles have different densities:

$$\delta(\text{left}) \neq \delta(\text{right})$$

What to do?

Redistribution of the densities:

Some triangles “share their density” with neighbors
Idea of the proof for Delaunay triangulation $\rightarrow$ weighted by the disc radii

Triangles have different densities:

$$\delta(\text{red}) \neq \delta(\text{blue})$$

What to do?

Redistribution of the densities:

Some triangles “share their density” with neighbors
Idea of the proof for Delaunay triangulation → weighted by the disc radii

Triangles have different densities:
\[ \delta(\text{red}) \neq \delta(\text{blue}) \]

What to do?

Redistribution of the densities:

Some triangles “share their density” with neighbors
Proof for

\( \mathcal{T}^\ast \) – saturated triangulated packing of density \( \delta \)

\( \mathcal{T} \) – any other saturated packing with the same discs

The sparsity of a triangle \( \triangle \in \mathcal{T} \):

\[
S(\triangle) = \delta \times \text{area}(\triangle) - \text{cov}(\triangle)
\]

\( S(\triangle) > 0 \) iff the density of covering of \( \triangle \) is less than \( \delta \)

\( S(\triangle) < 0 \) iff the density of covering of \( \triangle \) is greater than \( \delta \)

To prove that \( \mathcal{T}^\ast \) is no denser than \( \mathcal{T} \), we show that

\[
\sum_{\triangle \in \mathcal{T}} S(\triangle) \geq 0
\]

1: Introduce a potential \( U \) such that for any triangle \( \triangle \in \mathcal{T} \),

\[
S(\triangle) \geq U(\triangle)
\]

and

\[
\sum_{\triangle \in \mathcal{T}} U(\triangle) \geq 0
\]
$\mathcal{T}^*$ – saturated triangulated packing of density $\delta$

$\mathcal{T}$ – any other saturated packing with the same discs

The **sparsity** of a triangle $\triangle \in \mathcal{T}$: $S(\triangle) = \delta \times \text{area}(\triangle) - \text{cov}(\triangle)$

- $S(\triangle) > 0$ iff the density of covering of $\triangle$ is less than $\delta$
- $S(\triangle) < 0$ iff the density of covering of $\triangle$ is greater than $\delta$

To prove that $\mathcal{T}$ is no denser than $\mathcal{T}^*$, we show that $\sum_{\mathcal{T}} S(\triangle) \geq 0$
\( \mathcal{T}^* \) – saturated triangulated packing of density \( \delta \)

\( \mathcal{T} \) – any other saturated packing with the same discs

The sparsity of a triangle \( \triangle \in \mathcal{T} \): \( S(\triangle) = \delta \times \text{area}(\triangle) - \text{cov}(\triangle) \)

\( S(\triangle) > 0 \) iff the density of covering of \( \triangle \) is less than \( \delta \)

\( S(\triangle) < 0 \) iff the density of covering of \( \triangle \) is greater than \( \delta \)

To prove that \( \mathcal{T} \) is no denser than \( \mathcal{T}^* \), we show that \( \sum_{\mathcal{T}} S(\triangle) \geq 0 \)

1: Introduce a potential \( U \) such that for any triangle \( \triangle \in \mathcal{T} \),

\[ S(\triangle) \geq U(\triangle) \quad (\Delta) \]

and

\[ \sum_{\Delta \in \mathcal{T}} U(\Delta) \geq 0 \quad (U) \]
2: Instead of proving a global inequality

\[ \sum_{\Delta \in \mathcal{T}} U(\Delta) \geq 0 \quad (U) \]

we define the vertex potential: for a triangle \( \Delta \) with vertices \( A, B \) and \( C \),

\[ U(\Delta) = \dot{U}_\Delta^A + \dot{U}_\Delta^B + \dot{U}_\Delta^C \]

and prove a local inequality for each vertex \( v \in \mathcal{T} \):

\[ \sum_{\Delta \in \mathcal{T} | v \in \Delta} \dot{U}_\Delta^v \geq 0 \quad (\bullet) \]
Proof for 2: Instead of proving a global inequality

\[ \sum_{\triangle \in \mathcal{T}} U(\triangle) \geq 0 \]  \hfill (U)

we define the vertex potential: for a triangle \( \triangle \) with vertices \( A, B \) and \( C \),

\[ U(\triangle) = \dot{U}_\triangle^A + \dot{U}_\triangle^B + \dot{U}_\triangle^C \]

and prove a local inequality for each vertex \( v \in \mathcal{T} \):

\[ \sum_{\triangle \in \mathcal{T} \mid v \in \triangle} \dot{U}_\triangle^v \geq 0 \]  \hfill (\bullet)

Delaunay triangulation properties \( \rightarrow \) finite number of cases \( \rightarrow \) verification by computer

4\( \dot{U}_{\triangle_1}^v \) + 2\( \dot{U}_{\triangle_2}^v \) + \( \dot{U}_{\triangle_3}^v \) = 0

\( \dot{U}_{\triangle_1}^{v'} + \dot{U}_{\triangle_2}^{v'} + \dot{U}_{\triangle_3}^{v'} + \dot{U}_{\triangle_4}^{v'} > 0 \)
To store and perform computations on transcendental numbers (like $\pi$), we use intervals.

A representation of a number $x$ is an interval $I$ whose endpoints are exact values representable in a computer memory and such that $x \in I$.

```
sage: x = RIF(0,1)  # Interval [0,1]
sage: (x+x).endpoints()  # [0,1]+[0,1]
(0.0, 2.0)
sage: x < 2  # ∀t ∈ [0,1], t < 2
True
```
To store and perform computations on transcendental numbers (like $\pi$), we use intervals.

A representation of a number $x$ is an interval $I$ whose endpoints are exact values representable in a computer memory and such that $x \in I$.

```
sage: x = RIF(0,1)  # Interval $[0,1]$
sage: (x+x).endpoints()  # $[0,1]+[0,1]$
(0.0, 2.0)
sage: x < 2  # $\forall t \in [0,1], t < 2$
True
```

```
sage: Ipi = RIF(pi)  # Interval for $\pi$
(3.14159265358979, 3.14159265358980)
sage: sin(Ipi).endpoints()  # Interval for $\sin(\pi)$
(-3.21624529935328e-16, 1.22464679914736e-16)
sage: sin(Ipi) >= 0  # Interval for $\sin(\pi)$ contains 0
False
```
Defining $U$, we try to make it as small as possible keeping it locally positive around any vertex ($\bullet$).

3: How to check

$$S(\triangle) \geq U(\triangle)$$

(\triangle) on each triangle $\triangle$? (There is a continuum of them).
Proving a continuum of inequalities with interval arithmetic

Defining $U$, we try to make it as small as possible keeping it locally positive around any vertex ($\bullet$).

3: How to check

$$S(\triangle) \geq U(\triangle)$$

on each triangle $\triangle$? (There is a continuum of them).

Interval arithmetic!

Delaunay triangulation properties $\rightarrow$ uniform bound on edge length:

Verify $S(\triangle_{e_1,e_2,e_3}) \geq U(\triangle_{e_1,e_2,e_3})$ where

$$e_1 = [r_a+r_b, r_a+r_b+2s] \quad e_2 = [r_c+r_b, r_c+r_b+2s] \quad e_3 = [r_a+r_c, r_a+r_c+2s]$$

Not precise enough $\rightarrow$ dichotomy
Conclusion

What was done and what will be done...

- 14 cases proved
- 133 cases to prove (Connelly’s conjecture)
- Maximal density for other disc sizes (which do not allow triangulated packings)

Various techniques: computer-assisted proofs, interval arithmetic, optimisation, combinatorics, discrete geometry

For this: good comprehension of the density redistribution, more optimisation

Deformations of triangulated packings keep the density high → good lower bound on the maximal density