Statistics on permutation tableaux

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parts based on joint work with Sylvie Corteel (Paris-Sud)
and parts with Svante Janson (Uppsala)

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Permutation tableaux

Permutation tableau $T$ : a Ferrers diagram of a partition $\lambda$ filled with 0’s and 1’s such that:

1. Each column contains at least one 1.
2. There is no 0 which has a 1 above it in the same column and a 1 to its left in the same row.

\[
\begin{array}{ccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 \\
1
\end{array}
\]
Previous work

- introduced by Postnikov (2001)

- subsequently studied by Williams (2004), Steingrímsson and Williams (2005) (bijections with permutations)


- additional combinatorial work Corteel and Nadeau (2007) (more bijections), Burstein (2006) (some properties of permutation tableaux)
Statistics on $T$

- Length $\ell(T)$: no. rows plus no. columns

$$
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1 \mid 1 \\
0 & 0 & 1 & 0 & 1 & \\
0 & 1 & 1 & 1 & 1 & \\
0 & 1 & 1 & 1 & 1 & \\
0 & 0 & 0 & \\
1 & \\
\end{array}
\quad \ell(T) = 12
$$

Number of permutation tableaux of length $n = n!$. $\mathcal{T}_n$ is the set of all permutation tableaux with $\ell(T) = n$. 
Statistics on $T$

- Length $\ell(T)$: no. rows plus no. columns
- $U(T)$: number of unrestricted rows (a row is restricted if it has a 0 that has 1 above it)

$$
\begin{array}{cccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 \\
1 & \\
\end{array}
$$

$U(T) = 4$
Statistics on $T$

- Length $\ell(T)$: no. rows plus no. columns
- $U(T)$: number of unrestricted rows (a row is restricted if it has a 0 that has 1 above it)
- $F(T)$: number of 1’s in the first row

$F(T) = 3$
Statistics on $T$

- Length $\ell(T)$: no. rows plus no. columns
- $U(T)$: number of unrestricted rows (a row is restricted if it has a 0 that has 1 above it)
- $F(T)$: number of 1’s in the first row
- $R(T)$: number of rows

\[
\begin{array}{llllllllll}
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 \\
1 \\
\end{array}
\]

$R(T) = 5$
Statistics on $T$

- Length $\ell(T)$: no. rows plus no. columns
- $U(T)$: number of unrestricted rows (a row is restricted if it has a 0 that has 1 above it)
- $F(T)$: number of 1’s in the first row
- $R(T)$: number of rows
- $S(T)$: number of superfluous 1’s (1’s below the top one in the column)

\[
\begin{array}{cccccccc}
0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 \\
1 & \\
\end{array}
\]

$S(T) = 3$
From tableaux of length $n - 1$ to tableaux of length $n$

Let $T \in \mathcal{T}_{n-1}$ and suppose that it has $U_{n-1}$ unrestricted rows. From the SW corner of the tableau we can extend its length by one by either:

```
0 0 0 1 0 0 1 1
0 0 1 0 1
0 1 1 1 1
0 1 1 1
0 0 0
1
```
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- moving W; this adds a column that has to be filled

```
0 0 1 0 0 1 1
0 0 1 0 1
0 1 1 1 1
0 0 0
1
```
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  - Put zero in restricted rows

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From tableaux of length $n - 1$ to tableaux of length $n$

Let $T \in \mathcal{T}_{n-1}$ and suppose that it has $U_{n-1}$ unrestricted rows. From the SW corner of the tableau we can extend its length by one by either:

- moving S; this adds a row (unrestricted)
- moving W; this adds a column that has to be filled
  - Put zero in restricted rows
  - Put zero or one in unrestricted rows
Distribution of the number of unrestricted rows

So, there are $2^{U_{n-1}}$ extensions of $T$ (all equally likely).
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Let $U_n$ be the number of unrestricted rows in the extension of $T$. Elementary calculations based on these earlier observations yield that for $1 \leq k \leq U_{n-1} + 1$

$$P(U_n = k) = \frac{1}{2^{U_{n-1}}} \binom{U_{n-1}}{k-1} = P(\text{Bin}(U_{n-1}) = k - 1).$$
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$$P(U_n = k) = \frac{1}{2^{U_{n-1}}} \binom{U_{n-1}}{k - 1} = P(\text{Bin}(U_{n-1}) = k - 1).$$

This means that,

$$\mathcal{L}(U_n|U_{n-1}) = 1 + \text{Bin}(U_{n-1}).$$
Change of measure

Let $P_n$ be the uniform probability on $I_n$ and $E_n$ integration w.r.t. $P_n$. 
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$$E_n f(U_n) = E_n E(f(U_n)|U_{n-1}) = E_n E(f(1 + \text{Bin}(U_{n-1}))|U_{n-1}) = E_n \tilde{f}(U_{n-1}).$$
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The last integral over $\mathcal{I}_{n-1}$ is not w.r.t. the uniform measure but w.r.t. the measure induced by $P_n$. 
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Let $P_n$ be the uniform probability on $\mathcal{T}_n$ and $E_n$ integration w.r.t. $P_n$. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$ we have

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\]

The last integral over $\mathcal{T}_{n-1}$ is not w.r.t. the uniform measure but w.r.t. the measure induced by $P_n$. The relation is: for $T \in \mathcal{T}_{n-1}$ with $U_{n-1}$ unrestricted rows

\[
P_n(T) = \frac{2^{U_{n-1}}}{|\mathcal{T}_n|} = \frac{2^{U_{n-1}}}{|\mathcal{T}_n|} |\mathcal{T}_{n-1}| P_{n-1}(T).
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\]

So,

\[
E_n f(U_n) = \frac{|\mathcal{I}_{n-1}|}{|\mathcal{I}_n|} E_{n-1} 2^{U_{n-1}} \tilde{f}(U_{n-1}).
\]
Change of measure

Let $P_n$ be the uniform probability on $\mathcal{T}_n$ and $E_n$ integration w.r.t. $P_n$. For a function $f : R \rightarrow R$ we have

$$E_n f(U_n) = E_n E(f(U_n)\mid U_{n-1}) = E_n E(f(1 + \text{Bin}(U_{n-1}))\mid U_{n-1}) = E_n \tilde{f}(U_{n-1}).$$

The last integral over $\mathcal{T}_{n-1}$ is not w.r.t. the uniform measure but w.r.t. the measure induced by $P_n$. The relation is: for $T \in \mathcal{T}_{n-1}$ with $U_{n-1}$ unrestricted rows

$$P_n(T) = \frac{2^{U_{n-1}}}{|\mathcal{T}_n|} = \frac{2^{U_{n-1}}}{|\mathcal{T}_n|} |\mathcal{T}_{n-1}| P_{n-1}(T).$$

So,

$$E_n f(U_n) = \frac{|\mathcal{T}_{n-1}|}{|\mathcal{T}_n|} E_{n-1} 2^{U_{n-1}} \tilde{f}(U_{n-1}).$$

We know $\frac{|\mathcal{T}_{n-1}|}{|\mathcal{T}_n|} = \frac{1}{n}$ but we don’t want to use it yet.
Theorem: For every $n \geq 0$ $|\mathcal{T}_{n+1}| = (n + 1)!$. 

Illustration
Theorem: For every $n \geq 0$ $|\mathcal{I}_{n+1}| = (n + 1)!$.
Proof: Count the elements of $\mathcal{I}_{n+1}$ as follows

$$|\mathcal{I}_{n+1}| = \sum_{T \in \mathcal{I}_n} 2^{U_n(T)}.$$
**Theorem:** For every \( n \geq 0 \) \(|\mathcal{T}_{n+1}| = (n + 1)!\).

**Proof:** Count the elements of \( \mathcal{T}_{n+1} \) as follows

\[
|\mathcal{T}_{n+1}| = \sum_{T \in \mathcal{T}_n} 2^{U_n(T)}.
\]

Then

\[
|\mathcal{T}_{n+1}| = |\mathcal{T}_n| \cdot \sum_{T \in \mathcal{T}_n} 2^{U_n(T)} = |\mathcal{T}_n| \cdot E_n 2^{U_n} = |\mathcal{T}_n| \cdot E_n E(2^{U_n} | U_{n-1}).
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\[
|\mathcal{T}_{n+1}| = |\mathcal{T}_n| \frac{1}{|\mathcal{T}_n|} \sum_{T \in \mathcal{T}_n} 2^{U_n(T)} = |\mathcal{T}_n| \cdot E_n 2^{U_n} = |\mathcal{T}_n| \cdot E_n E(2^{U_n} | U_{n-1}).
\]

And

\[
E(2^{U_n} | U_{n-1}) = E(2^{1+\text{Bin}(U_{n-1})} | U_{n-1}) = 2 \left( \frac{3}{2} \right)^{U_{n-1}},
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\]

Hence, by the change of measure

\[
E_n 2^{U_n} = 2 E_n \left( \frac{3}{2} \right)^{U_{n-1}} = 2 \frac{|\mathcal{T}_{n-1}|}{|\mathcal{T}_n|} E_{n-1} 2^{U_{n-1}} \left( \frac{3}{2} \right)^{U_{n-1}}
\]

\[
= 2 \frac{|\mathcal{T}_{n-1}|}{|\mathcal{T}_n|} E_{n-1} 3^{U_{n-1}}.
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This can be iterated and gives

\[ |\mathcal{T}_{n+1}| = 2 \cdot 3 \cdot \ldots \cdot n \cdot |\mathcal{T}_1| \cdot E_1 (n+1)^U_1 = (n+1)! \]
So,

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Methodology:
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**Methodology:**

- **condition:** $\mathcal{L}(U_n | U_{n-1}) = 1 + \text{Bin}(U_{n-1})$
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▶ compute: expectation of a function of \( \text{Bin}(U_{n-1}) \)
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  - change the measure and reduce from \( n \) to \( n - 1 \).
So,
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- change the measure and reduce from \( n \) to \( n - 1 \).
- iterate.
Some results

We can get, in a unified and elementary way
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**Theorem:** For a random tableau of length $n$:

- The number of unrestricted rows is distributed like $\sum_{k=1}^{n} J_k$, where $J_k$ are independent indicators and $P(J_k = 1) = 1/k$. 
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- For superfluous one’s: $ES_n = (n - 1)(n - 2)/12$, $\text{var}(S_n) = (n - 2)(2n^2 + 11n - 1)/360$, and

$$
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  \[
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  \]

There is convergence to $N(0, 1)$ in the first three cases, too.
Remarks:

- The first three results can also be deduced from bijections between permutation tableaux and permutations, an involution on permutation tableaux, and known properties of permutations.
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- But, the combinatorics behind those results is not simple and all three required different methods.
- The results about superfluous ones rely on a (bijectively proved) fact that the number of superfluous ones is equidistributed with the number of occurrences of the generalized pattern \( 31-2 \) (\( i < j \) such that \( \sigma_{i-1} > \sigma_j > \sigma_i \)).
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The results about superfluous ones rely on a (bijectively proved) fact that the number of superfluous ones is equidistributed with the number of occurrences of the generalized pattern 31-2 \((i < j \text{ such that } \sigma_{i-1} > \sigma_j > \sigma_i)\).

But, there is no proof independent of the bijections between permutation tableaux and permutations. One writes

\[
S = \sum_{2 \leq i < j \leq n} I_{\sigma_{i-1} > \sigma_j > \sigma_i}
\]

and proves the Central Limit Theorem for dependent random variables (Janson).
Sample easy proof (unrestricted rows)

For the characteristic function of $U_n$ we have:

$$E_n e^{itU_n} = E_n E \left( e^{itU_n} | U_{n-1} \right) = E_n E \left( e^{it(1+\text{Bin}(U_{n-1}))} | U_{n-1} \right)$$

$$= e^{it} E_n \left( \frac{e^{it} + 1}{2} \right)^{U_{n-1}} = \frac{e^{it}}{n} E_{n-1} 2^{U_{n-1}} \left( \frac{e^{it} + 1}{2} \right)^{U_{n-1}}$$

$$= \frac{e^{it}}{n} E_{n-1} \left( e^{it} + 1 \right)^{U_{n-1}},$$

where we have used (in that order) conditioning, distributional properties of $U_n$, an obvious fact that for a complex number $z$, $E_z \text{Bin}(m) = \left( \frac{z+1}{2} \right)^m$, and the change of measure.
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Applying the same procedure to the last expectation, this time with $z = 1 + e^{it}$ we see that

$$E_n e^{itU_n} = \frac{e^{it}(e^{it} + 1)}{n(n-1)} E_{n-2} \left(e^{it} + 2\right)^{U_{n-2}}.$$
Further iterations yield

\[ E_n e^{it} U_n = \left( \prod_{k=0}^{n-2} \frac{e^{it} + k}{n - k} \right) E_1 (e^{it} + n - 1) U_1 = \prod_{k=0}^{n-1} \frac{e^{it} + k}{k + 1} \]

\[ = \prod_{k=1}^{n} \left( \frac{e^{it}}{k} + 1 - \frac{1}{k} \right). \]
Further iterations yield

\[ E_n e^{it} U_n = \left( \prod_{k=0}^{n-2} \frac{e^{it} + k}{n - k} \right) E_1(e^{it} + n - 1) U_1 = \prod_{k=0}^{n-1} \frac{e^{it} + k}{k + 1} \]

\[ = \prod_{k=1}^{n} \left( \frac{e^{it}}{k} + 1 - \frac{1}{k} \right). \]

The factor in the last term is the characteristic function of a random variable that is 1 with probability $1/k$ and 0 with probability $1 - 1/k$. 
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\[ E_n e^{it} U_n = \left( \prod_{k=0}^{n-2} \frac{e^{it} + k}{n-k} \right) E_1(e^{it} + n - 1) U_1 = \prod_{k=0}^{n-1} \frac{e^{it} + k}{k + 1} \]

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The factor in the last term is the characteristic function of a random variable that is 1 with probability \(1/k\) and 0 with probability \(1 - 1/k\). Since the product corresponds to summing independent random variables, we get that the characteristic function of \(U_n\) is equal to that of \(\sum_{k=1}^{n} J_k\).