Graph Properties of Graph Associahedra

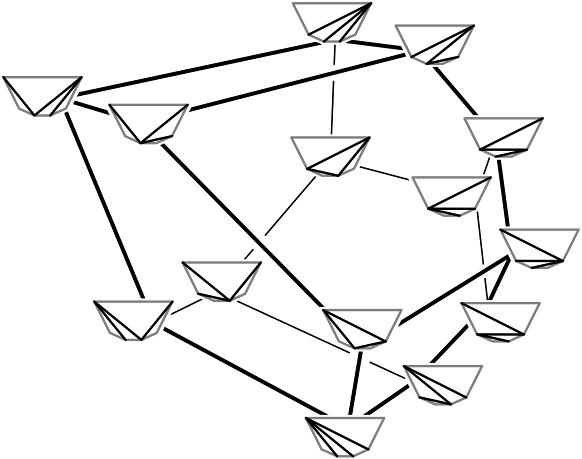
Thibault Manneville (LIX, Polytechnique)

joint work with **Vincent Pilaud** (CNRS, LIX Polytechnique)

April 14th, 2015

Definition

An ***associahedron*** is a polytope whose graph is the flip graph of triangulations of a convex polygon.

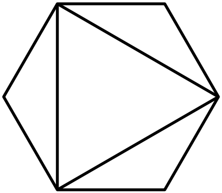


Faces $\leftrightarrow$ dissections of the polygon

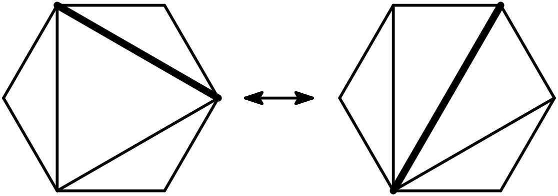
Focus on graphs

Flip graph on the triangulations of the polygon:

Vertices: *triangulations*



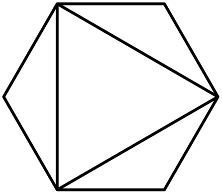
Edges: *flips*



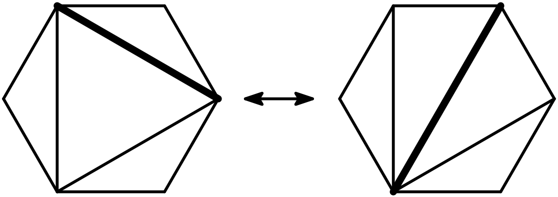
Focus on graphs

Flip graph on the triangulations of the polygon:

Vertices: *triangulations*



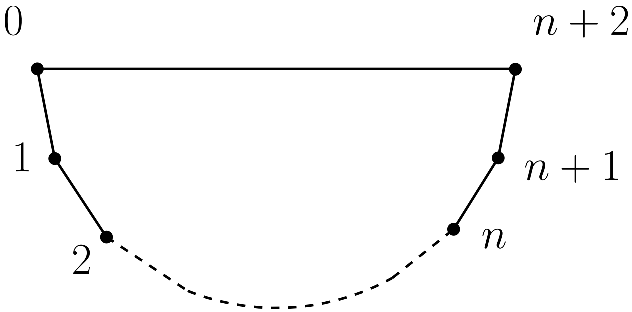
Edges: *flips*



n diagonals $\Rightarrow$ the flip graph is n-regular.

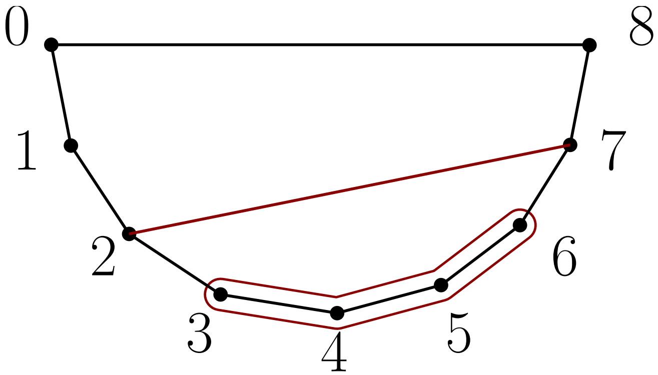
Useful configuration (Loday's)

$G_{n+3} =$



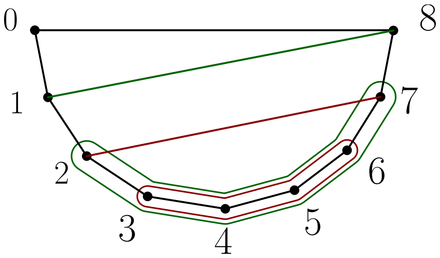
Graph point of view

$$\{\text{diagonals of } G_{n+3}\} \longleftrightarrow \{\text{strict subpaths of the path } [n+1]\}$$

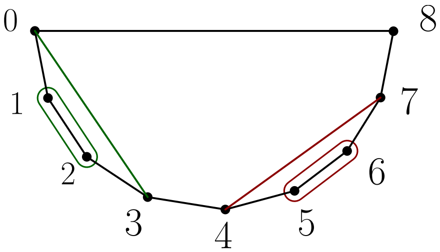


Non-crossing diagonals

Two ways to be non-crossing in Loday's configuration:



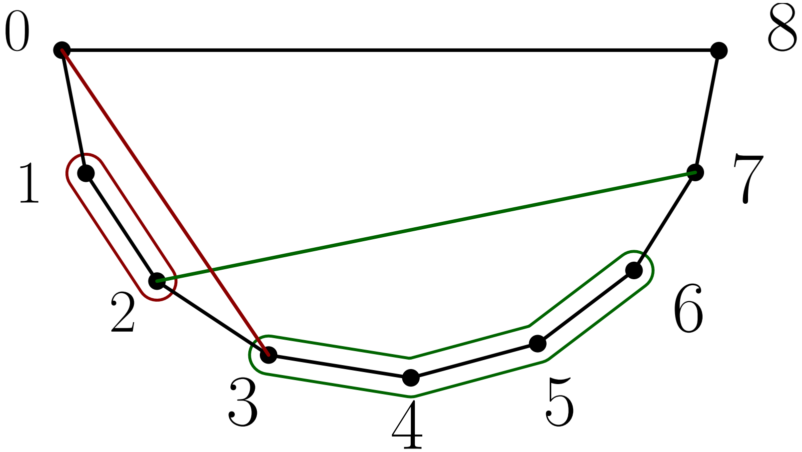
nested subpaths



non-adjacent subpaths

Pay attention to the second case:

The right condition is indeed *non-adjacent*, disjoint is not enough!



Now do it on graphs

$G = (V, E)$ a (connected) graph.

Definition

Now do it on graphs

$G = (V, E)$ a (connected) graph.

Definition

- A ***tube*** of G is a proper subset $t \subseteq V$ inducing a connected subgraph of G;

Now do it on graphs

$G = (V, E)$ a (connected) graph.

Definition

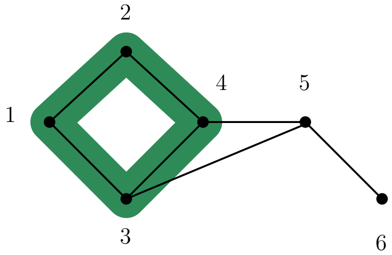
- A ***tube*** of G is a proper subset $t \subseteq V$ inducing a connected subgraph of G;
- t and t' are ***compatible*** if they are nested or non-adjacent;

Now do it on graphs

$G = (V, E)$ a (connected) graph.

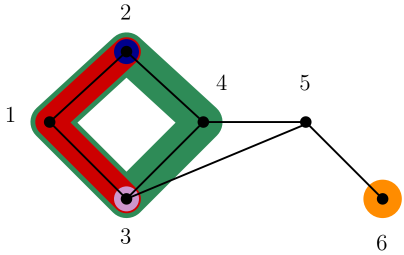
Definition

- A ***tube*** of G is a proper subset $t \subseteq V$ inducing a connected subgraph of G;
- t and t' are ***compatible*** if they are nested or non-adjacent;
- A ***tubing*** of G is a set of pairwise compatible tubes of G.



A tube

(generalizes a diagonal)



A maximal tubing

(generalizes a triangulation)

Graph associahedra

The simplicial complex of tubings is spherical

Graph associahedra

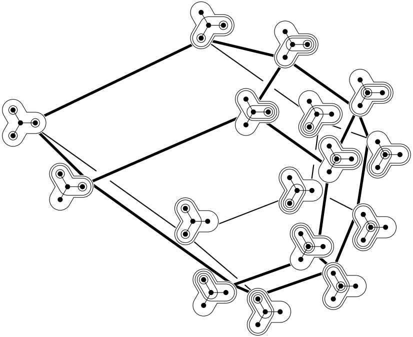
The simplicial complex of tubings is spherical $\Rightarrow$ **flip graph**!

Graph associahedra

The simplicial complex of tubings is spherical $\Rightarrow$ **flip graph**!

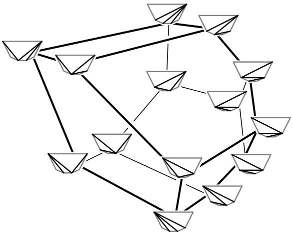
Theorem (Carr-Devadoss '06)

There exists a polytope called ***graph associahedron*** *of* G*, denoted* $\mathbf{Asso}_G$*, whose graph is this flip graph.*

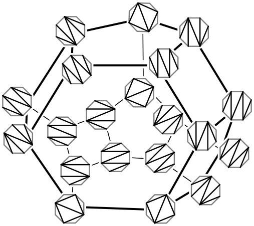


Faces $\leftrightarrow$ tubings of G.

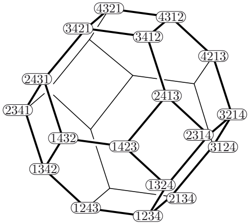
Classical polytopes...



The associahedron

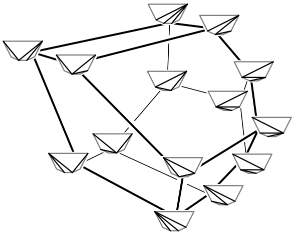


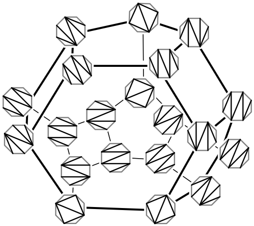
The cyclohedron

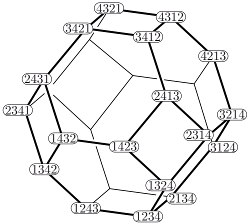


The permutahedron

...can be seen as graph associahedra



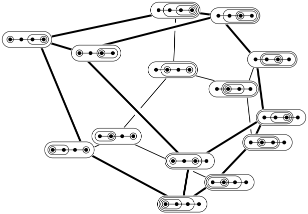


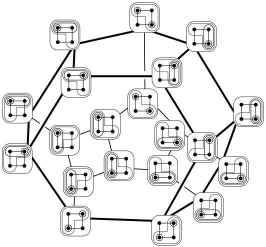


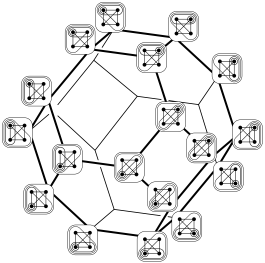
The associahedron

The cyclohedron

The permutahedron







Diameter of flip graphs

Lemma

The diameter of the n-dimensional permutahedron is $\binom{n+1}{2}$.

Diameter of flip graphs

Lemma

The diameter of the n-dimensional permutahedron is $\binom{n+1}{2}$.

Theorem (Sleator-Trajan-Thurston '88, Pournin '12)

The diameter of the n-dimensional associahedron is $2n-4$ *for* $n \geq 10$.

Diameter

$\delta(\mathcal{F}(G))$ = diameter of the flip graph $\mathcal{F}(G)$ on tubings on G.

Diameter

$\delta(\mathcal{F}(G))$ = diameter of the flip graph $\mathcal{F}(G)$ on tubings on G.

Theorem (M.-Pilaud '14)

$\delta(\mathcal{F}(.))$ is a non-decreasing function:
G partial subgraph of $G' \Longrightarrow \delta(\mathcal{F}(G)) \leq \delta(\mathcal{F}(G'))$.

Diameter

$\delta(\mathcal{F}(G))$ = diameter of the flip graph $\mathcal{F}(G)$ on tubings on G.

Theorem (M.-Pilaud '14)

$\delta(\mathcal{F}(.))$ is a non-decreasing function:
G partial subgraph of $G' \Longrightarrow \delta(\mathcal{F}(G)) \leq \delta(\mathcal{F}(G'))$.

Idea:

$\rightarrow$ Carr and Devadoss: iterated truncations of a simplex.

Diameter

$\delta(\mathcal{F}(G))$ = diameter of the flip graph $\mathcal{F}(G)$ on tubings on G.

Theorem (M.-Pilaud '14)

$\delta(\mathcal{F}(.))$ is a non-decreasing function:
G partial subgraph of $G' \implies \delta(\mathcal{F}(G)) \leq \delta(\mathcal{F}(G'))$.

Idea:

$\rightarrow$ Carr and Devadoss: iterated truncations of a simplex.
$\rightarrow$ If $G \subseteq G'$, $\mathbf{Asso}_{G'}$ is obtained by truncations of $\mathbf{Asso}_G$.

Diameter

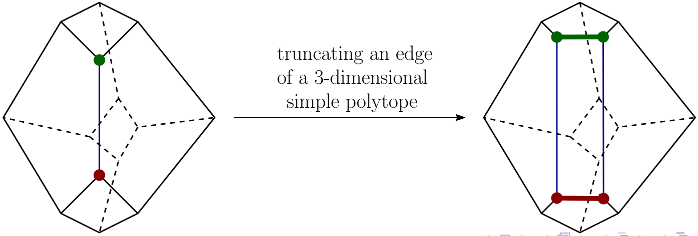
$\delta(\mathcal{F}(G))$ = diameter of the flip graph $\mathcal{F}(G)$ on tubings on G.

Theorem (M.-Pilaud '14)

$\delta(\mathcal{F}(.))$ is a non-decreasing function:
G partial subgraph of $G' \Longrightarrow \delta(\mathcal{F}(G)) \leq \delta(\mathcal{F}(G'))$.

Idea:

$\rightarrow$ Carr and Devadoss: iterated truncations of a simplex.
$\rightarrow$ If $G \subseteq G'$, $\mathbf{Asso}_{G'}$ is obtained by truncations of $\mathbf{Asso}_G$.
$\rightarrow$ Truncating $\iff$ replacing vertices by complete graphs.

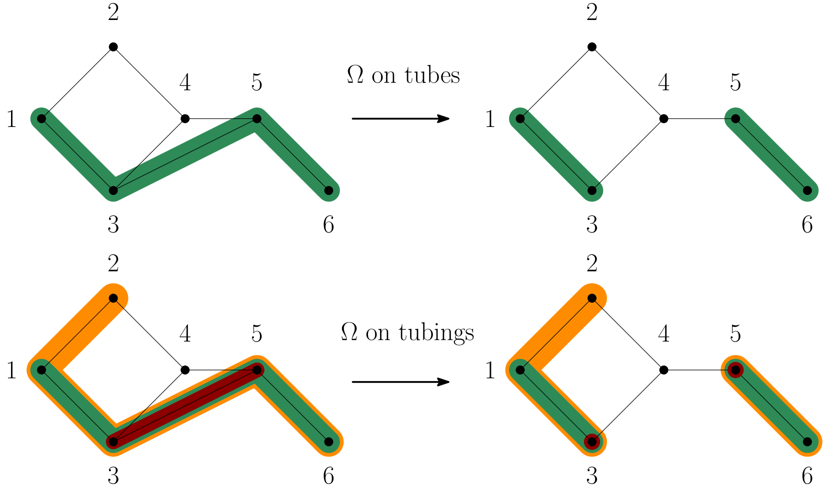


$G \subseteq G'$ two graphs,

$G \subseteq G'$ two graphs,
$\rightarrow$ Define a map Ω from tubings on G' to tubings on G.

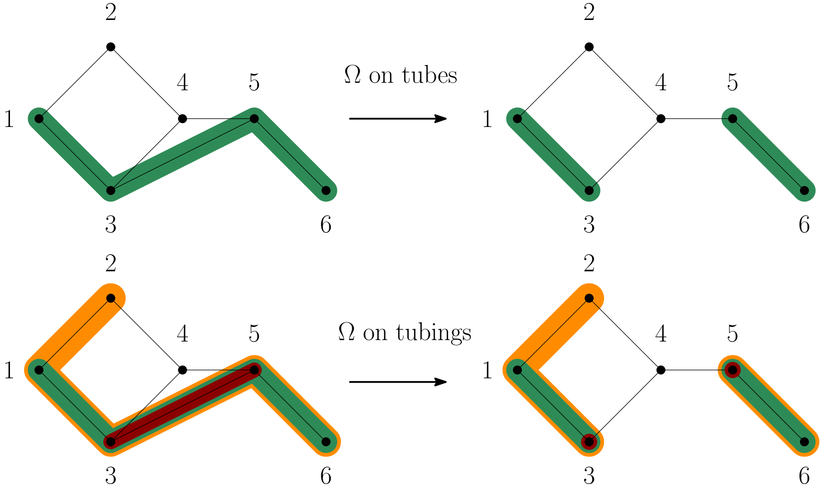
$G \subseteq G'$ two graphs,

$\rightarrow$ Define a map Ω from tubings on G' to tubings on G.



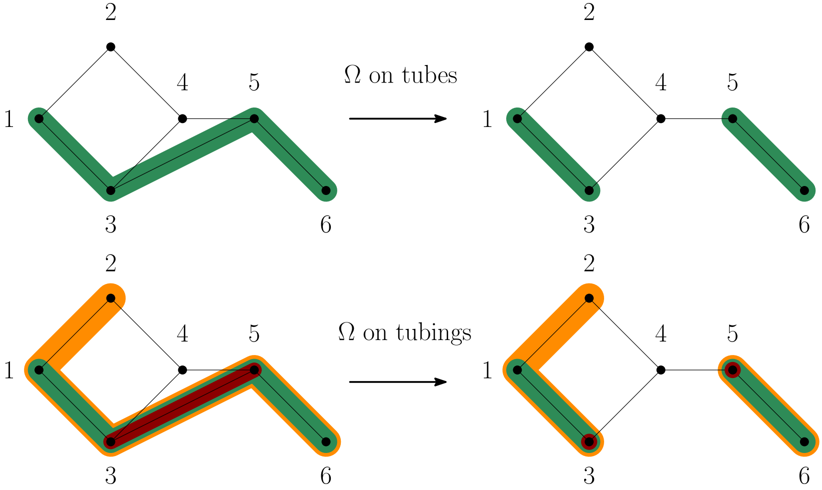
$G \subseteq G'$ two graphs,

$\rightarrow$ Define a map Ω from tubings on G' to tubings on G.



$\rightarrow$ Ω is surjective.

$G \subseteq G'$ two graphs,
$\rightarrow$ Define a map Ω from tubings on G' to tubings on G.



$\rightarrow$ Ω is surjective.
$\rightarrow$ Ω sends a flip either on a flip or on an empty step.

Inequalities for the diameter

Corollary

For any graph G, $\delta(\mathcal{F}(G)) \leq \binom{|V(G)|}{2}$.

Inequalities for the diameter

Corollary

For any graph G, $$\delta(\mathcal{F}(G)) \leq \binom{|V(G)|}{2}.$$

G is included in the complete graph on its vertices... ■

Inequalities for the diameter

Corollary

For any graph G, $\delta(\mathcal{F}(G)) \leq \binom{|V(G)|}{2}$.

G is included in the complete graph on its vertices... ■

Theorem (M.-Pilaud 14)

For any graph G, $2|V(G)| - 18 \leq \delta(\mathcal{F}(G))$.

Inequalities for the diameter

Corollary

For any graph G, $\delta(\mathcal{F}(G)) \leq \binom{|V(G)|}{2}$.

G is included in the complete graph on its vertices... ■

Theorem (M.-Pilaud 14)

For any graph G, $2|V(G)| - 18 \leq \delta(\mathcal{F}(G))$.

Ingredients of the proof:

Inequalities for the diameter

Corollary

For any graph G, $\delta(\mathcal{F}(G)) \leq \binom{|V(G)|}{2}$.

G is included in the complete graph on its vertices... ■

Theorem (M.-Pilaud 14)

For any graph G, $2|V(G)| - 18 \leq \delta(\mathcal{F}(G))$.

Ingredients of the proof:

- $\delta(\mathcal{F}(.))$ is non-decreasing;

Inequalities for the diameter

Corollary

For any graph G, $\delta(\mathcal{F}(G)) \leq \binom{|V(G)|}{2}$.

G is included in the complete graph on its vertices... ■

Theorem (M.-Pilaud 14)

For any graph G, $2|V(G)| - 18 \leq \delta(\mathcal{F}(G))$.

Ingredients of the proof:

- $\delta(\mathcal{F}(.))$ is non-decreasing;
- Non-leaving-face property;

Inequalities for the diameter

Corollary

For any graph G, $\delta(\mathcal{F}(G)) \leq \binom{|V(G)|}{2}$.

G is included in the complete graph on its vertices... ■

Theorem (M.-Pilaud 14)

For any graph G, $2|V(G)| - 18 \leq \delta(\mathcal{F}(G))$.

Ingredients of the proof:

- $\delta(\mathcal{F}(.))$ is non-decreasing;
- Non-leaving-face property;
- Pournin's result for the classical associahedron.

Non-leaving-face property (NLFP)

Definition (NLFP)

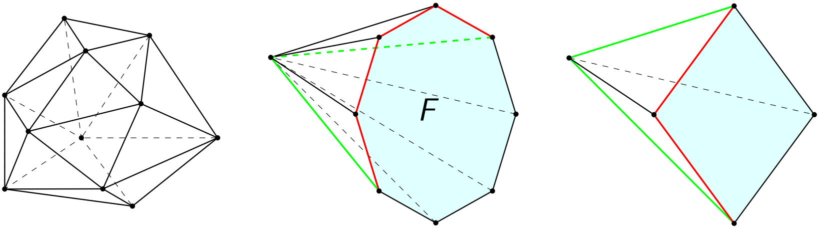
A face F of a polytope P has the *non-leaving-face property* (**NLFP**) if all geodesics in P between vertices of F stay in F.

Non-leaving-face property (**NLFP**)

Definition (NLFP)

A face F of a polytope P has the *non-leaving-face property* (**NLFP**) if all geodesics in P between vertices of F stay in F.

$\rightarrow$ How "round" is the polytope?



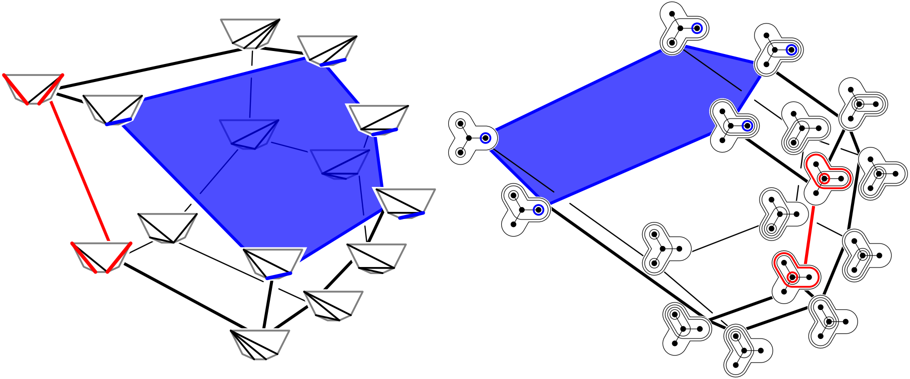
simplicial$\Rightarrow$**NLFP** F does not have **NLFP** Limit case

Non-leaving-face property (NLFP)

Idea: S set of compatible tubes of $G \longleftrightarrow$ face F_S of $\mathbf{Asso}_G$.

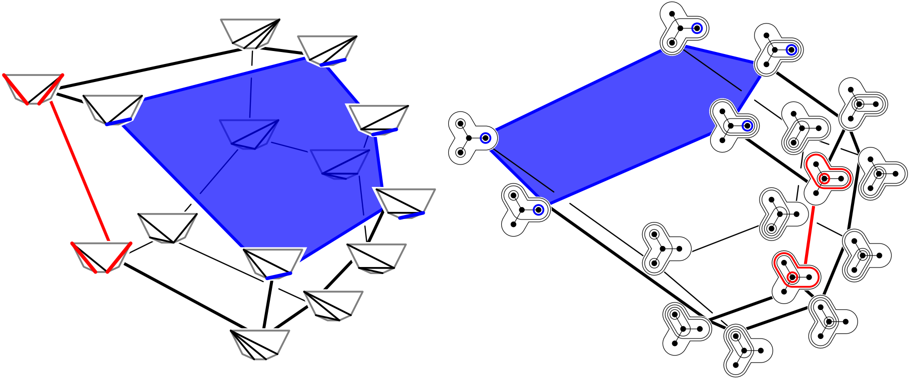
Non-leaving-face property (NLFP)

Idea: S set of compatible tubes of $G \longleftrightarrow$ face F_S of $\mathbf{Asso}_G$.



Non-leaving-face property (NLFP)

Idea: S set of compatible tubes of $G \longleftrightarrow$ face F_S of $\mathbf{Asso}_G$.



$\rightarrow$ Do faces of graph associahedra have **NLFP**?

Non-leaving-face property (NLFP)

Proposition (M.-Pilaud 14)

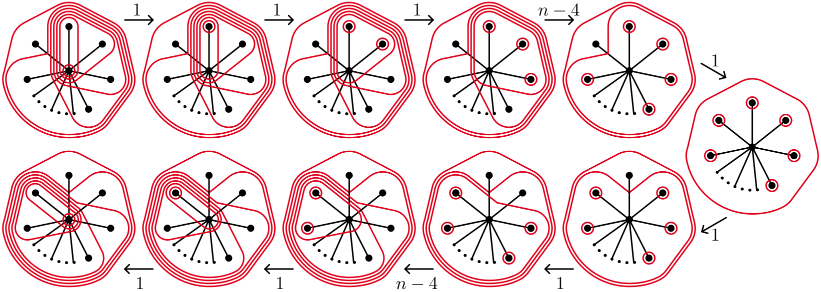
S upper set of a tubing on $G \Rightarrow F_S$ has **NLFP** *in $Asso_G$.*

Non-leaving-face property (NLFP)

Proposition (M.-Pilaud 14)

S upper set of a tubing on G $\Rightarrow$ F_S has **NLFP** *in $Asso_G$.*

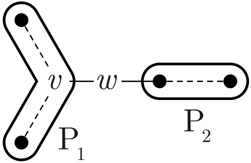
$\rightarrow$ Not all faces have **NLFP**.



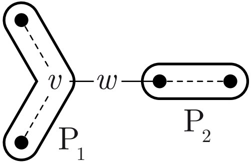
$\rightarrow$ restriction to trees T with set of leaves $L = \{\ell_1, \ldots, \ell_k\}$.

$\rightarrow$ restriction to trees T with set of leaves $L = \{\ell_1, \ldots, \ell_k\}$.
$\rightarrow$ if $k \leq 4 \quad \Longrightarrow \quad$ **NLFP** + Pournin's result.

→ restriction to trees T with set of leaves $L = \{\ell_1, \ldots, \ell_k\}$.
→ if $k \leq 4 \quad \Longrightarrow \quad$ **NLFP** + Pournin's result.



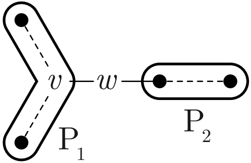
→ restriction to trees T with set of leaves $L = \{\ell_1, \ldots, \ell_k\}$.
→ if $k \leq 4 \implies$ **NLFP** + Pournin's result.



$$\delta(\mathcal{F}(T)) \geq \delta(\mathcal{F}(P_1)) + \delta(\mathcal{F}(P_2))$$

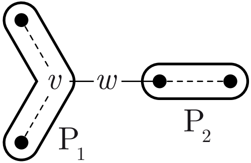
$\rightarrow$ restriction to trees T with set of leaves $L = \{\ell_1, \ldots, \ell_k\}$.

$\rightarrow$ if $k \leq 4 \quad \Longrightarrow \quad$ **NLFP** + Pournin's result.



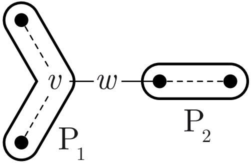
$$\begin{aligned} \delta(\mathcal{F}(T)) &\geq \delta(\mathcal{F}(P_1)) + \delta(\mathcal{F}(P_2)) \\ &\geq (2p_1 - 4) + (2p_2 - 4) \end{aligned}$$

→ restriction to trees T with set of leaves $L = \{\ell_1, \ldots, \ell_k\}$.
→ if $k \leq 4 \quad \Longrightarrow \quad$ **NLFP** + Pournin's result.

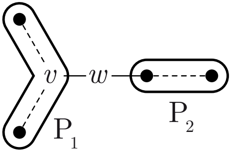


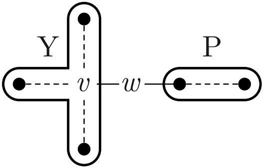
$$\begin{aligned} \delta(\mathcal{F}(T)) &\geq \delta(\mathcal{F}(P_1)) + \delta(\mathcal{F}(P_2)) \\ &\geq (2p_1 - 4) + (2p_2 - 4) \\ &= 2(p_1 + p_2 + 2) - 12 \\ &= 2n - 12 \end{aligned}$$

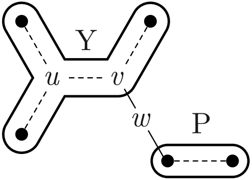
→ restriction to trees T with set of leaves $L = \{\ell_1, \dots, \ell_k\}$.
→ if $k \leq 4 \quad \Longrightarrow \quad$ **NLFP** + Pournin's result.



$$\begin{aligned} \delta(\mathcal{F}(T)) &\geq \delta(\mathcal{F}(P_1)) + \delta(\mathcal{F}(P_2)) \\ &\geq (2p_1 - 4) + (2p_2 - 4) \\ &= 2(p_1 + p_2 + 2) - 12 \\ &= 2n - 12 \end{aligned}$$

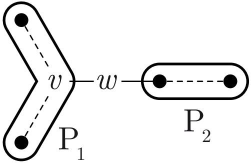




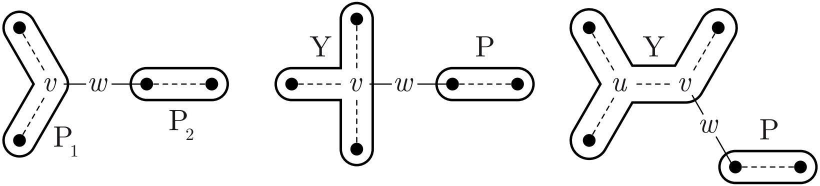


→ restriction to trees T with set of leaves $L = \{\ell_1, \dots, \ell_k\}$.

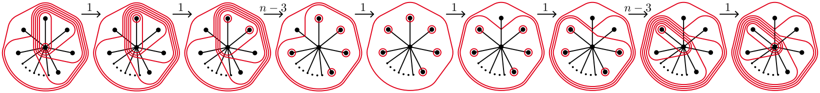
→ if $k \leq 4 \implies$ **NLFP** + Pournin's result.



$$\begin{aligned} \delta(\mathcal{F}(T)) &\geq \delta(\mathcal{F}(P_1)) + \delta(\mathcal{F}(P_2)) \\ &\geq (2p_1 - 4) + (2p_2 - 4) \\ &= 2(p_1 + p_2 + 2) - 12 \\ &= 2n - 12 \end{aligned}$$



→ $k \geq 5 \implies \delta(\mathcal{F}(T)) \geq 2.k + \delta(\mathcal{F}(T \setminus L))$.



Hamiltonicity of flip graphs

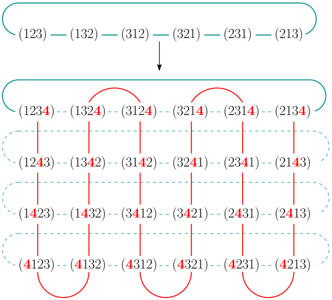
Theorem (Trotter '62, Johnson '63, Steinhaus '64)

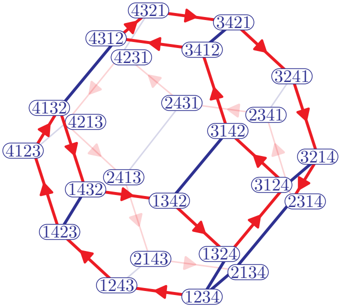
The n-dimensional permutahedron is hamiltonian for $n \geq 2$.

Hamiltonicity of flip graphs

Theorem (Trotter '62, Johnson '63, Steinhaus '64)

The n-dimensional permutahedron is hamiltonian for $n \geq 2$.

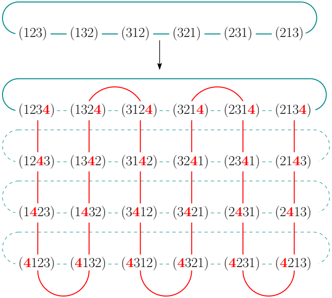


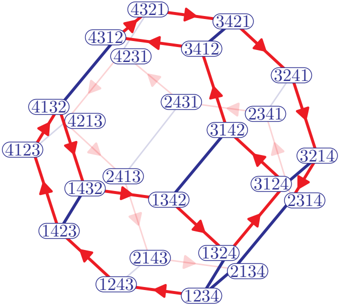


Hamiltonicity of flip graphs

Theorem (Trotter '62, Johnson '63, Steinhaus '64)

The n-dimensional permutahedron is hamiltonian for $n \geq 2$.





Theorem (Lucas 87, Hurtado-Noy '99)

The n-dimensional associahedron is hamiltonian for $n \geq 2$.

Hamiltonicity

Theorem (M.-Pilaud '14)

Any graph associahedron $\mathcal{F}(G)$ is hamiltonian.

Hamiltonicity

Theorem (M.-Pilaud '14)

Any graph associahedron $\mathcal{F}(G)$ is hamiltonian.

Idea:

$\rightarrow$ Carr and Devadoss: iterated truncations of a simplex.

Hamiltonicity

Theorem (M.-Pilaud '14)

Any graph associahedron $\mathcal{F}(G)$ is hamiltonian.

Idea:

$\rightarrow$ Carr and Devadoss: iterated truncations of a simplex.

$\rightarrow$ Truncation hyperplanes correspond to tubes.

Hamiltonicity

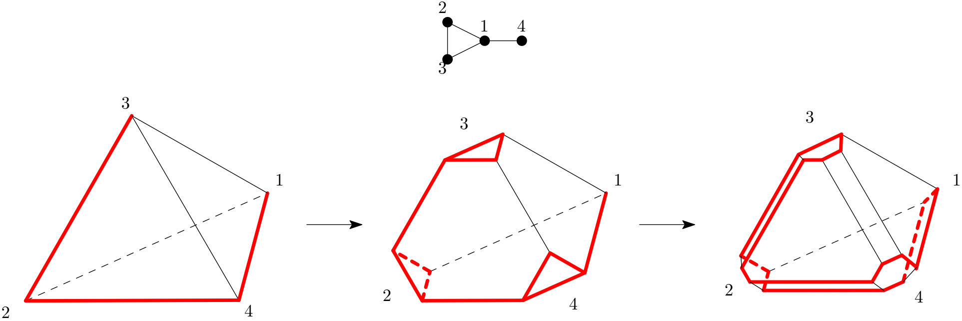
Theorem (M.-Pilaud '14)

Any graph associahedron $\mathcal{F}(G)$ is hamiltonian.

Idea:

→ Carr and Devadoss: iterated truncations of a simplex.

→ Truncation hyperplanes correspond to tubes.



Discussion

Diameter

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

  The cyclohedron has a diameter equivalent to $\frac{5}{2}n$ (Pournin).

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

  The cyclohedron has a diameter equivalent to $\frac{5}{2}n$ (Pournin).
  Correlation between number of edges and diameter of the flip graph?

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

  The cyclohedron has a diameter equivalent to $\frac{5}{2}n$ (Pournin).
  Correlation between number of edges and diameter of the flip graph?
- Hardness of $\delta(\mathcal{F}(G))$?

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

  The cyclohedron has a diameter equivalent to $\frac{5}{2}n$ (Pournin).
  Correlation between number of edges and diameter of the flip graph?
- Hardness of $\delta(\mathcal{F}(G))$?

Hamiltonicity

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

  The cyclohedron has a diameter equivalent to $\frac{5}{2}n$ (Pournin).
  Correlation between number of edges and diameter of the flip graph?
- Hardness of $\delta(\mathcal{F}(G))$?

Hamiltonicity

- Algorithmic inefficience of the proof.

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

  The cyclohedron has a diameter equivalent to $\frac{5}{2}n$ (Pournin).
  Correlation between number of edges and diameter of the flip graph?
- Hardness of $\delta(\mathcal{F}(G))$?

Hamiltonicity

- Algorithmic inefficience of the proof.
- How many Hamiltonian cycles?

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

  The cyclohedron has a diameter equivalent to $\frac{5}{2}n$ (Pournin).
  Correlation between number of edges and diameter of the flip graph?
- Hardness of $\delta(\mathcal{F}(G))$?

Hamiltonicity

- Algorithmic inefficience of the proof.
- How many Hamiltonian cycles?

Other problems

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

  The cyclohedron has a diameter equivalent to $\frac{5}{2}n$ (Pournin).
  Correlation between number of edges and diameter of the flip graph?
- Hardness of $\delta(\mathcal{F}(G))$?

Hamiltonicity

- Algorithmic inefficience of the proof.
- How many Hamiltonian cycles?

Other problems

- How many tubings ?

Discussion

Diameter

- What happens between $2n$ and $\binom{n}{2}$?

  The cyclohedron has a diameter equivalent to $\frac{5}{2}n$ (Pournin).
  Correlation between number of edges and diameter of the flip graph?
- Hardness of $\delta(\mathcal{F}(G))$?

Hamiltonicity

- Algorithmic inefficience of the proof.
- How many Hamiltonian cycles?

Other problems

- How many tubings ?
- $\vdots$

THANK YOU FOR LISTENING SO FERVENTLY!