Combinatorial aspects of renormalization in QFT

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Introduction

- **Renormalization** ↔ physics, combinatorics, algebra, number theory,…
- Particles physics described by renormalizable quantum field theory (Standard Model).
- Interpretation: physical constants depend on the observation scale.

Noncommutative space:

- definition of a new class of renormalization group (harmonic term).
- Topical problem of physics: compatibility between quantum physics and general relativity.

⇒ At high energy scale, space-time could be noncommutative.
→ Existence of renormalizable noncommutative QFT is a crucial question.
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1. Commutative scalar theory
2. Power counting
3. Renormalization
4. Hopf algebra interpretation
5. Noncommutative QFT
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Definition of the theory

- Action with parameters $m$ and $\lambda$:

$$ S[\phi] = \int d^D x \left( \frac{1}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 + \lambda \phi^4 \right) $$

- Feynman graphs: arbitrary graphs whose vertices are of coordination 4 (internal) or 1 (external).
- 1PI graphs: connected and still connected after cutting any internal line.

Amplitudes of the graphs:
- Each line carries an oriented impulsion $k \in \mathbb{R}^D$.
- Conservation of impulsion for every vertex.
- Remaining internal impulsions are integrated over in the amplitude.
- Contribution of a vertex: $\lambda$.
- Contribution of an internal line: $\frac{1}{k^2 + m^2}$.
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Physical quantities

Physical quantities: correlation functions

\[ \Gamma_N(p_1, \ldots, p_N) : \text{sum of the amplitudes of all 1PI Feynman graphs with } N \text{ external legs carrying the impulsions } p_i. \]

Particles interpretation: Feynman graphs represent particles of a certain impulsion propagating along the lines and interacting at the vertices.

Some coefficients of \( \lambda \) are divergent. Example: the tadpole.

\[ \int d^D k \frac{1}{k^2 + m^2} \]

is quadratically divergent for \( D = 4 \) in the UV sector (\( |k| \to \infty \)).
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Superficial degree of divergence

Let $G$ be a 1PI Feynman graph with $V$ vertices, $L$ loops and $N$ external legs.

- Amplitude:
  \[
  A_G(p_1,..,p_N) = \delta(p_1+..+p_N) \int \prod_{i=1}^{L} dk_i \ I_G(p_2,..,p_N,k_1,..,k_L)
  \]

- Euler characteristic $\Rightarrow L = V + 1 - \frac{N}{2}$.
- Scale transformation: $p_i \mapsto \rho p_i$ and $k_i \mapsto \rho k_i$
  \[
  A_G^{(\rho)} \propto \rho^\omega(G).
  \]

- Superficial degree of divergence of the theory:
  \[
  \omega(G) = D + (D - 4)V + (2 - D)\frac{N}{2}.
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Superficial degree of divergence

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$$\omega(G) = D + (D-4)V + (2-D)\frac{N}{2}.$$
A graph $G$ is said **primitively divergent** if $\omega(G) \geq 0$.

**Theorem**

The amplitude of a graph $G$ is absolutely convergent if and only if $G$ and each of its 1PI subgraphs are not primitively divergent.

\[
\omega(G) = D + (D - 4)V + (2 - D)\frac{N}{2}.
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- $D > 4$: $\forall N, \exists V, \omega(G) \geq 0$: non-renormalizable.
- $D < 4$: finite number of $(N, V)$ such that $\omega(G) \geq 0$: super-renormalizable.
- $D = 4$: $N = 2, 4 \Leftrightarrow \omega(G) \geq 0$: renormalizable.
Renormalizability

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Subtraction scheme

- **Dimensional regularization**: analytic continuation $D \in \mathbb{C}$. Singularity of the amplitudes for $D = 4$.

- Subtraction operator: Taylor

  $$\tau A_G(p_1,..,p_N) = \delta(p_1+..+p_N) \sum_{j=0}^{\omega(G)} \frac{1}{j!} \frac{d^j}{dt^j} A_G(tp_2,..,tp_N)|_{t=0}$$

- $G$: prim. div. graph without prim. div. subgraph

  $$A_G^R = (1 - \tau) A_G$$: renormalized amplitude
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General formula

- **Contracted graph**: let $g$ be a subgraph of $G$. \( G/g \): graph $G$ where $g$ is contracted to a point.

- $G$: graph with only one prim. div. subgraph $g$

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  A_G^R = A_G - \tau A_G - (\tau A_g)(A_{G/g} - \tau A_{G/g})
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- **General case**:

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- Solution of the recursive equations: forest formula (Zimmermann).
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Properties:

- **The renormalized amplitudes are convergent for** $D \rightarrow 4$.
- **Locality:** All the divergent counterterms $c_G = A_G - A^R_G$ are of the form of the action, so that they can be included in the constants: $\lambda \mapsto \lambda_R$, $m \mapsto m_R$...
- The correlation function $\Gamma_N(p_1, \ldots, p_N)$ is the sum of the renormalized amplitudes of all 1PI Feynman graphs with $N$ external legs carrying the impulsions $p_i$ for the renormalized constants.

→ experimental verification.
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Hopf algebra of graphs

- Complex vector space associated to 1PI Feynman graphs. Empty graph = 1 (unit).
- Product $\mu$: (disconnected) juxtaposition of graphs. $H$: generated algebra. Graded by number of loops.
- Counit is trivial: $\epsilon : H \to \mathbb{C}$, $\epsilon(1) = 1$.
- Coproduct: $\Delta : H \to H \otimes H$

$$\Delta G = G \otimes 1 + 1 \otimes G + \sum_{g \subset G} g \otimes G/g$$

where the sum is over the 1PI prim. div. subgraphs $g$ of $G$.
- Antipode: $S(G) = -G - \sum_g S(g)(G/g)$, $S(1) = 1$.

**Theorem** (Connes Kreimer)

Endowed with the coproduct $\Delta$, $H$ is a graded Hopf algebra.
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Endowed with the coproduct $\Delta$, $H$ is a graded Hopf algebra.
Hopf algebra of graphs

- Complex vector space associated to 1PI Feynman graphs.
  Empty graph = 1 (unit).
- Product $\mu$: (disconnected) juxtaposition of graphs.
  $\rightarrow \mathcal{H}$: generated algebra. Graded by number of loops.
- Counit is trivial: $\varepsilon : \mathcal{H} \rightarrow \mathbb{C}, \varepsilon(1) = 1$.
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\[ \Delta G = G \otimes 1 + 1 \otimes G + \sum_{g \subset G} g \otimes G/g \]

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Endowed with the coproduct \(\Delta\), \(\mathcal{H}\) is a graded Hopf algebra.
Renormalized amplitudes

- $A_\varepsilon$: algebra of Laurent series in $\varepsilon$.
- Amplitude $A : \mathcal{H} \to A_\varepsilon$ is a homomorphism.
- Taylor operator is a projection $\tau : A_\varepsilon \to A_\varepsilon$.
- Convolution product: if $f, g \in Hom(\mathcal{H}, A_\varepsilon)$,
  \[
  f \ast g := \mu_{A_\varepsilon} \circ (f \otimes g) \circ \Delta.
  \]
- Counterterm: twisted antipode
  \[
  c_G = -\tau \left( A_G + \sum_{g \subset G} c_g A_{G/g} \right)
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- Renormalized amplitude:
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The Moyal space

- **Space of Schwartz functions** \( f, g \in \mathcal{S}(\mathbb{R}^D, \mathbb{C}) \).
- **Deformed product:**

\[
(f \star g)(x) = \frac{1}{\pi^D \theta^D} \int d^Dy d^Dz \ f(x + y) g(x + z) e^{-2iy\Theta^{-1}z}
\]

\[
\Theta = \theta \Sigma, \quad \Sigma = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
& & & \ddots
\end{pmatrix}
\]

- For \( \theta = 0 \): \((f \star g)(x) = f(x) \cdot g(x)\).
- Extension to the multiplier algebra: \( \mathcal{M}_\theta \).
- **Tracial property:**

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Action $\phi^4$ on the Moyal space:

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UV-IR mixing for this theory (Minwalla et al. '00).

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$$\lambda \int d^4 k \frac{e^{ik\Theta p}}{k^2 + m^2} \propto |p| \to 0 \frac{1}{\theta^2 p^2}$$

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• Addition of a harmonic term to the action:

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• Power counting \((D = 4):\) renormalizable).

• Form of the counterterms \(\text{(structure of the Moyal product)}\).

\[ \Rightarrow \text{Renormalizability of the theory to all orders} \ (D = 2, 4) \]
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  \Rightarrow Renormalizability of the theory to all orders ($D = 4$)
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- Invariance under translations.
- Same properties of the flow as in the commutative theory.
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$$\tilde{S}[\phi] = S[\phi] + \int d^D p \frac{a}{2\theta^2 p^2} \hat{\phi}(-p)\hat{\phi}(p).$$

- **Power counting** ($D = 4$: renormalizable).
- **Form of the counterterms** (structure of the Moyal product).

$\Rightarrow$ **Renormalizability** of the theory to all orders ($D = 4$)

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- Ingredients of the renormalization: power counting and locality.
- BPHZ subtraction scheme has a Hopf algebra structure.
- Noncommutative field theory exhibits a new divergence: UV-IR mixing.
- First solution: with harmonic term. It defines a new class of renormalization group.
- Second solution: with term $1/p^2$. Translation-invariant but same properties as in the commutative theory.
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