Bijections for planar maps with boundaries

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Planar maps

• **Planar map** = connected graph embedded on the **sphere**, considered up to continuous deformation

![Planar maps diagram]

• **Rooted map** = map with a marked corner

![Rooted map diagram]
**Counting formulas for rooted maps**

- **Beautiful counting formulas** discovered by Tutte

<table>
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2-connected maps with $n$ edges

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\frac{4 \cdot (3n - 3)!}{(n - 1)!(2n)!}
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- Tutte’s slicings formula (1962):

Let $B[n_1, n_2, \ldots, n_k]$ be the number of rooted bipartite maps with $n_i$ faces of degree $2i$ for $i \in [1..k]$. Then

\[
B[n_1, \ldots, n_k] = 2 \frac{e!}{v!} \prod_{i=1}^{k} \frac{1}{n_i!} \left( \frac{2i - 1}{i - 1} \right)^{n_i}
\]

where $e = \#\text{edges} = \sum_i in_i$ and $v = \#\text{vertices} = e - k + 2$
Counting formulas for rooted maps

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- **Tutte’s slicings formula (1962):**

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$$B[n_1, \ldots, n_k] = \frac{2 e!}{v!} \prod_{i=1}^{k} \frac{1}{n_i!} \left(\frac{2i - 1}{i - 1}\right)^{n_i}$$

where $e = \#\text{edges} = \sum_i in_i$ and $v = \#\text{vertices} = e - k + 2$

**Counting methods:** recursive method, matrix integrals, bijections
The BDG bijection for pointed bipartite maps

[Bouttier, Di Francesco, Guitter'04]
Label vertices by distance from the marked vertex.

The BDG bijection for pointed bipartite maps

[Bouttier, Di Francesco, Guitter'04]
Construction of a labeled mobile

(i) Add a black vertex in each face

The BDG bijection for pointed bipartite maps

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Construction of a labeled mobile

(i) Add a black vertex in each face

(ii) Each map-edge gives a mobile-edge using the local rule

\[
i-1 \\
i
\]
The BDG bijection for pointed bipartite maps
[Bouttier, Di Francesco, Guitter’04]

Conditions:
(i) $\exists$ vertex of label 1
(ii) $j \leq i+1$

remove the map-edges and the marked vertex 0
The BDG bijection for pointed bipartite maps
[Bouttier, Di Francesco, Guitter’04]

![Diagram of the BDG bijection]

**Local rule**

(i) ∃ vertex of label 1
(ii) $j \leq i+1$

**Conditions:**

Theorem: The mapping is a **bijection**.

- face of degree $2i$ ↔ black vertex of degree $i$
Reformulation with orientations

Distance-labeling

Geodesic orientation

Local rule

\[ \delta = i - j \geq -1 \]

\[ \delta + 1 \text{ buds} \]
Mobile conditions in the two formulations

Formulation with labels

gives a labeled mobile

with the conditions:
(i) \( \exists \) node of label 1
(ii) \( j \leq i + 1 \)

Formulation with orientations

gives a “blossoming” mobile

with the condition:
each black vertex has as many buds as neighbors
Definition of blossoming mobiles

- **Blossoming mobile** = bipartite tree (black/white vertices) where each corner at a black vertex carries \( i \geq 0 \) buds

\[
\text{excess} = \text{number of edges} - \text{number of buds}
\]

A blossoming mobile of excess \(-2\)
Definition of blossoming mobiles

- **Blossoming mobile** = bipartite tree (black/white vertices) where each corner at a black vertex carries \( i \geq 0 \) buds

\[
\text{excess} = \text{number of edges} - \text{number of buds}
\]

- A blossoming mobile is called **balanced** iff each black vertex has as many buds as neighbors

\[
\text{Rk: implies that the excess is } 0
\]
Summary of the reformulation

Theorem: The mapping is a bijection between pointed bipartite maps and balanced blossoming mobiles

Local rule:

Condition:
Each black vertex has as many buds as neighbors

\[
\text{face of degree } 2^i \quad \leftrightarrow \quad \text{black vertex of degree } 2^i
\]
Proof of Tutte’s slicing formula

+marked edge

(rooted mobile)
Proof of Tutte’s slicings formula

Let $B[n_1, n_2, \ldots, n_k]$ be the number of rooted bipartite maps with $n_i$ faces of degree $2i$ for $i \in [1..k]$

- **Bijection** gives:

$$v \cdot B[n_1, \ldots, n_k] = 2 \cdot \text{coeff } t_1^{n_1} \cdots t_k^{n_k} \text{ in } R(t_1, t_2, \ldots)$$

where $R \equiv R(t_1, t_2, \ldots)$ is the GF of rooted mobiles given by the equation

$$R = 1 + \sum_{i \geq 1} \frac{(2i-1)}{(i-1)!} t_i R^i$$

- **Lagrange inversion formula** gives:

$$[t_1^{n_1} \cdots t_k^{n_k}] R = \frac{e!}{(v-1)!} \prod_{i=1}^{k} \frac{1}{n_i!} \left(\frac{2i-1}{i-1}\right)^{n_i}$$
• More generally, we obtain a blossoming mobile (of excess 0) if we start from a vertex-pointed orientation such that:
  - the marked vertex $v_0$ is a “source” (no incoming edge)
  - every vertex is accessible from $v_0$ by a directed path
  - there is no ccw cycle (with $v_0 \in$ outer face)
More generally, we **obtain a blossoming mobile** (of excess 0) if we start from a vertex-pointed orientation such that:

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**Theorem**: Let $O_0$ be this family of orientations, then the correspondence is a bijection with mobiles of excess 0.
Proof that it gives a tree

Start from an oriented map $M \in O_0$ and apply the local rule

Let $G$ be the graph of red edges and their incident vertices
Proof that it gives a tree

Start from an oriented map $M \in O_0$ and apply the local rule

Let $G$ be the graph of red edges and their incident vertices. $G$ has $|V_M| - 1$, white vertices, $|F_M|$ black vertices, and $|E_M|$ edges.
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**Euler relation:** $|E_M| = |V_M| + |F_M| - 2$

$\Rightarrow G$ has one more vertices than edges

hence $G$ is a tree iff $G$ is acyclic
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Assume $G$ has a cycle:

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\begin{array}{c}
v_0 \\
e_1 \\
e_2
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$v_0$ e1 e2
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\[ \Rightarrow G \text{ has one more vertices than edges} \]

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Assume $G$ has a cycle:

![Diagram of a graph with labeled vertices and edges]
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Start from an oriented map $M \in O_0$ and apply the local rule

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Assume $G$ has a cycle :

⇒ contradiction
Extension for mobiles of negative excess

More generally the “source” can be a \(d\)-gon, for any \(d \geq 0\)

Example for \(d = 3\)

For \(d > 0\), we take the \(d\)-gonal source as the outer face
Extension for mobiles of negative excess

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Example for \(d = 3\)

For \(d > 0\), we take the \(d\)-gonal source as the outer face

Let \(O_{-d}\) be the family of these orientations, still with the conditions
- the \(d\)-gonal **source** has no ingoing edge
- **accessibility** of every vertex from the source
- no ccw cycle
Theorem [Bernardi-F’10]: For $\delta \leq 0$, the correspondence $\Phi$ is a bijection between $O_\delta$ and mobiles of excess $\delta$.

degrees of the inner faces $\leftrightarrow$ degrees of the black vertices
indegrees of internal vertices $\leftrightarrow$ degrees of white vertices

cf [Bernardi’07], [Bernardi-Chapuy’10]
Extension for mobiles of negative excess

- Inverse mapping (tree → cactus → closure operations)
Specializing the correspondence

The correspondence $\Phi$ is a bijection between the family $\mathcal{O} = \bigcup_{d \geq 0} \mathcal{O}_{-d}$ of oriented maps and mobiles of nonpositive excess

**Idea:** Let $\mathcal{F}$ be the family of planar maps we consider

(e.g. bipartite maps, simple triangulations, etc.)

Prove that a map is in $\mathcal{F}$ iff it admits a **canonical orientation** in $\mathcal{O}$ specified by face-degrees and vertex-indegrees conditions

Specialize $\Phi$ to the corresponding subfamily $\mathcal{O}_\mathcal{F} \subseteq \mathcal{O}$

Gives a bijection between $\mathcal{F}$ and a well characterized family of mobiles
Application to simple triangulations

For a triangulation $T$, a \textbf{3-orientation} of $T$ is an orientation of the inner edges of $T$ such that every inner vertex has \textbf{indegree 3}.

\[\text{Rk: If a triangulation } T \text{ admits a 3-orientation, then } T \text{ is simple}\]

Assume there is a 2-cycle $C$

If there are $k$ vertices inside $C$ then there are $3k - 1$ edges inside $C$

$\Rightarrow$ total indegree is too large compared to the number of edges
Existence of a canonical 3-orientation

Theorem (Schnyder’89): Any simple triangulation admits a 3-orientation.

Theorem: Let $T$ be a simple triangulation. Then $T$ has a unique 3-orientation with no ccw cycle, the **minimal 3-orientation** (set of 3-orientations is a lattice, flip = reverse cw to ccw).

[Ossona de Mendez’94], [Brehm’03], [[Felsner’03]]
**Bijection for simple triangulations**

- From the lattice property **(taking the min)** we have:
  
  family $\mathcal{T}$ of simple triangulations $\leftrightarrow$ subfamily $\mathcal{O}_\mathcal{T}$ of $\mathcal{O}$ where:
  - faces have degree 3
  - inner vertices have indegree 3

- From the bijection $\Phi$ specialized to $\mathcal{F}$, we have:
  $\mathcal{F} \leftrightarrow$ mobiles where all vertices have degree 3

[Bernardi, F’10], other bijection in [Poulalhon, Schaeffer’03]
Counting simple triangulations

**Counting:** The generating function of mobiles with vertices of degree 3 rooted on a white corner is $T(x) = U(x)^3$, where $U(x) = 1 + xU(x)^4$.

Consequently, the number of (rooted) simple triangulations with $2n$ faces is $\frac{1}{n(2n - 1)} \binom{4n - 2}{n - 1}$.
Extension to any girth and face-degrees

\( \text{girth} = \text{length shortest cycle} \)

**Rk:** \( \text{girth} \leq \text{minimal face-degree} \)

Our approach works in any girth \( d \), with control on the face-degrees

Other approach using slice decompositions [Bouttier, Guitter’15]
Maps with boundaries

- Sphere with $k$ holes = sphere where $k$ disks have been removed

- Map with $k$ boundaries = graph embedded on the sphere with $k$ holes
  the boundaries are occupied by cycles of edges

A quadrangulations with 2 boundaries
of lengths 8 and 6, and 5 internal vertices
Maps with boundaries

- Sphere with $k$ holes = sphere where $k$ disks have been removed

![Sphere with 3 holes](image)

- Map with $k$ boundaries = graph embedded on the sphere with $k$ holes
  the boundaries are occupied by cycles of edges

![Map with 2 boundaries](image)

A quadrangulations with 2 boundaries of lengths 8 and 6, and 5 internal vertices

(also = planar map with $k$ distinguished faces whose contours are vertex-disjoint simple cycles)
Counting triangulations with boundaries

- **Formula**: \( t_n^{(k)} = \frac{2^{n+1}(2k-3)!}{(k-2)!^2} \frac{(3n+2k-3)!}{n!(2n+2k-2)!} \)

- **Formula**: \( s_n^{(k)} = \frac{2(2k-3)!}{(k-1)!(k-3)!} \frac{(4n+2k-5)!}{n!(3n+2k-3)!} \)

- **Formula**: \( a_n^{(k_1,...,k_b)} = \frac{4^{n-1}(2k + 3n - 5)!!}{(n - b + 1)!(2k + n - 1)!!} \prod_{j=1}^{b} k_j \binom{2k_j}{k_j} \)

**Notations**:
- **Counting triangulations**
- **Boundaries**
- **Internal vertices**
- **Without loops and multiple edges**, formula only for \( b = 1 \)
- **With loops and multiple edges**, nice factorized formula

**References**:
- [Mullin’65] (recursive method)
- [Brown’64] (recursive method)
- Bijective proofs in [Poulalhon, Schaeffer’06]
- Bijective proofs in [Bernardi, F’10]
- Bijective proof in [Bernardi, F’15]
Orientations for maps with boundaries

For maps with boundaries we consider orientations such that every inner boundary is a \textit{cw cycle} and the outer cycle is a boundary. These are called \textit{boundary-orientations}

To apply the mobile construction we still require the orientations to satisfy:

- the outer \( d \)-gon is a \textit{source} (no ingoing edge)
- every vertex can be \textit{reached} by a directed path starting from the source
- there is no \textit{ccw cycle}
Orientations for maps with boundaries

For maps with boundaries we consider orientations such that every inner boundary is a **cw cycle** and the outer cycle is a boundary. These are called **boundary-orientations**.

To apply the mobile construction we still require the orientations to satisfy:

- the outer $d$-gon is a **source** (no ingoing edge)
- every vertex can be **reached** by a directed path starting from the source
- there is no ccw cycle

**indegree of a boundary** $B$: total number of edges toward $B$

$B_1$ has indegree 4
$B_2$ has indegree 2
Extension of the bijection $\Phi$ to this setting
Extension of the bijection $\Phi$ to this setting

$\Downarrow$ contraction of boundaries
Extension of the bijection $\Phi$ to this setting

- **Vertex** $\circ$ of indegree $k$
- **Internal boundary**
- **Internal face** of degree $p$
- **Degré** $r$
- **Entrantes** $b$

$\downarrow$ **Contraction of boundaries**

- **Vertex** $\circ$ of degree $k$
- **Legs** $r$
- **Neighbours** $b$

- **Vertex** $\bullet$ of degree $p$
Orientations for simple triangulations with boundaries
Orientations for simple triangulations with boundaries

Triangulate each inner boundary of length $> 3$
Orientations for simple triangulations with boundaries

Triangulate each inner boundary of length $> 3$
and compute the minimal 3-orientation
Orientations for simple triangulations with boundaries

delete the added edges inside boundaries
and reorient the inner boundaries as cw cycles
Orientations for simple triangulations with boundaries

Each inner boundary of length $i$ has indegree $i + 3$
Each internal vertex has indegree 3

Such a boundary-orientation is called a **pseudo-3-orientation**
Orientations for simple triangulations with boundaries

Each inner boundary of length $i$ has indegree $i + 3$
Each internal vertex has indegree 3

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Take the **minimal** such orientation (no ccw cycle)
Orientations for simple triangulations with boundaries

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Such a boundary-orientation is called a pseudo-3-orientation

Take the **minimal** such orientation (no ccw cycle)
Mobiles for simple triangulations with boundaries

Apply the bijection $\Phi$ to the minimal pseudo-3-orientation

white vertices have
$\#\text{neighbours} - \#\text{legs} = 3$

black vertices have degree 3
Obstacles for the existence of pseudo-3-orientations

Not all 2-cycles are forbidden!

- **Contractible 2-cycle**
  - 5 edges inside
  - Total indegree 6 inside
  - Forbidden

- **Non-contractible 2-cycle not touching any boundary from the inside**
  - 9 non-boundary edges inside
  - Total indegree 10 inside
  - Forbidden

- **Non-contractible 2-cycle touching a boundary from the inside**
  - 8 non-boundary edges inside
  - Total indegree 6 inside
  - Not forbidden
Pseudo-girth parameter

For a map with boundaries that is planarly embedded

**pseudo-girth** = length of a shortest curve of the form

(curve that is the outer border of a region consisting of non-boundary faces)

\[ R_k: \quad \text{girth} \leq \text{pseudo-girth} \leq \text{contractible girth} \]
Pseudo-girth parameter

For a map with boundaries that is planarly embedded

\[ \text{pseudo-girth} = \text{length of a shortest curve of the form} \]

\( \text{curve that is the outer border of a region consisting of non-boundary faces} \)

\[ \text{Rk: } \text{girth} \leq \text{pseudo-girth} \leq \text{contractible girth} \]

The map is called **pseudo-simple** if the pseudo-girth is \( \geq 3 \)
Results for pseudo-simple triangulations with boundaries

A triangulation with boundaries
(outer face being a triangular boundary-face)
is pseudo simple iff admits a pseudo 3–orientation

bijection with explicit mobiles

- internal face (degree 3) ↔ black vertex of degree 3
- inner boundary of length $i$ ↔ white vertex with $i$ legs and $i + 3$ neighbours

Counting formula:
Let $N[n; a, k_1, \ldots, k_r]$ be the number of pseudo-simple triangulations where:
- the outer boundary has length $a$
- the inner boundaries $B_1, \ldots, B_r$ have lengths $k_1, \ldots, k_r$
- there are $n$ internal vertices
- in every boundary, a vertex is distinguished

\[
N[n; a, k_1, \ldots, k_r] = \frac{2(2a - 3)!}{(a - 3)!(a - 1)!} \frac{(4n + 4r + 2L - 5)!}{(n - 1)!(3m + 4r + 2L - 3)!} \prod_{i=1}^{r} \binom{2i + 2}{i}
\]

where $L = a + \sum_{i=1}^{r} k_i$ (total boundary length)
Results in any given pseudo-girth

We have a bijection in each pseudo-girth $d \geq 1$
for maps with boundaries, with inner face degrees in $\{d, d + 1, d + 2\}$
Results in any given pseudo-girth

We have a bijection in each pseudo-girth $d \geq 1$ for maps with boundaries, with inner face degrees in $\{d, d + 1, d + 2\}$

Pseudo-girth-constraint is void for

$\begin{align*}
&d = 1 \text{ (recover Krikun’s formula)} \\
&d = 2 \text{ bipartite case (new formula for quadrangulations with boundaries)}
\end{align*}$
Factorized counting formulas

- Let $m \geq 0$ and $\ell_1, \ldots, \ell_r$ positive integers
- Let $T[m; \ell_1, \ldots, \ell_r]$ (resp. $Q[m; \ell_1, \ldots, \ell_r]$) be the set of triangulations (resp. quadrangulations) with $r$ boundaries $B_1, \ldots, B_r$ s.t.
  - there are $m$ internal vertices
  - every boundary $B_i$ has length $\ell_i$ and a marked corner

### Triangulations : Krikun’s formula (2007)

$$|T[m; a_1, \ldots, a_r]| = \frac{4^k(e - 2)!!}{m!(2b + k)!!} \prod_{i=1}^{r} a_i \binom{2a_i}{a_i}$$

with $b = \sum_i a_i$, $k = r + m - 2$, and $e = 2b + 3k$

### Quadrangulations : [Bernardi, F’15]

$$|Q[m; 2a_1, \ldots, 2a_r]| = \frac{3^k(e - 1)!!}{m!(3b + k)!!} \prod_{i=1}^{r} 2a_i \binom{3a_i}{a_i}$$

with $b = \sum_i a_i$, $k = r + m - 2$, and $e = 2b + 3k$
Solution of the dimer model on quadrangulations

Map with dimers = pair \((M, X)\) where \(M\) is a map and \(X\) is a subset of edges giving a partial-matching.
Solution of the dimer model on quadrangulations

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- Dimer model on (rooted) quadrangulations

**Generating function**: \(F(t, w) = \sum \text{t}^{\text{#faces}} \text{w}^{\text{#dimers}}\) configurations

\(\text{map with 2 dimers}\)
Solution of the dimer model on quadrangulations

Map with dimers = pair \((M, X)\) where \(M\) is a map and \(X\) is a subset of edges giving a partial-matching

- Dimer model on (rooted) quadrangulations
  
  **Generating function** : 
  \[ F(t, w) = \sum \text{ configurations } t^{\# \text{ faces}} w^{\# \text{ dimers}} \]

  \[ F(t, w) = R - 1 - t R^3 - 6 w t^2 R^6 \]

  où \( R = 1 + 3t R^2 + 9w t^2 R^5 \)
Solution of the dimer model on quadrangulations

Map with dimers = pair \((M, X)\) where \(M\) is a map and \(X\) is a subset of edges giving a partial-matching

- Dimer model on (rooted) quadrangulations

**Generating function**: \[ F(t, w) = \sum_{\text{configurations}} t^{\# \text{faces}} w^{\# \text{dimers}} \]

- Dimer model on (rooted) quadrangulations

Asymptotics: for \(w \in \mathbb{R}\) fixed, \([t^n] F \sim c_w \gamma_w^n n^{-5/2}\) except at critical weight \(w_0 = -3/125\) where \([t^n] F \sim c_0 \gamma_0^n n^{-7/3}\)

\[ F(t, w) = R - 1 - t R^3 - 6 w t^2 R^6 \]

\[ \text{où } R = 1 + 3 t R^2 + 9 w t^2 R^5 \]
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- Solution of the dual model in [Bouttier, Di Francesco, Guitter’03]

**Generating function**: \(F(t, w) = R - 1 - t R^3 - 6wt^2 R^6\)

\(\text{où } R = 1 + 3tR^2 + 9wt^2 R^5\)

- Solution of the dual model in [Bouttier, Di Francesco, Guitter’03]

\(F(t, w) = R - 1 - t R^3 - 15wt^2 R^4\)

\(\text{où } R = 1 + 3tR^2 + 30wt^2 R^3\)

critical weight \(w_0 = -1/10\)

where typical distance \(\approx n^{1/6}\)

weight \(t^7 w^3\)

weight \(3t^2 w\) per hexagonal face

bijection