Forbidden subgraph characterizations of some classes of intersection graphs

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   - Characterizations

2. Circle Graphs
   - Preliminaries
   - Characterizations
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Definitions

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- A family of graphs is hereditary if given any $\mathcal{F}$-graph, then all its induced subgraphs are $\mathcal{F}$-graphs.
Definitions

- Given a family of graphs \( \mathcal{F} \). An \( \mathcal{F} \)-graph is a graph belonging to \( \mathcal{F} \).
- A family of graphs is hereditary if given any \( \mathcal{F} \)-graph, then all its induced subgraphs are \( \mathcal{F} \)-graphs.
- A graph is minimally non-\( \mathcal{F} \) (or minimal forbidden subgraph for the class \( \mathcal{F} \)) if it is not an \( \mathcal{F} \)-graph and all its induced are \( \mathcal{F} \)-graph.
Let $\mathcal{F}$ be a finite family of non-empty sets. The *intersection graph* of $\mathcal{F}$ is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.
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Let $\mathcal{F}$ be a finite family of non-empty sets. The **intersection graph** of $\mathcal{F}$ is obtained by representing each set by a vertex, two vertices being connected by an edge if and only if the corresponding sets intersect.

An **interval graph** is the intersection graph of a finite family of open intervals on the real line (such a family of intervals is called an **interval model** for the graph).

- The class of interval graphs is a **hereditary class**.
A graph $G$ is an interval graph if and only if $G$ does not contain any of the following graphs as induced subgraphs:

- Bipartite claw
- $n$-net, $n \geq 2$
- Umbrella
- $n$-tent, $n \geq 3$
- $C_n$, $n \geq 4$
A **unit interval** is an interval graph having an interval model with all its intervals having the same length, such an interval model is called **unit interval model**. The class of unit interval graph is denoted by $\mathcal{U}$.
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**Theorem (Roberts, 1969)**

Let $G$ be an interval graph. $G$ is proper interval if and only if $G$ does not contain an induced claw $\{a, b, c, d\}$.
A **unit interval** is an interval graph having an interval model with all its intervals having the same length, such an interval model is called **unit interval model**. The class of unit interval graph is denoted by $\mathcal{U}$.

**Theorem (Roberts, 1969)**

Let $G$ be an interval graph. $G$ is proper interval if and only if $G$ does not contain an induced claw.

**Corollary**

A graph is a unit interval graph if and only if it contains no induced claw, 2-net, 3-tent, or $C_n$ for any $n \geq 4$. 

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Forbidden subgraph characterizations
A circular-arc graph (CA graph) is the intersection graph of a finite family of arcs on a circle (such a family of arcs is called a circular-arc model of the graph).
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Circular-arc graphs are a generalization of interval graphs.
Circular-arc graphs

- A circular-arc graph (CA graph) is the intersection graph of a finite family of arcs on a circle (such a family of arcs is called a circular-arc model of the graph).

- Circular-arc graphs are a generalization of interval graphs.
- They can be recognized in linear time (McConnell, 2003).
Tucker (1974) characterized the following subclasses of CA graphs by minimal forbidden subgraphs:

- **proper CA graphs** (ie, those that have a CA model in which no arc contains another), and
- **unit CA graphs** (ie, those that have a CA model with all arcs of equal length).
Tucker (1974) characterized the following subclasses of CA graphs by minimal forbidden subgraphs:
- **proper CA graphs** (ie, those that have a CA model in which no arc contains another), and
- **unit CA graphs** (ie, those that have a CA model with all arcs of equal length).

Trotter and Moore (1976) characterized, by minimal forbidden subgraphs, those CA graphs that are complements of bipartite graphs.
Basic minimally non-CA graphs

Lemma (Trotter and Moore, 1976)

The following are minimally non-CA graphs

- Bipartite claw
- $n$-net, $n \geq 3$
- Net $\cup K_1$
- Umbrella $\cup K_1$
- $n$-tent $\cup K_1$, $n \geq 3$
- $C_n \cup K_1$, $n \geq 4$

We refer to these graphs as basic minimally non-CA graphs.
In this work we present new characterizations of circular-arc graphs by minimal forbidden subgraphs for graphs that belong to one of the following classes:

- diamond-free graphs
- cographs (ie, $P_4$-free graphs)
- paw-free graphs
- claw-free chordal graphs
Nonbasic minimally non-CA graphs

**Proposition**

Let $G$ be a minimally non-CA graph. Then at least one of the following conditions hold:

1. $G$ is a basic minimally non-CA graph, or
2. $G$ contains at least one induced subgraph $H$ isomorphic to one of the following graphs

- net
- umbrella
- $n$-tent, $n \geq 3$
- $C_n$, $n \geq 4$

Moreover, all vertices $v$ of $G - H$ are adjacent to at least one vertex of $H$. 
A hole is a chordless cycle of length \( \geq 4 \).

**Theorem**

Let \( G \) be a minimally non-CA graph. Then exactly one of the following conditions hold:

1. For each hole \( H \) of \( G \) and for each vertex \( v \) of \( G - H \), either \( v \) is complete to \( H \) or \( N_{H}(v) \) induces a non-empty path in \( H \), or
2. \( G \) is isomorphic to \( C_{j} \cup K_{1} \) for some \( j \geq 4 \), or to one of the following graphs

![Diagram of graphs](attachment:image.png)
Cographs are those graphs not containing $P_4$ as induced subgraph.

**Theorem**

Let $G$ be a cograph. The following conditions are equivalent:

1. $G$ is a CA graph,
2. $G$ is $\{C_4 \cup K_1, K_{2,3}\}$-free graph.

![Diagram](image-url)
Paw-free graphs

The graph \( \bullet \longrightarrow \bullet \) is called a paw. A graph is paw-free if it does not contain an induced paw.

**Theorem**

Let \( G \) be a paw-free graph. The following conditions are equivalent:

1. \( G \) is a CA graph,
2. \( G \) contains neither an induced \( C_j \cup K_1 \) for any \( j \geq 4 \), nor a bipartite claw, nor any of the following graphs as induced subgraphs:
The graph \( \begin{array}{c} \text{\includegraphics[height=1cm]{diamond.png}} \end{array} \) is called a **diamond**. A graph is **diamond-free** if it does not contain an induced diamond.

**Theorem**

Let \( G \) be a diamond-free graph. The following are equivalent:

- \( G \) is a CA graph,
- \( G \) contains neither an induced \( C_j \cup K_1 \) for any \( j \geq 4 \), nor any of the following graphs as induced subgraphs:
Claw-free chordal graphs

The graph $\bullet - \bullet - \bullet$ is called a claw. A graph is claw-free if it does not contain an induced claw.

Theorem

Let $G$ be a claw-free chordal graph. The following conditions are equivalent:

1. $G$ is a CA graph,
2. $G$ does not contain any of the following graphs as induced subgraphs.
A circle graph is the intersection graph of a family \( \{L_v\}_{v \in V} \) of chords of a circle; i.e., \( v \) and \( w \) are adjacent if and only if \( L_v \cap L_w \neq \emptyset \). The family \( \{L_v\}_{v \in V} \) is called a circle model of \( G \).

Example:
The local complement of a graph $G = (V, E)$ with respect to a vertex $u \in V$ is the graph $G * u$ that arises from $G$ by replacing the induced subgraph $G[N_G(u)]$ by its complement.

Example:
The **local complement** of a graph $G = (V, E)$ with respect to a vertex $u \in V$ is the graph $G \ast u$ that arises from $G$ by replacing the induced subgraph $G[N_G(u)]$ by its complement. Two graphs $G$ and $H$ are **locally equivalent** if and only if $G$ arises from $H$ by a sequence of local complementations.

**Example:**

![Graph Example](image-url)
Bouchet’s Characterization

Theorem, Bouchet (1994)

Let $G$ be a graph. Then $G$ is a circle graph if and only if no graph locally equivalent to $G$ contains $W_5$, $W_7$ or $BW_3$ as induced subgraph.

5-Wheel

7-Wheel

$BW_3$
Split Decomposition

Let $G_1$ and $G_2$ be two graphs such that $|V(G_i)| \geq 3$, $i = 1, 2$. Let
$v_i \in G_i$ (mark vertex of $G_i$), $i = 1, 2$. The split composition with
respect to $v_1$ and $v_2$ is the graph $G_1 * G_2$, where

$$V(G_1 * G_2) = (V(G_1) \cup V(G_2)) \setminus \{v_1, v_2\}$$
and

$$E(G_1 * G_2) = E(G_1 - \{v_1\}) \cup E(G_2 - \{v_2\}) \cup \{uv : u \in N_{G_1}(v_1) \text{ and } v \in N_{G_2}(v_2)\}.$$ 

Example:
We say that $G$ has a **split decomposition** if there exist two graphs $G_1$ and $G_2$ with $|V(G_i)| \geq 3$, $i = 1, 2$, such that $G = G_1 \ast G_2$. $G_1$ and $G_2$ are called the **factors** of the split decomposition. Those graphs that do not have a split decomposition are called **prime graphs**.

**Example:**

- $G = G_1 \ast G_2$
- $G_1$
- $G_2$
We say that $G$ has a **split decomposition** if there exist two graphs $G_1$ and $G_2$ with $|V(G_i)| \geq 3$, $i = 1, 2$, such that $G = G_1 \ast G_2$. $G_1$ and $G_2$ are called the **factors** of the split decomposition. Those graphs that do not have a split decomposition are called **prime graphs**.

If any of the factors of a split decomposition has a split decomposition we can continue the process until every factor is prime, a star or a complete.

**Example:**

$$G = G_1 \ast G_2$$
**Theorem (Bouchet, 1987)**

Let $G$ be a graph such that $G = G_1 \ast G_2$. Then, $G$ is a circle graph if and only if $G_1$ and $G_2$ are circle graphs.

**Example:** The following figure shows two circle graphs and their circle models.
Split Decomposition on Circle Graphs

Theorem (Bouchet, 1987)

Let $G$ be a graph such that $G = G_1 * G_2$. Then, $G$ is a circle graph if and only if $G_1$ and $G_2$ are circle graphs.

Example: The following figure shows two circle graphs and their circle models.
Theorem

Let $G$ be a graph. If $G$ is not a circle graph, then any graph $H$ that arises from $G$ by edge subdivisions is not a circle graph.

Steps of the proof:
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The Prisms

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- If the edges linking the triangles of $\overline{C_6}$ are subdivided, then the resulting graph is also called a prism.
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- $\overline{C_6}$ is called a prism.
- If the edges linking the triangles of $\overline{C_6}$ are subdivided, then the resulting graph is also called a **prism**.
- Since $\overline{C_6}$ is locally equivalent to $W_5$, it is not a circle graph. So, prisms are not circle graphs.
The graph $G$ is **domino** if each of its vertices belong to at most two cliques. In addition, if each of its edges belongs to at most one clique, $G$ is **linear domino**. Linear domino graphs coincide with $\{\text{claw, diamond}\}$-free graphs.
Linear Domino Prime Graphs

**Theorem**

Let $G$ be a linear domino connected prime graph. Then, $G$ is a circle graph if and only if $G$ does not contain prisms as induced subgraphs.

**Example of linear domino prime graph**

![Prime linear domino graph](image1)

![Circle Model](image2)
Theorem

Let $G$ be a linear domino connected prime graph. Then, $G$ is a circle graph if and only if $G$ does not contain prisms as induced subgraphs.

Corollary

Let $G$ be a linear domino graph. Then, $G$ is a circle graph if and only if $G$ contains no induced prism.

Example of linear domino prime graph
Permutation graphs

- Comparability graphs were characterized by Gallai in 1967. This characterization implies the characterization by forbidden induced subgraphs for permutation graphs.
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Given two graphs $G$ and $H$. The join of $G$ and $H$ is the graph denoted by $G + H$ whose vertex set is $V(G) \cup V(H)$ and whose edge set is $E(G) \cup E(H) \cup \{vw : v \in V(G), w \in V(H)\}$.

**Lemma**

The join $G = G_1 + G_2$ is a circle graph if and only if both $G_1$ and $G_2$ are permutation graphs.
Let $G$ be a graph and let $A$ be a vertex set inducing a $P_4$ in $G$. A vertex $v$ of $G$ is said a partner of $A$ if $G[A \cup \{v\}]$ contains at least two induced $P_4$'s. Finally, $G$ is called $P_4$-tidy if each vertex set $A$ inducing a $P_4$ in $G$ has at most one partner.
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Tree-cographs are a generalization of cographs. They are defined recursively as follows: trees are tree-cographs; the disjoint union of tree-cographs is a tree-cograph; and the complement of a tree-cograph is also a tree-cograph.
A spider $H$ is a graph whose vertex set can be partitioned into three sets $S$, $C$, and $R$, where $S = \{s_1, \ldots, s_k\}$ ($k \geq 2$) is a stable set; $C = \{c_1, \ldots, c_k\}$ is a complete set; $s_i$ is adjacent to $c_j$ if and only if $i = j$ (a thin spider, denoted by $\text{thin}_k(H[R])$), or $s_i$ is adjacent to $c_j$ if and only if $i \neq j$ (a thick spider, denoted by $\text{thick}_k(H[R])$); $R$ is allowed to be empty and if it is not, then all the vertices in $R$ are adjacent to all the vertices in $C$ and nonadjacent to all the vertices in $S$. The triple $(S, C, R)$ is called the spider partition. A fat spider is obtained from a spider by adding a true or false twin of a vertex $v \in S \cup C$. 

![thin4](image1), ![thick4](image2)
Theorem (V. Giakoumakis et al, 1997).

Let $G$ be a $P_4$-tidy graph with at least two vertices. Then, exactly one of the following conditions holds:

1. $G$ is disconnected.
2. $\overline{G}$ is disconnected.
3. $G$ is isomorphic to $P_5$, $\overline{P_5}$, $C_5$, a spider, or a fat spider.
Superclasses of cographs

- $G^+$ stands for the graph $G$ plus a universal vertex.

**Theorem**

Let $G$ be a $P_4$-tidy graph. Then, $G$ is a circle graph if and only if $G$ contains no $W_5$, net$^+$, tent$^+$, or tent-with-center as induced subgraph.

![Diagram of net, tent, and tent con centro subgraphs]
Superclasses of cographs

- $G^+$ stands for the graph $G$ plus a universal vertex.

**Theorem**

Let $G$ be a $P_4$-tidy graph. Then, $G$ is a circle graph if and only if $G$ contains no $W_5$, net$^+$, tent$^+$, or tent-with-center as induced subgraph.

**Theorem**

Let $G$ be a tree-cograph. Then, $G$ is a circle graph if and only if $G$ contains no induced bipartite-claw$^+$ and no induced co-(bipartite-claw).
Definitions

- A graph is **Helly circle** if it has a circle model whose chords are all different and every subset of pairwise intersecting chords has a point in common.
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A graph is **unit circle** if it has a circle model such that every chord has the same length.
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A graph is **unit circle** if it has a circle model such that every chord has the same length.

A graph is **unit Helly circle (UHC)** if it has a circle model such that every chord has the same length and every subset of chords pairwise intersecting has a point in common.
Theorem

Let $G$ be a graph. Then the following assertions are equivalent:

1. $G$ is a unit Helly circle graph.
2. $G$ contains no induced paw, no induced claw, no induced diamond and no induced $C_n \cup K_1$ for any $n \geq 3$.
3. $G$ is a chordless cycle, a complete graph, or a disjoint union of chordless paths.

\begin{center}
\begin{tikzpicture}
  \draw (0,0) -- (0.5,0.866) -- (1,0) -- (0,0);
  \node at (0.2,0.4) {paw};
\end{tikzpicture}
\quad
\begin{tikzpicture}
  \draw (0,0) -- (0,1) -- (1,1) -- (1,0) -- (0,0);
  \node at (0.5,0.5) {diamond};
\end{tikzpicture}
\end{center}