A survey on Riordan arrays

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Outline

1. Some history
2. Main properties of Riordan arrays
3. Riordan arrays and binary words avoiding a pattern
4. Riordan arrays, combinatorial sums and recursive matrices
I’m very sorry to have not met P. Flajolet in the recent years.
I’m very sorry to have not met P. Flajolet in the recent years. I remember with pleasure my seminar at INRIA on October 10, 1994: *Riordan arrays and their applications*


References -2-


References -2-


Some history

Main properties of Riordan arrays

Riordan arrays and binary words avoiding a pattern

Riordan arrays, combinatorial sums and recursive matrices

References -2-


The bibliography on the subject is vast and still growing.
A **Riordan array** is a pair

\[ D = R(d(t), h(t)) \]

in which \( d(t) \) and \( h(t) \) are formal power series such that \( d(0) \neq 0 \) and \( h(0) = 0 \); if \( h'(0) \neq 0 \) the Riordan array is called **proper**.
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The pair defines an infinite, lower triangular array \( (d_{n,k})_{n,k \in \mathbb{N}} \) where:

\[ d_{n,k} = [t^n]d(t)(h(t))^k \]
An example: the Pascal triangle

\[ P = \mathcal{R} \left( \frac{1}{1-t}, \frac{t}{1-t} \right) \]

\[ d_{n,k} = [t^n] \frac{1}{1-t} \cdot \frac{t^k}{(1-t)^k} = [t^{n-k}](1-t)^{-k-1} = \binom{n}{k} \]

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An example: the Catalan triangle

\[
C = \mathcal{R} \left( \frac{1 - \sqrt{1 - 4t}}{2t}, \frac{1 - \sqrt{1 - 4t}}{2} \right)
\]

\[
d_{n,k} = [t^n]d(t)(h(t))^k = [t^{n+1}] \left( \frac{1 - \sqrt{1 - 4t}}{2} \right)^{k+1} = \frac{k + 1}{n + 1} \binom{2n - k}{n - k}
\]

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The Group structure

Product: \( \mathcal{R}(d(t), h(t)) \ast \mathcal{R}(a(t), b(t)) = \mathcal{R}(d(t)a(h(t)), b(h(t))) \)

Identity: \( \mathcal{R}(1, t) \)

Inverse: \( \mathcal{R}(d(t), h(t))^{-1} = \mathcal{R} \left( \frac{1}{d(h(t))}, \overline{h}(t) \right) \)

\( h(\overline{h}(t)) = \overline{h}(h(t)) = t \)
Pascal triangle: product and inverse

\[ P = \mathcal{R} \left( \frac{1}{1-t}, \frac{t}{1-t} \right) \]

\[ P \ast P = \mathcal{R} \left( \frac{1}{1-t}, \frac{t}{1-t} \right) \ast \mathcal{R} \left( \frac{1}{1-t}, \frac{t}{1-t} \right) = \]

\[ = \mathcal{R} \left( \frac{1}{1-t} \frac{1-t}{1-2t}, \frac{t}{1-t} \frac{1-t}{1-2t} \right) = \mathcal{R} \left( \frac{1}{1-2t}, \frac{t}{1-2t} \right). \]

\[ P^{-1} = \mathcal{R} \left( \frac{1}{1+t}, \frac{t}{1+t} \right) \]
Subgroups

**APPELL**

\[
\mathcal{R}(d(t), t) \ast \mathcal{R}(a(t), t) = \mathcal{R}(d(t)a(t), t)
\]

\[
\mathcal{R}(d(t), t)^{-1} = \mathcal{R}\left(\frac{1}{d(t)}, t\right)
\]

**LAGRANGE**

\[
\mathcal{R}(1, h(t)) \ast \mathcal{R}(1, b(t)) = \mathcal{R}(1, h(b(t)))
\]

\[
\mathcal{R}(1, h(t))^{-1} = \mathcal{R}(1, \overline{h(t)})
\]

**RENEWAL**

\[
d(t) = h(t)/t
\]

**HITTING – TIME**

\[
d(t) = \frac{th'(t)}{h(t)}
\]
Inversion of Riordan arrays

\[ R(d(t), h(t))^{-1} = R \left( \frac{1}{d(h(t))}, \bar{h}(t) \right) \]

Every Riordan array is the product of an Appell and a Lagrange Riordan array

\[ R(d(t), h(t)) = R(d(t), t) \ast R(1, h(t)) \]

From this fact we obtain the formula for the inverse Riordan array
Pascal triangle: construction by columns

\[
d(t)h(t)^k \text{ is the g.f. of column } k
\]

\[
\begin{array}{c}
\frac{1}{1-t} \quad \frac{t}{(1-t)^2} \quad \frac{t^2}{(1-t)^3} \quad \cdots
\end{array}
\]

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Pascal triangle: construction by rows

\[
\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}
\]
The \(A\) and \(Z\) sequences

An alternative definition, is in terms of the so-called \(A\)-sequence and \(Z\)-sequence, with generating functions \(A(t)\) and \(Z(t)\) satisfying the relations:

\[
h(t) = tA(h(t)), \quad d(t) = \frac{d_0}{1 - tZ(h(t))} \quad \text{with} \quad d_0 = d(0).
\]

\[
d_{n+1,k+1} = a_0 d_{n,k} + a_1 d_{n,k+1} + a_2 d_{n,k+2} + \cdots
\]

\[
d_{n+1,0} = z_0 d_{n,0} + z_1 d_{n,1} + z_2 d_{n,2} + \cdots
\]

Pascal triangle: \(A\)-sequence \(1, 1, 0, 0, \cdots \implies A(t) = 1 + t\)
### The $A$-sequence for the Catalan triangle

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$A$-sequence $1, 1, 1, 1, \cdots \implies A(t) = \frac{1}{1-t}$
Rogers’ Theorem - 1978

The A-sequence is unique and only depends on $h(t)$

$$h(t) = tA(h(t))$$

Pascal

$$h(t) = t(1 + h(t))$$

$$h_P(t) = \frac{t}{1 - t}$$

Catalan

$$h(t) = t\frac{1}{1 - h(t)}$$

$$h_C(t) = \frac{1 - \sqrt{1 - 4t}}{2}.$$
The \( B \)-sequence: \( B(t) = A(t)^{-1} \)

\( d_{n,k} \) linearly depends on the elements of row \( n + 1 \)

\[
\begin{array}{c|cccccc}
 n/k & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
 0 & 1 \\
 1 & 1 & 1 \\
 2 & 1 & 2 & 1 \\
 3 & 1 & 3 & 3 & 1 \\
 4 & 1 & 4 & 6 & 4 & 1 \\
 5 & 1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

\[
\sum_{j=0}^{n} (-1)^j \binom{n+1}{k+j+1} = \binom{n}{k}
\]
A-approach to R.a.’s

Product \[ A_3(t) = A_2(t)A_1 \left( \frac{t}{A_2(t)} \right) \]

Inverse \[ A^*(t) = \left[ \frac{1}{A(y)} \mid y = tA(y) \right] \]

\[ A_{P* C}(t) = \frac{1}{1 - t} \left[ 1 + y \mid y = t(1 - t) \right] = \frac{1 + t - t^2}{1 - t} \]

\[ A_{C* P}(t) = (1 + t) \left[ \frac{1}{1 - y} \mid y = \frac{t}{1 + t} \right] = (1 + t)^2 \]

\[ A_{P^{-1}}(t) = \left[ \frac{1}{1 + y} \mid y = t(1 + y) \right] = 1 - t \]
Pascal triangle: the $A$-matrix (not unique)

$$
\begin{array}{c|cccccc}
 n/k & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
 0 & 1 \\
 1 & 1 & 1 \\
 2 & 1 & 2 & 1 \\
 3 & 1 & 3 & 3 & 1 \\
 4 & 1 & 4 & 6 & 4 & 1 \\
 5 & 1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
$$

\begin{align*}
P[0](t) &= 1 \\
P[1](t) &= 1 + t \\
A(t) &= \frac{P[0](t) + \sqrt{P[0](t)^2 + 4tP[1](t)}}{2} \\
A(t) &= \frac{1 + \sqrt{1 + 4t + 4t^2}}{2} = 1 + t
\end{align*}
The A-matrix in general

\[ d_{n+1,k+1} = \sum_{i \geq 0} \sum_{j \geq 0} \alpha_{i,j} d_{n-i,k+j} + \sum_{j \geq 0} \rho_j d_{n+1,k+j+2}. \]

Matrix \((\alpha_{i,j})_{i,j \in \mathbb{N}}\) is called the A-matrix of the Riordan array. If, for \(i \geq 0:\)

\[ P[i](t) = \alpha_{i,0} + \alpha_{i,1} t + \alpha_{i,2} t^2 + \alpha_{i,3} t^3 + \ldots \]

and \(Q(t)\) is the generating function for the sequence \((\rho_j)_{j \in \mathbb{N}}\), then we have:

\[ \frac{h(t)}{t} = \sum_{i \geq 0} t^i P[i](h(t)) + \frac{h(t)^2}{t} Q(h(t)). \]

\[ A(t) = \sum_{i \geq 0} t^i A(t)^{-i} P[i](t) + tA(t)Q(t). \]
A graphical representation of the $A$-matrix

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot \\
\alpha_{1,0} & \alpha_{1,1} & \cdots \\
\alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} & \cdots \\
\rho_0 & \rho_1 & \cdots \\
\end{array}
\]
We consider the language of binary words with no occurrence of a pattern $p = p_0 \cdots p_{h-1}$. 
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The problem of determining the generating function counting the number of words with respect to their length has been studied by several authors.
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The fundamental notion is that of the *autocorrelation vector* of bits $c = (c_0, \ldots, c_{h-1})$ associated to a given $p$. 

The pattern $p = 00011$
The pattern $p = 00011$
The pattern $p = 00011$

\[
\begin{array}{ccccc|c|cc}
0 & 0 & 0 & 1 & 1 & \text{Tails} \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
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\]
The pattern $p = 00011$
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The autocorrelation vector is then $c = (1, 0, 0, 0, 0, 0)$
Let $F_{n,k}^\mathcal{P}$ denote the number of words excluding the pattern and having $n$ bits 1 and $k$ bits 0, then we have

$$F^\mathcal{P}(x, y) = \sum_{n,k \geq 0} F_{n,k}^\mathcal{P} x^n y^k = \frac{C^\mathcal{P}(x, y)}{(1 - x - y)C^\mathcal{P}(x, y) + x^{n_1^\mathcal{P}} y^{n_0^\mathcal{P}}},$$

where $n_1^\mathcal{P}$ and $n_0^\mathcal{P}$ correspond to the number of ones and zeroes in the pattern and $C^\mathcal{P}(x, y)$ is the bivariate autocorrelation polynomial.
An example with $p = 110011$

We have $C[p](x, y) = 1 + x^2 y^2 + x^3 y^2$, and:

$$F[p](x, y) = \frac{1 + x^2 y^2 + x^3 y^2}{(1 - x - y)(1 + x^2 y^2 + x^3 y^2) + x^4 y^2}.$$
...the lower and upper triangular parts

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<td>122</td>
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</table>
Let $R_{n,k}^{[p]} = F_{n,n-k}^{[p]}$ with $k \leq n$. More precisely, $R_{n,k}^{[p]}$ counts the number of words avoiding $p$ with $n$ bits one and $n - k$ bits zero.
Matrices $R^{[p]}$ and $R^{[\bar{p}]}$

- Let $R_{n,k}^{[p]} = F_{n,n-k}$ with $k \leq n$. More precisely, $R_{n,k}^{[p]}$ counts the number of words avoiding $p$ with $n$ bits one and $n-k$ bits zero.
- Let $\bar{p} = \bar{p}_0 \ldots \bar{p}_{h-1}$ be the conjugate pattern.
Matrices $\mathcal{R}^{[p]}$ and $\mathcal{R}^{[\bar{p}]}$

- Let $R^{[p]}_{n,k} = F^{[p]}_{n,n-k}$ with $k \leq n$. More precisely, $R^{[p]}_{n,k}$ counts the number of words avoiding $p$ with $n$ bits one and $n-k$ bits zero.
- Let $\bar{p} = \bar{p}_0 \ldots \bar{p}_{h-1}$ be the conjugate pattern.
- We obviously have $R^{[\bar{p}]}_{n,k} = F^{[\bar{p}]}_{n,n-k} = F^{[p]}_{n-k,n}$, therefore, the matrices $\mathcal{R}^{[p]}$ and $\mathcal{R}^{[\bar{p}]}$ represent the lower and upper triangular part of the array $F^{[p]}$, respectively.
Riordan patterns

- When matrices $\mathcal{R}^{[p]}$ and $\mathcal{R}^{[\bar{p}]}$ are both Riordan arrays?
When matrices $\mathcal{R}^p$ and $\mathcal{R}^{p^\top}$ are both Riordan arrays?

Riordan patterns

- When matrices $R[p]$ and $R[\bar{p}]$ are both Riordan arrays?
- We say that $p = p_0...p_{h-1}$ is a Riordan pattern if and only if

$$C[p](x, y) = C[p](y, x) = \sum_{i=0}^{\lfloor (h-1)/2 \rfloor} c_{2i} x^i y^i, \ |n^1[p] - n^0[p]| \in \{0, 1\}. $$
Main Theorem -1-

The matrices $\mathcal{R}^p$ and $\mathcal{R}^{\overline{p}}$ are both Riordan arrays
$\mathcal{R}^p = (d^p(t), h^p(t))$ and $\mathcal{R}^{\overline{p}} = (d^{\overline{p}}(t), h^{\overline{p}}(t))$ if and only if $p$ is a Riordan pattern. Moreover we have:

$$d^p(t) = d^{\overline{p}}(t) = [x^0] F \left( x, \frac{t}{x} \right) = \frac{1}{2\pi i} \oint F \left( x, \frac{t}{x} \right) \frac{dx}{x}$$

and

$$h^p(t) = \frac{1 - \sum_{i=0}^{n_1^p-1} \alpha_{i,1} t^{i+1} - \sqrt{(1 - \sum_{i=0}^{n_1^p-1} \alpha_{i,1} t^{i+1})^2 - 4 \sum_{i=0}^{n_1^p-1} \alpha_{i,0} t^{i+1} (\sum_{i=0}^{n_1^p-1} \alpha_{i,2} t^{i+1} + 1)}}{2(\sum_{i=0}^{n_1^p-1} \alpha_{i,2} t^{i+1} + 1)}$$
Main Theorem -2-

... where $\delta_{i,j}$ is the Kronecker delta,

$$\sum_{i=0}^{n_1^p-1} \alpha_{i,0} t^i = \sum_{i=0}^{n_1^p-1} c_{2i} t^i - \delta_{1,0} n_0^p - n_1^p t^{n_1^p-1},$$

$$\sum_{i=0}^{n_1^p-1} \alpha_{i,1} t^i = -\sum_{i=0}^{n_1^p-1} c_{2(i+1)} t^i - \delta_{0,0} n_0^p - n_1^p t^{n_1^p-1},$$

$$\sum_{i=0}^{n_1^p-1} \alpha_{i,2} t^i = \sum_{i=0}^{n_1^p-1} c_{2(i+1)} t^i - \delta_{1,0} n_0^p - n_1^p t^{n_1^p-1},$$

and the coefficients $c_i$ are given by the autocorrelation vector of $p$. An analogous formula holds for $h^{[\bar{p}]}(t)$. 
A Corollary

Let $p$ be a Riordan pattern. Then the Riordan array $\mathcal{R}^{[p]}$ is characterized by the $A$-matrix defined by the following relation:

$$\begin{align*}
R^{[p]}_{n+1,k+1} &= R^{[p]}_{n,k} + R^{[p]}_{n+1,k+2} - R^{[p]}_{n+1-n_1^p,k+1+n_0^p-n_1^p} + \\
& - \sum_{i \geq 1} c_{2i} \left( R^{[p]}_{n+1-i,k+1} - R^{[p]}_{n-i,k} - R^{[p]}_{n+1-i,k+2} \right),
\end{align*}$$

where the $c_i$ are given by the autocorrelation vector of $p$. 
The coefficients in the gray circles are negative, \( s = 2n_1^p \), \( q = 2(n_1^p - 1) \). Moreover, we have to consider the contribution of

\[-R^{[p]}_{n+1-n_1^p,k+1+n_0^p-n_1^p}.\]
The case $n_1^p - n_0^p = 1$

By specializing the main result to the cases $|n_1^p - n_0^p| \in \{0, 1\}$ and by setting $C[p](t) = C[p](\sqrt{t}, \sqrt{t}) = \sum_{i \geq 0} c_{2i} t^i$, we have the following explicit generating functions:

$$d[p](t) = \frac{C[p](t)}{\sqrt{C[p](t)^2 - 4t C[p](t)(C[p](t) - t n_0^p)}},$$

$$h[p](t) = \frac{C[p](t) - \sqrt{C[p](t)^2 - 4t C[p](t)(C[p](t) - t n_0^p)}}{2 C[p](t)}.$$
The case $n_{1}^{[p]} - n_{0}^{[p]} = 0$

\[ d^{[p]}(t) = \frac{C^{[p]}(t)}{\sqrt{(C^{[p]}(t) + t^{n_{0}^{p}})^2 - 4tC^{[p]}(t)^2}}, \]

\[ h^{[p]}(t) = \frac{C^{[p]}(t) + t^{n_{0}^{p}} - \sqrt{(C^{[p]}(t) + t^{n_{0}^{p}})^2 - 4tC^{[p]}(t)^2}}{2C^{[p]}(t)}. \]
The case $n_0^{[p]} - n_1^{[p]} = 1$

\[
d^{[p]}(t) = \frac{C^{[p]}(t)}{\sqrt{C^{[p]}(t)^2 - 4tC^{[p]}(t)(C^{[p]}(t) - t^n_1^{[p]})}},
\]

\[
h^{[p]}(t) = \frac{C^{[p]}(t) - \sqrt{C^{[p]}(t)^2 - 4tC^{[p]}(t)(C^{[p]}(t) - t^n_1^{[p]})}}{2(C^{[p]}(t) - t^n_1^{[p]})}.
\]
### An example with $p = 00011$

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\[
d^{[p]}(t) = \frac{1}{\sqrt{1 - 4t + 4t^3}}
\]

\[
h^{[p]}(t) = \frac{1 - \sqrt{1 - 4t + 4t^3}}{2(1 - t^2)}
\]

\[
R^{[p]}_{n+1,k+1} = R^{[p]}_{n,k} + R^{[p]}_{n+1,k+2} - R^{[p]}_{n-1,k+2}.
\]
The $A$-sequence for $p = 00011$

For $p = 00011$, we find after setting $R(t) = \sqrt{1 + 4t^4 - 4t^3}$:

$$A(t) = \frac{(2t^3 - t^2 - t - 1 - (t^2 + t + 1)R(t)) (2t^3 - \sqrt{2}\sqrt{2t^6 + 8t^4 - 12t^3 + 4 - (4 - 4t^3)R(t)})}{8t^4(t - 1)(t + 1)}$$

$$= 1 + t + t^2 + t^4 + t^5 + 2t^7 + t^8 - t^9 + 5t^{10} - t^{11} - 4t^{12} + 16t^{13} - 14t^{14} - 8t^{15} + 57t^{16} - 83t^{17} + 15t^{18} + 197t^{19} + O(t^{20}).$$
The $A$-sequence for $p = 00011$

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In general, the Riordan arrays for binary words avoiding $p$ are characterized by a complex $A$-sequence, while the $A$-matrix is quite simple. However, the presence of negative coefficients leads to non trivial combinatorial interpretations.
The $A$-sequence for $p = 00011$

- For $p = 00011$, we find after setting $R(t) = \sqrt{1 + 4t^4 - 4t^3}$:

$$A(t) = \frac{\left(2t^3 - t^2 - t - 1 - (t^2 + t + 1)R(t)\right) \left(2t^3 - \sqrt{2}\sqrt{2t^6 + 8t^4 - 12t^3 + 4 - (4 - 4t^3)R(t)}\right)}{8t^4(t - 1)(t + 1)}$$

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- In general, the Riordan arrays for binary words avoiding $p$ are characterized by a complex $A$-sequence, while the $A$-matrix is quite simple. However, the presence of negative coefficients leads to non trivial combinatorial interpretations.

- S. Bilotta, D. Merlini, E. Pergola, R. Pinzani. Pattern $1^{j+1}0^j$ avoiding binary words. To appear in *Fundamenta Informaticae*. 
Formulas relative to whole classes of patterns

- \( p = 1^j + 1 \cdot 0^j \)
  
  \[
  d^{[p]}(t) = \frac{1}{\sqrt{1 - 4t + 4t^{j+1}}}, \quad h^{[p]}(t) = \frac{1 - \sqrt{1 - 4t + 4t^{j+1}}}{2}
  \]

- \( p = 0^j + 1 \cdot 1^j \)
  
  \[
  d^{[p]}(t) = \frac{1}{\sqrt{1 - 4t + 4t^{j+1}}}, \quad h^{[p]}(t) = \frac{1 - \sqrt{1 - 4t + 4t^{j+1}}}{2(1 - t^j)}
  \]

- \( p = 1^j \cdot 0^j \) and \( p = 0^j \cdot 1^j \)
  
  \[
  d^{[p]}(t) = \frac{1}{\sqrt{1 - 4t + 2t^j + t^{2j}}}, \quad h^{[p]}(t) = \frac{1 + t^j - \sqrt{1 - 4t + 2t^j + t^{2j}}}{2}
  \]

- \( p = (10)^j \cdot 1 \)
  
  \[
  d^{[p]}(t) = \frac{\sum_{i=0}^{j} t^i}{\sqrt{1 - 2 \sum_{i=1}^{j} t^i - 3 \left( \sum_{i=1}^{j} t^i \right)^2}}, \quad h^{[p]}(t) = \frac{\sum_{i=0}^{j} t^i - \sqrt{1 - 2 \sum_{i=1}^{j} t^i - 3 \left( \sum_{i=1}^{j} t^i \right)^2}}{2 \sum_{i=0}^{j} t^i}
  \]
Riordan array summation

\[ \sum_{k=0}^{n} d_{n,k} f_k = [t^n] d(t)f(h(t)) \]

Partial sum theorem:

\[ \sum_{k=0}^{n} f_k = [t^n] \frac{f(t)}{1-t} \]

Euler transformation:

\[ \sum_{k=0}^{n} \binom{n}{k} f_k = [t^n] \frac{1}{1-t} f \left( \frac{t}{1-t} \right). \]
A simple example: Harmonic numbers

\[ G\left(\frac{1}{n}\right) = \ln \frac{1}{1-t} \]

\[ G\left(\sum_{k=1}^{n} \frac{1}{k}\right) = G(H_n) = \frac{1}{1-t} \ln \frac{1}{1-t} \]

\[ \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k-1}}{k} = \]

\[ = [t^n] \frac{1}{1-t} \left[ \ln \frac{1}{1+w} \bigg| w = \frac{t}{1-t} \right] = \]

\[ = [t^n] \frac{1}{1-t} \ln \frac{1}{1-t} = H_n. \]
General rules for binomial coefficients

\begin{align*}
\sum_k \left( \binom{n + ak}{m + bk} \right) f_k &= [t^n] \frac{t^m}{(1 - t)^{m+1}} f \left( \frac{t^{b-a}}{(1 - t)^{b}} \right) \quad b > a \\
\sum_k \left( \binom{n + ak}{m + bk} \right) f_k &= [t^m](1 + t)^n f(t^{-b}(1 + t)^a) \quad b < 0
\end{align*}

\begin{align*}
\sum_k \left( \binom{n + k}{m + 2k} \binom{2k}{k} \frac{(-1)^k}{k + 1} \right) &= [t^n] \frac{t^m}{(1 - t)^{m+1}} \left[ \frac{\sqrt{1 + 4y} - 1}{2y} \right]_{y = \frac{t}{(1 - t)^2}} \\
&= [t^{n-m}] \frac{1}{(1 - t)^{m+1}} \left( \sqrt{1 + \frac{4t}{(1 - t)^2}} - 1 \right) \frac{(1 - t)^2}{2t} = [t^{n-m}] \frac{1}{(1 - t)^m} = \binom{n-1}{m-1}.
\end{align*}
General rules for binomial coefficients

\[
\sum_k \binom{n + ak}{m + bk} f_k = [t^n] \frac{t^m}{(1 - t)^{m+1}} f \left( \frac{t^{b-a}}{(1 - t)^b} \right) \quad b > a
\]

\[
\sum_k \binom{n + ak}{m + bk} f_k = [t^m](1 + t)^n f(t^{-b}(1 + t^a)) \quad b < 0
\]

\[
\sum_k \binom{n + k}{m + 2k} \binom{2k}{k} \frac{(-1)^k}{k + 1} = [t^n] \frac{t^m}{(1 - t)^{m+1}} \left[ \frac{\sqrt{1 + 4y} - 1}{2y} \right] \bigg| y = \frac{t}{(1 - t)^2} = [t^{n-m}] \frac{1}{(1 - t)^m} = \binom{n-1}{m-1}.
\]

Recursive matrices

Recursive matrices


\[ D = \mathcal{X}(d(t), h(t)) \]

\[ d_{n,k} = [t^n]d(t)h(t)^k \quad n, k \in \mathbb{Z} \]
Recursive matrices


\[
D = \mathcal{X}(d(t), h(t))
\]

\[
d_{n,k} = [t^n]d(t)h(t)^k \quad n, k \in \mathbb{Z}
\]

- The introduction of recursive matrices simply extends the properties of Riordan arrays.
### The Pascal recursive matrix

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Generalized Sums

**Identities with three parameters** \( k, n, m \in \mathbb{Z} \)

\[
d_{n+m,k+m} = \sum_{j=0}^{n-k} a_j^{(m)} d_{n,k+j} = \sum_{j=0}^{n-k} h_{j+m}^{(m)} d_{n-j,k}
\]

\[
a_j^{(m)} = [t^j] A(t)^m
\]

\[
h_{j+m}^{(m)} = [t^{j+m}] h(t)^m = [t^j] (h(t)/t)^m
\]
Generalized Sums for the Catalan triangle

\[
\sum_{j=0}^{n-k} \binom{m+j-1}{j} \frac{k+j+1}{n+1} \binom{2n-j-k}{n-j-k} = \\
= \frac{k+m+1}{n+m+1} \binom{2n+m-k}{n-k}.
\]

\[
\sum_{j=0}^{n-k} \frac{m}{m+2j} \binom{m+2j}{j} \frac{k+1}{n-j+1} \binom{2n-2j-k}{n-j-k} = \\
= \frac{k+m+1}{n+m+1} \binom{2n+m-k}{n-k}.
\]
Specializing the parameters

\[ n \mapsto n, \ m \mapsto n, \ k \mapsto 0 \]

\[ \sum_{j=0}^{n} \frac{j + 1}{n + 1} \binom{n + j - 1}{j} \binom{2n - j}{n - j} = \frac{n + 1}{2n + 1} \binom{3n}{n} \]

\[ \sum_{j=0}^{n} \frac{n}{n + 2j} \binom{n + 2j}{j} \frac{1}{n - j + 1} \binom{2n - 2j}{n - j} = \frac{n + 1}{2n + 1} \binom{3n}{n} \]

\[ n \mapsto 2n, \ m \mapsto n, \ k \mapsto n \]

\[ \sum_{j=0}^{n} \frac{n + j + 1}{2n + 1} \binom{n + j - 1}{j} \binom{3n - j}{n - j} = \frac{2n + 1}{3n + 1} \binom{4n}{n} \]

\[ \sum_{j=0}^{n} \frac{n}{n + 2j} \binom{n + 2j}{j} \frac{n + 1}{2n - j + 1} \binom{3n - 2j}{n - j} = \frac{2n + 1}{3n + 1} \binom{4n}{n} \]
Work in progress: the complementary Riordan array

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\[ D^\perp = \mathcal{R}(d(\bar{h}(t))\bar{h}'(t), \bar{h}(t)) = \mathcal{R}(\frac{1 - 2t}{1 - t}, t(1 - t)) \]
Thank you for your attention and for the invitation
Exercise: find the identities induced by Pascal triangle.
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\[ d_{n+m, k+m} = \sum_{j=0}^{n-k} a_j^{(m)} d_{n, k+j} = \sum_{j=0}^{n-k} h_{j+m}^{(m)} d_{n-j, k} \]
Exercise: find the identities induced by Pascal triangle.

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- \( a_j^{(m)} = [t^j](1 + t)^m = \binom{m}{j} \)

- \( h_m^{m+j} = [t^{j+m}]{\left(\frac{t}{1-t}\right)^m} = \binom{m+j-1}{j} \)
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Well! You have proved Vandermonde's identity.
Exercise: find the identities induced by Pascal triangle.

- \( d_{n+m,k+m} = \sum_{j=0}^{n-k} a_j^{(m)} d_{n,k+j} = \sum_{j=0}^{n-k} h_j^{(m)} d_{n-j,k} \)

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a_j^{(m)} = [t^j](1 + t)^m = \binom{m}{j}
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\]

Well! You have proved Vandermonde’s identity

\[
\binom{n+m}{k+m} = \sum_{j=0}^{n} \binom{m+j-1}{j} \binom{n-j}{k}.
\]
Exercise: find $A^{[p]}(t)$ for $p = 10101$

$$C^{[p]}(x, y) = 1 + xy + x^2 y^2 \Rightarrow Q(t) = 1, \quad P^{[0]}(t) = P^{[1]}(t) = 1 - t + t^2$$

Moreover, we have to consider the contribution of $-R^{[p]}_{n+1,n_1^{p},k+1+n_0^{p} - n_1^{p}} = -R^{[p]}_{n-2,k}$. 
Exercise: find $A^{[p]}(t)$ for $p = 10101$

$$C^{[p]}(x, y) = 1 + xy + x^2y^2 \Rightarrow Q(t) = 1, \quad P^{[0]}(t) = P^{[1]}(t) = 1 - t + t^2$$

Moreover, we have to consider the contribution of $-R^{[p]}_{n+1-n_1, k+1+n_0-n_1} = -R^{[p]}_{n-2, k}$.

$$A(t) = \sum_{i \geq 0} t^i A(t)^{-i} p[i](t) + tA(t) Q(t) = 1 - t + t^2 + tA(t)^{-1}(1 - t + t^2) + tA(t)$$

$$A(t) = \frac{1 - t + t^2 - \sqrt{1 + 2t - 5t^2 + 6t^2 - 3t^4}}{2(1 - t)} = 1 + t + 3t^3 - 3t^4 + 12t^5 - 30t^6 + 93t^7 - 282t^8 + O(t^9)$$