Universal Asymptotics for Positive Catalytic Equations

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Journée MathStic “Combinatoire et Probabilités”
One Functional Equation

Unrestricted paths

\[ B(z) = 1 + 2zB(z) \]

\[ B(z) = \frac{1}{1 - 2z} \quad \text{(polar singularity)} \]

\[ b_n = [z^n]B(z) = 2^n \]
One Functional Equation

Dyck paths

\[ B(z) = 1 + z^2 B(z)^2 \]

\[ B(z) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2} \] (square root singularity)

\[ b_{2n} = [z^{2n}]B(z) = \frac{1}{n} \binom{2n}{n} \sim \sqrt{\frac{8}{\pi}} n^{-3/2} 2^n \]
One Functional Equation

Non-negative lattice paths

\[ f_{n,i} \text{ ... number of non-negative paths from (0, 0) → (n, i)} \]

\[ f_i(z) = \sum_{n \geq 0} f_{n,i} z^i \]

\[ F(z, u) = \sum_{i \geq 0} f_i(z) u^i = \sum_{n,i \geq 0} f_{n,i} z^n u^i \]

\[ f_0(z) = 1 + zf_1(z), \]
\[ f_i(z) = zf_{i-1}(z) + zf_{i+1}(z) \quad (i \geq 1) \]

\[ F(z, u) = 1 + zuF(z, u) + z \frac{F(z, u) - F(z, 0)}{u} \]

\[ u \text{ ... “catalytic variable”} \]
Non-negative lattice paths

\[ F(z, 0) = \frac{1 - \sqrt{1 - 4z^2}}{2z^2} \quad \text{(squareroot singularity)} \]

\[ f_{2n,0} = [z^{2n}]F(z, 0) = \frac{1}{n} \binom{2n}{n} \sim \sqrt{\frac{8}{\pi}} n^{-3/2} 2^n \]
One Functional Equation

Planar Maps

\[ M_{n,k} \quad \text{... number of planar maps with } n \text{ edges and outer face valency } k \]

\[ M(z, u) = \sum_{n,k} M_{n,k} z^n u^k \]
One Functional Equation

Planar Maps

\[ M(z, u) = 1 + zu^2M(z, u)^2 + uz \frac{uM(z, u) - M(z, 1)}{u - 1}. \]

u ... “catalytic variable”

\[ M(z, 1) = -\frac{1}{54z^2} \left( 1 - 18z - \left( 1 - 12z \right)^{3/2} \right) \quad (3/2\text{-singularity}) \]

\[ M_n = [z^n]M(z, 1) = \frac{2(2n)!}{(n + 2)!n!} 3^n \sim \frac{2}{\sqrt{\pi}} \cdot n^{-5/2} 12^n \]
One Functional Equation

One positive linear equation

Theorem 1. Polar singularity:

\[ Q_0(z), \, Q_1(z) \ldots \text{polynomials with non-negative coefficients.} \]

\[ B(z) = Q_0(z) + zQ_1(z)B(z) \]

\[ \implies b_n = [z^n]B(z) \sim c_j \cdot z_0^{-n}, \quad n \equiv j \mod m \]

for \( j \in \{0, 1, \ldots, m - 1\} \) and some \( m \geq 1. \)

\( z_0 > 0 \) is given by \( z_0Q_1(z_0) = 1. \)

Remark. Proof is simple analysis of \( B(z) = Q_0(z)/(1 - zQ_1(z)). \)
One Functional Equation

One positive non-linear equation

Theorem 2. [Bender, Canfield, Meir+Moon, …] \textbf{Squareroot sing.:}

\[ Q(z, y) \ldots \text{polynomial with non-negative coefficients} \text{ and } Q(0, 0) = 0 \text{ and } Q_{yy} \neq 0. \]

\[ B(z) = Q(z, B(z)) \]

\[ \implies b_n = [z^n]B(z) \sim c \cdot n^{-3/2}z_0^{-n}, \quad n \equiv j_0 \mod m, \]

and \[ b_n = 0 \] for \( n \not\equiv j_0 \mod m, \) where \( m \geq 1. \)

\( z_0 > 0 \) satisfies \( b_0 = Q(z_0, b_0) \) and \( 1 = Q_y(z_0, b_0) \) for some \( b_0 > 0. \)

\textbf{Remark.} Proof is based on the analysis of the singular point \((z_0, b_0)\) of the curve \( b = Q(z, b) \) that leads to the squareroot singularity \( B(z) = g(z) - h(z)\sqrt{1 - z/z_0}. \)
One Functional Equation

One positive linear catalytic equation

Theorem 3. [D. + Noy + Yu] Squareroot singularity:

\( Q_0(z, u), Q_1(z, u), Q_2(z, u) \ldots \) polynomials with non-negative coefficients such that \( Q_1, u \neq 0 \) and \( u \not| Q_2 \).

\[
M(z, u) = Q_0(z, u) + z M(z, u) Q_1(z, u) + z \frac{M(z, u) - M(z, 0)}{u} Q_2(z, u)
\]

\[\Rightarrow \quad M_n = [z^n] M(z, 0) \sim c \cdot n^{-3/2} z_0^{-n}, \quad n \equiv j_0 \text{ mod } m,\]

(for some constants \( c, z_0 > 0 \)) and \( M_n = 0 \) for \( n \not\equiv j_0 \text{ mod } m \), where \( m \geq 1 \).
One Functional Equation

One positive non-linear catalytic equation

Theorem 4. [D.+Noy+Yu] 3/2-Singularity:

\[ Q(y_0, y_1, z, u) \] ... polynomial with non-negative coefficients that is non-linear in \( y_0, y_1 \) (and depends on \( y_0, y_1 \)) and \( Q_0(u) \) a non-negative polynomial in \( u \).

\[
M(z, u) = Q_0(u) + zQ\left(M(z, u), \frac{M(z, u) - M(z, 0)}{u}, z, u\right)
\]

\[
\implies M_n = [z^n] M(z, 0) \sim c \cdot n^{-5/2} z_0^{-n}, \quad n \equiv j_0 \mod m,
\]

(for some constants \( c, z_0 > 0 \)) and \( M_n = 0 \) for \( n \not\equiv j_0 \mod m \), where \( m \geq 1 \).
System of Functional Equations

$Q_1, \ldots, Q_d$ ... polynomials with non-negative coefficients.
y_1 = y_1(z), \ldots, y_d = y_d(z)$ ... solution of the system:

\[
y_1 = Q_1(z, y_1, \ldots, y_d), \\
\vdots \\
y_d = Q_d(z, y_1, \ldots, y_d).
\]

Recall that if $d = 1$ then the single equation $y = Q(z, y)$ has either a polar singularity (if it is linear) or a squareroot singularity (if it is non-linear).

Question. What happens for $d > 1$ ??
Systems of functional equations

Strongly connected dependency graph

**Theorem 5** [D., Lalley, Woods]

\[ y = Q(z, y) \] \[ \quad \text{non-negative} \] (and well defined) polynomial system of \( d \geq 1 \) equations such that the dependency graph is strongly connected.

Then the situation is the same as for a single equation.

It the system is linear then we have a common polar singularity and

\[ [z^n] y_1(z) \sim c_j \cdot z_0^{-n}, \quad n \equiv j \mod m \]

whereas if it is non-linear then we have a square root singularity and

\[ [z^n] y_1(z) \sim c \cdot n^{-3/2} z_0^{-n}, \quad n \equiv j_0 \mod m. \]
Systems of functional equations

General dependency graph

**Theorem 6** [Banderier+D.]

\[ y = Q(z, y) \quad \text{... non-negative} \quad \text{(and well defined) polynomial system of equations.} \]

\[ \implies [z^n] y_1(z) \sim c_j n^{\alpha_j} \rho_j^{-n} \quad (n \equiv j \mod m), \]

for \( j \in \{0, 1, \ldots, m - 1\} \) for some \( m \geq 1 \), where

\[ \alpha_j \in \{-2^{-k} - 1 : k \geq 1\} \cup \{m2^{-k} - 1 : m \geq 1, k \geq 0\}. \]
Theorem 3: Kernel Method

\[ M(z, u) = Q_0(z, u) + zM(z, u)Q_1(z, u) + z\frac{M(z, u) - M(z, 0)}{u}Q_2(z, u) \]

rewrites to

\[ M(z, u) \left( 1 - zQ_1(z, u) - \frac{z}{u}Q_2(z, u) \right) = Q_0(z, u) - \frac{z}{u}M(z, 0)Q_2(z, u). \]

If \( u = u(z) \) satisfies the kernel equation

\[ 1 - zQ_1(z, u(z)) - \frac{z}{u(z)}Q_2(z, u(z)) = 0 \]

Then the right hand side is also zero and we obtain

\[ M(z, 0) = \frac{Q_0(z, u(z))}{1 - zQ_1(z, u(z))} \]
Theorem 3: Kernel Method

The kernel equation

\[ 1 - zQ_1(z, u(z)) - \frac{z}{u(z)}Q_2(z, u(z)) = 0 \]

rewrites to

\[ u(z) = zQ_2(z, u(z)) + zu(z)Q_1(z, u(z)) \]

By Theorem 2 we, thus, obtain a square root singularity for \( u(z) \) which implies a square root singularity for

\[ M(z, 0) = \frac{Q_0(z, u(z))}{1 - zQ_1(z, u(z))}. \]
Theorem 4: Bousquet-Melou–Jehanne Method

Let $P(x_0, x_1, z, u)$ be an analytic function such that $(y(z) = M(z, 0))$

$$P(M(z, u), y(z), z, u) = 0.$$  

By taking the derivative with respect to $u$ we get

$$P_{x_0}(M(z, u), y(z), z, u) M_u(z, u) + P_u(M(z, u), y(z), z, u) = 0.$$  

Key observation:

$$\exists u(z) : P_{x_0}(M(z, u(z)), y(z), z, u(z)) = 0 \implies P_u(M(z, u(z)), y(z), z, u(z)) = 0$$

Thus, with $f(z) = M(z, u(z))$ we get the system for $f(z), y(z), u(z)$

$$P(f(z), y(z), z, u(z)) = 0$$

$$P_{x_0}(f(z), y(z), z, u(z)) = 0$$

$$P_u(f(z), y(z), z, u(z)) = 0.$$
Theorem 4: Bousquet-Melou–Jehanne Method

Set (as given in our case)

\[ P(x_0, x_1, z, u) = Q_0(u) + zQ(x_0, (x_0 - x_1)/u, z, u) - x_0. \]

Then the system \( P = 0, \ P_{x_0} = 0, \ P_u = 0 \) rewrites to

\[
\begin{align*}
  f(z) &= Q_0(u(z)) + zQ(f(z), w(z), z, u(z)), \\
  u(z) &= uz(z)Qy_0(f(z), w(z), z, u(z)) + zQy_1(f(z), w(z), z, u(z)), \\
  w(z) &= Q_{0,u}(u(z)) + zQv(f(z), w(z), z, u(z)) + zw(z)Qy_0(f(z), w(z), z, u(z)),
\end{align*}
\]

where

\[ w(z) = \frac{f(z) - y(z)}{u(z)}. \]

This is a positive strongly connected polynomial system.
Thus, by Theorem 5 the solution functions $f(z), u(z), w(z)$ have a \textbf{squareroot singularity} at some common singularity $z_0$:

\[
\begin{align*}
  f(z) &= g_1(z) - h_1(z) \sqrt{1 - \frac{z}{z_0}}, \\
  u(z) &= g_2(z) - h_2(z) \sqrt{1 - \frac{z}{z_0}}, \\
  w(z) &= g_3(z) - h_3(z) \sqrt{1 - \frac{z}{z_0}}.
\end{align*}
\]

\[\implies y(z) = f(z) - u(z)w(z) \text{ has also a squareroot singularity at } z_0\]

\[
y(z) = g_4(z) - h_4(z) \sqrt{1 - \frac{z}{z_0}} = a_0 + a_1 \sqrt{1 - \frac{z}{z_0}} + a_2 \left(1 - \frac{z}{z_0}\right) + a_3 \left(1 - \frac{z}{z_0}\right)^{3/2} + \cdots
\]

but maybe there are \textbf{cancellations of coefficients} $a_j$ (and actually this happens!!!): we have $a_1 = 0$ and $a_3 > 0$. 
Bousquet-Melou–Jehanne Method – General Case

1\textsuperscript{st} difference

\[ M(z, u) = Q_0(u) + zQ \left( M(z, u), \frac{M(z, u) - M(z, 0)}{u}, z, u \right) \]

Higher differences

\[ M(z, u) = Q_0(u) + zQ \left( M(z, u), \Delta^{(1)}(z, u), \ldots, \Delta^{(d)}(z, u), z, u \right) \]

where

\[ \Delta^{(j)}(z, u) = \frac{M(z, u) - M(z, 0) - M_u(z, 0)u - \cdots - M_{uj-1}(z, 0)u^{j-1}}{u^j} \]

Theorem (Bousquet-Melou–Jehanne). Such an equation has always an algebraic solution.
Kernel Method for the Linear Case ($d = 2$)

\[ M(z, u) = Q_0(z, u) + zM(z, u)Q_1(z, u) + z\frac{M(z, u) - M(z, 0)}{u}Q_2(z, u) \]
\[ + z\frac{M(z, u) - M(z, 0) - Mu(z, 0)u}{u^2}Q_3(z, u) \]

rewrites to

\[ M(z, u) \left( 1 - zQ_1(z, u) - zQ_2(z, u) - z\frac{z}{u}Q_3(z, u) \right) \]
\[ = Q_0(z, u) - M(z, 0) \left( \frac{z}{u}Q_2(z, u) + \frac{z}{u^2}Q_3(z, u) \right) - Mu(z, 0)\frac{z}{u}Q_3(z, u) \]

Here two functions $u = u_1(z)$ and $u = u_2(z)$ satisfy the kernel equation

\[ 1 - zQ_1(z, u(z)) - z\frac{z}{u(z)}Q_2(z, u(z)) - z\frac{z}{u(z)^2}Q_3(z, u(z)) = 0 \]

The right hand side is then zero for $u = u_1(z)$ and $u = u_2(z)$ which is a linear system for $M(z, 0)$ and $Mu(z, 0)$.
Kernel Method for the Linear Case ($d = 2$)

The kernel equation for $u = u_1(z)$ and $u = u_2(z)$

$$1 - zQ_1(z, u(z)) - \frac{z}{u(z)}Q_2(z, u_1, 2(z)) - \frac{z}{u(z)^2}Q_3(z, u(z)) = 0$$

rewrites to

$$u(z)^2 = u(z)^2 zQ_1(z, u(z)) + z u(z) Q_2(z, u_1, 2(z)) + z Q_3(z, u(z))$$

or to

$$u_1(z) = \sqrt{zu_1(z)^2 Q_1(z, u_1(z)) + z u_1(z) Q_2(z, u_1(z)) + Q_3(z, u_1(z))}$$

$$u_2(z) = -\sqrt{zu_2(z)^2 Q_1(z, u_2(z)) + z u_2(z) Q_2(z, u_2(z)) + Q_3(z, u_2(z))}$$

We lose the property that $u_1(z)$ and $u_2(z)$ have just non-negative coefficients and it is not clear that there is a squareroot singularity.
Bousquet-Melou–Jehanne Method for the Non-linear Case

Let $P(x_0, x_1, x_2, z, u)$ be an analytic function such that

$$P(M(z, u), y_0(z), y_1(z), z, u) = 0.$$ 

By taking the derivative with respect to $u$ we get

$$P_{x_0}(M(z, u), y_0(z), y_1(z), z, u) \cdot M_u(z, u) + P_u(M(z, u), y_0(z), y_1(z), z, u) = 0.$$ 

**Key observation:**

$$P_{x_0}(M(z, u(z)), y_0(z), y_1(z), z, u(z)) = 0 \implies P_u(M(z, u(z)), y_0(z), y_1(z), z, u(z)) = 0.$$ 

We need **two functions** $u_1(z)$ and $u_2(z)$. Setting $f_j(z) = M(z, u_j(z))$ we get the system for $f_1(z), f_2(z), y_0(z), y_1(z), u_1(z), u_2(z)$

\[ P(f_1(z), y_0(z), y_1(z), z, u_1(z)) = 0, \quad P(f_2(z), y_0(z), y_1(z), z, u_2(z)) = 0 \]
\[ P_{x_0}(f_1(z), y_0(z), y_1(z), z, u_1(z)) = 0, \quad P_{x_0}(f_2(z), y_0(z), y_1(z), z, u_2(z)) = 0 \]
\[ P_u(f_1(z), y_0(z), y_1(z), z, u_1(z)) = 0, \quad P_u(f_2(z), y_0(z), y_1(z), z, u_2(z)) = 0 \]
Bousquet-Melou–Jehanne Method for the Nonlinear Case

Set (as given in our case)

\[ P(x_0, x_1, x_2, z, u) = Q_0(u) + zQ(x_0, (x_0 - x_1)/u, (x_0 - x_1 - ux_2)/u^2, z, u) - x_0. \]

Then the above system rewrites to

\[
\begin{align*}
    f_{1,2}(z) &= Q_0(u_{1,2}(z)) + + zQ(f_{1,2}(z), \frac{f_{1,2}(z) - M(z, 0)}{u_{1,2}(z)}, \frac{f_{1,2}(z) - M(z, 0) - u_{1,2}(z)M_u(z, 0)}{u_{1,2}(z)^2}, z, u_{1,2}(z)), \\
    u_{1,2}(z)^2 &= zu_{1,2}(z)^2Q_{y_0}(\cdots) + zu_{1,2}(z)Q_{y_1}(\cdots) + zQ_{y_2}(\cdots), \\
    Q_{0,u}(u_{1,2}(z)) &= \frac{f_{1,2}(z) - M(z, 0)}{u_{1,2}(z)}(1 - zQ_{y_0}(\cdots) \\
    &+ z\frac{f_{1,2}(z) - M(z, 0) - u_{1,2}(z)M_u(z, 0)}{u_{1,2}(z)^3}Q_{y_2}(\cdots)
\end{align*}
\]

This cannot be rewritten into a positive strongly connected polynomial system.
Second Differences: The Linear Case

Theorem 3’. [D.+Hainzl] Squareroot singularity:

\[ Q_0(z, u), Q_1(z, u), Q_2(z, u), Q_3(z, u) \ldots \] polynomials with non-negative coefficients (+ some technical conditions).

\[
M(z, u) = Q_0(z, u) + z M(z, u) Q_1(z, u) + z^2 \frac{M(z, u) - M(z, 0)}{u} Q_2(z, u)
+ z^3 \frac{M(z, u) - M(z, 0) - M_u(z, u) u}{u^2} Q_3(z, u)
\]

\[
\implies M_n = [z^n] M(z, 0) \sim c \cdot n^{-3/2} z_0^{-n}, \quad n \equiv j_0 \mod m,
\]

(for some constants \( c, z_0 > 0 \)) and \( M_n = 0 \) for \( n \not\equiv j_0 \mod m \), where \( m \geq 1 \).
Second Differences: The Non-linear Case

Theorem 4’. [D.+Hainzl] 3/2-Singularity:

\( Q(y_0, y_1, y_2, z, u) \) \ldots polynomial with non-negative coefficients that is \textbf{non-linear} in \( y_0, y_1, y_2 \) (+ some technical conditions).

\[ M(z, u) = Q_0(u) + zQ \left( M(z, u), \frac{M(z, u) - M(z, 0)}{u}, \frac{M(z, u) - M(z, 0) - M_u(z, 0)u}{u^2}, z, u \right) \]

\[ \implies M_n = [z^n] M(z, 0) \sim c \cdot n^{-5/2} z_0^{-n}, \quad n \equiv j_0 \mod m, \]

(for some constants \( c, z_0 > 0 \)) and \( M_n = 0 \) for \( n \neq j_0 \mod m \), where \( m \geq 1 \).
Applications

One-dimensional non-negative lattice path with steps $\pm 1$ and $\pm 2$

$E_0(z) = 1 + z(E_1(z) + E_2(z)),$

$E_1(z) = z(E_0(z) + E_1(z) + E_2(z)),$

$E_k(z) = z(E_{k-2}(z) + E_{k-1}(z) + E_{k+1}(z) + E_{k+2}(z)) \quad (k \geq 2),$

which gives for $E(z, u) = \sum_{k \geq 0} E_k(z) u^k$

$$E(z, u) = 1 + z(u + u^2)E(z, u) + z\frac{E(z, u) - E(z, 0)}{u} + z\frac{E(z, u) - E(z, 0) - uE_v(u, 0)}{u^2}.$$
Applications

3-Constellations in Eulerian Maps

\[ M(z, u) = 1 + zuM(z, u)^3 + zu(2M(z, u) + M(z, 1)) \frac{M(z, u) - M(z, 1)}{u - 1} \]
\[ + zu \frac{M(z, u) - M(z, 1) - M_u(z, 1)(u - 1)}{(u - 1)^2} \]

Remark. There are many equations of this type in the context of map enumeration (even more generally with higher differences)
Higher Differences

Conjecture

Consider a catalytic equation with higher differences:

\[ M(z, u) = Q_0(u) + zQ\left(M(z, u), \Delta^{(1)}(z, u), \ldots, \Delta^{(d)}(z, u), z, u\right) \]

where \( Q_0 \) and \( Q \) have non-negative coefficients (+ some technical conditions)

- If \( Q \) is linear in \( y_0, y_1, \ldots, y_d \) then \( M(z, 0) \) has a squareroot singularity

- If \( Q \) is non-linear in \( y_0, y_1, \ldots, y_d \) then \( M(z, 0) \) has a \( 3/2 \)-singularity
Theorem 3’: Proof Ideas for the Linear Case

Set

\[ R(z, u) = zu^2Q_1(z, u) + zuQ_2(z, u) + Q_3(z, u) \]

Then the kernel equation for \( u = u_{1,2}(z) \) reads as

\[ u^2 = R(z, u) \]

Ansatz

\[ u_1(z) = g(z) + \sqrt{h(z)} \quad u_2(z) = g(z) - \sqrt{h(z)} \]
Proof Ideas for the Linear Case

\[ u^2 = (g \pm \sqrt{h})^2 = g^2 + h \pm \sqrt{h} \cdot 2g \]

\[ R(z, g \pm \sqrt{h}) = \sum_k R_k(z)(g \pm \sqrt{h})^k \]

\[ = \sum_k R_k(z) \sum_{j=0}^{k} \binom{k}{j} g^{k-j}(\pm 1)^j h^{j/2} \]

\[ = \sum_{k, \ell} R_k(z) \binom{k}{2\ell} g^{k-2\ell} h^\ell \pm \sqrt{h} \sum_{k, \ell} R_k(z) \binom{k}{2\ell+1} g^{k-2\ell-1} h^\ell \]

\[ = R^+(z, g, h) \pm \sqrt{h} \cdot R^-(z, g, h) \]

\[ u^2 = R(z, u) \implies g^2 + h = R^+(z, g, h), \quad 2g = R^-(z, g, h) \]
Proof Ideas for the Linear Case

The kernel equation

\[ u^2 = R(z, u) \]

rewrites to

\[ g^2 + h = R^+(z, g, h), \quad 2g = R^-(z, g, h) \]

or to

\[ h = R^+(z, g, h) - g^2, \quad g = \frac{1}{2}R^-(z, g, h) \]

This is not a positive system!
Proof Ideas for the Linear Case

Lemma

The functions $g(z)$, $h(z)$ have the following properties:

- they have non-negative coefficients
- they have a common *squareroot singularity* $z_0$
- the function $u_2(z) = g(z) - \sqrt{h(z)}$ is regular at $z_0$

Corollary. The functions $M(z, 0)$, $M_u(z, 0)$ have a *squareroot singularity* at $z_0$, too.
Additional Parameters

Number of vertices in planar maps

\[ M(z, x, u) \] generating function of rooted planar maps, where the variable \( z \) corresponds to the number of edges, \( x \) to the number of vertices and \( u \) to the root face valency.

\[
M(z, x, v) = x + zu^2M(z, x, u)^2 + zu \frac{M(z, x, 1) - uM(z, x, u)}{1 - u}
\]

\( X_n \) number of vertices in a random planar map with \( n \) edges

Central Limit Theorem

\( X_n \) satisfies a central limit theorem with \( \mathbb{E}[X_n] = \frac{1}{2}n + O(1) \) and \( \text{Var}[X_n] = \frac{5}{32}n + O(1) \).
Additional Parameters

Theorem 7

Suppose that $M(z, x, u)$ and $M_1(z, x)$ are the solutions of the catalytic equation

$$P(M(z, x, u), M_1(z, x), z, x, u) = 0,$$

where the function $P(x_0, x_1, z, x, u)$ is analytic and $M_1(z, 1)$ has a singularity at $z = z_0$ of the form

$$M_1(z, 1) = y_0 + y_2 \left(1 - \frac{z}{z_0}\right) + y_3 \left(1 - \frac{z}{z_0}\right)^{3/2} + \cdots,$$

with $y_3 \neq 0$ (+ some technical conditions)

Then $M_1(z, x)$ has a local singular representation of the form

$$M_1(z, x) = a_0(x) + a_2(x) \left(1 - \frac{z}{\rho(x)}\right) + a_3(x) \left(1 - \frac{z}{\rho(x)}\right)^{3/2} + \cdots$$

Corollary. Hwang’s Quasi-Power-Theorem leads then to a Central Limit Theorem
Additional Parameters

Vertices of degree $k$ in planar maps

$M(z, x, u) \ldots$ generating function for rooted planar maps, where $z$ corresponds to the number of edges, $x$ to the number of non-root faces of degree $k$, and $u$ to the root-face degree

$$M(z, x, u) \left( 1 - z(x - 1)u^{-k+2} \right)$$

$$= 1 + zu^2 M(z, x, u) + zu \frac{uM(z, x, u) - M(z, x, 1)}{u - 1}$$

$$- z(x - 1)u^{-k+2} G(z, x, M(z, x, 1), u),$$

where $G(z, x, y, u)$ is a polynomial of degree $k - 2$ in $u$ with coefficients that are analytic functions in $(z, x, y)$ for $|z| \leq 1/10$, $|x - 1| \leq 2^{1-k}$, and $|y| \leq 2$. 
Additional Parameters

Pure $k$-gons in planar maps

We say that a face is a pure $k$-gon ($k \geq 2$) if it is incident exactly to $k$ different edges and $k$ different vertices.

$P(z, x, u)$ ... generating function for rooted planar maps, where $z$ corresponds to the number of edges, $x$ to the number of non-root faces that are pure $k$-gons, and $u$ to the root-face degree.

$$P(z, x, u) = 1 + zu^2 P(z, x, u) + z u \frac{uP(z, x, u) - P(z, x, 1)}{u - 1}$$

$$- z(x - 1)u^{-k+2} \tilde{G}(z, x, P(z, x, 1), u),$$

where $\tilde{G}(z, x, y, u)$ is a polynomial of degree $k - 2$ in $u$ with coefficients that are analytic functions in $(z, x, y)$ for $|z| \leq 1/10$, $|x - 1| \leq 2^{1-k}$, and $|y| \leq 2$. 
Additional Parameters

Vertices of degree \( k \) in simple planar maps

\( S(z, x, u) \) ... generating function for simple rooted planar maps, where \( z \) corresponds to the number of edges, \( x \) to the number of non-root vertices of degree \( k \), and \( u \) to the root-face degree.

\[
S(z, x, u) = 1 + zu^2S(z, x, u) + zu \frac{uS(z, x, u) - S(z, x, 1)}{u - 1}
- zuS(z, x, u)S(z, x, 1) - (S(z, x, u) - 1)(S(z, x, 1) - 1)
+ (x - 1) \left( zu^{-k+2}S(z, x, u)G_1(z, x, S(z, x, 1), u)
- zuS(z, x, u)G_2(z, x, S(z, x, 1))
- (S(z, x, u) - 1)G_3(z, x, S(z, x, 1)) \right),
\]

where \( G_1(z, x, y, u) \) is a polynomial of degree \( k - 2 \) in \( u \) with coefficients that are analytic functions in \((z, x, y)\) for \( |z| \leq 2/25, |x - 1| \leq 2^{-k-5} \), and \( |y - 1| \leq 2/5 \). Similarly properties hold for the functions \( G_2(z, x, y) \) and \( G_3(z, x, y) \).
Thank You!