On the Polynomial Part of a Restricted Partition Function

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Joint work with

Christophe Vignat

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Le 21 novembre 2017 à 14h45 en B107, Christophe Vignat nous parlera de : ***Séries génératrices pour la fonction "somme des chiffres en base b"***

Résumé : La fonction $s_b(n)$ est égale à la somme des chiffres qui composent la représentation de l'entier n en base b. De nombreuses séries associées à cette séquence ont été calculées, par exemple dans J.P. Allouche, J. Shallit, Sums of digits and the Hurwitz Zeta function, Analytic Number Theory pp. 19-30, LNM 1434. On présentera de nouvelles séries génératrices associées à la suite $s_b(n)$, ainsi que certains produits infinis. L'obtention de ces résultats utilise des outils rencontrés dans le domaine du calcul symbolique et de la théorie des probabilités. Ce travail a été realisé en collaboration avec T. Wakhare.

1. Introduction

An interesting topic in the theory of partitions is that of **restricted partitions**:

Given a vector

$$\mathbf{d} := (d_1, d_2, \ldots, d_m)$$

of positive integers, let $W(s, \mathbf{d})$ be the number of partitions of the integer s with parts in $\mathbf{d}$,

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i.e., $W(s, \mathbf{d})$ is the number of solutions of

$$d_1x_1 + d_2x_2 + \cdots + d_mx_m = s \qquad (1)$$

in nonnegative integers $x_1, \dots, x_m$.

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So $W(6, (1, 2, 3)) = 7$.

Remark: When $\mathbf{d} = (1, 2, \ldots, m)$, one usually writes

$$W(s, \mathbf{d}) = p(s, m).$$

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In general:

$$F(t,\mathbf{d}) := \prod_{j=1}^{m}\frac{1}{1-t^{d_j}} = \sum_{s=0}^{\infty}W(s,\mathbf{d})t^s.$$

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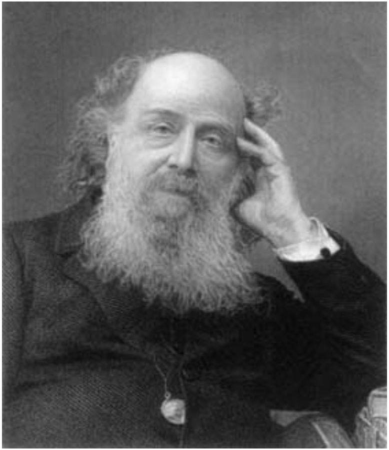

J. J. Sylvester (1814–1897)

He wrote $W(s,\mathbf{d})$ as a sum of “waves",

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- To give an elementary expression for $W_1(s, \mathbf{d})$.
- In the process, introduce a symbolic notation for Bernoulli numbers and polynomials.

2. The Main Result

Sylvester (1882) showed that for each such j, $W_j(s, \mathbf{d})$ is the residue of

$$F_j(s,t) := \sum_{\substack{0\le\nu<j\\ \gcd(\nu,j)=1}} \frac{\rho_j^{-\nu s} e^{st}}{\left(1-\rho_j^{\nu d_1} e^{-d_1 t}\right)\ldots\left(1-\rho_j^{\nu d_m} e^{-d_m t}\right)},$$

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In other words: Sum is taken over **all** j-th roots of unity ρ_j^ν.

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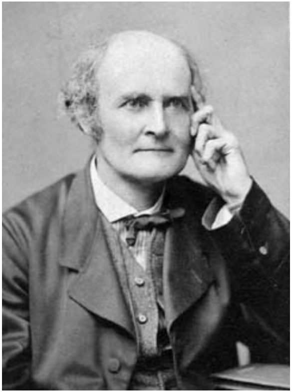
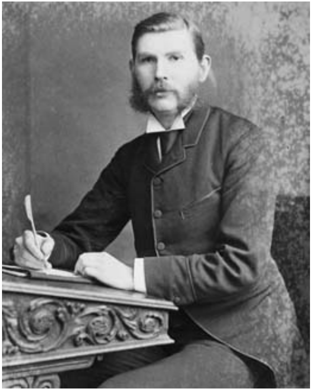


A. Cayley
(1821–1895)



J. W. L. Glaisher
(1848–1928)

More recently, restricted partitions and Sylvester waves were investigated in detail by

- M. Beck, I. M. Gessel, and T. Komatsu (2001),
- L. G. Fel and B. Y. Rubinstein (2002, 2006),
- B. Y. Rubinstein (2008),
- J. S. Dowker (preprints, 2011, 2013),
- A. V. Sills and D. Zeilberger (2012),
- C. O'Sullivan (2015),
- M. Cimpoeas and F. Nicolae (2017).

Consider Sylvester's formula

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This fact was used by Rubinstein and Fel (2006) to write $W_1(s, \mathbf{d})$ in a very compact form in terms of a single higher-order Bernoulli polynomial. (See later).

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A version of this result, given in two different forms, was earlier obtained by Beck, Gessel and Komatsu (2001).

Similarly, for $j = 2$ we have $\rho_j = -1$, and

$$\sum_{\substack{0 \le \nu < j \\ \gcd(\nu,j)=1}} \frac{\rho_j^{-\nu s} e^{st}}{\left(1 - \rho_j^{\nu d_1} e^{-d_1 t}\right) \dots \left(1 - \rho_j^{\nu d_m} e^{-d_m t}\right)}$$

leads to a convolution sum of

- higher-order Bernoulli polynomials and
- higher-order Euler polynomials.

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Rubinstein (2008):
All the $W_j(s, \mathbf{d})$ can be written as linear combinations of the first wave ($j = 1$) alone, with modified integers s and vectors $\mathbf{d}$.

This last result makes it worthwhile to give further consideration to $W_1(s, \mathbf{d})$.

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Theorem 1 (D & Vignat)

Let $\mathbf{d} := (d_1, d_2, \ldots, d_m)$*, and denote* $d := d_1 \ldots d_m$ *and* $\widetilde{d}_i := d/d_i$*,* $i = 1, \ldots, m$*.*

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$$W_1(s, \mathbf{d}) = \frac{1}{(m-1)!\,d^m} \times \sum_{\substack{0 \le \ell_1 \le \widetilde{d}_1 - 1 \\ \cdots \\ 0 \le \ell_m \le \widetilde{d}_m - 1}} \prod_{j=1}^{m-1} \left(s + jd - \ell_1 d_1 - \cdots - \ell_m d_m\right).$$

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Note: New in this identity:
It does not contain Bernoulli numbers or polynomials.

Examples:

We can obtain some well-known small cases, e.g.,

$$W_1(s,(d_1,d_2)) = \frac{1}{d_1 d_2}\, s + \frac{d_1 + d_2}{2 d_1 d_2},$$

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or, for $m=3$,

$$\begin{aligned} W_1(s,(d_1,d_2,d_3)) &= \frac{1}{2d_1 d_2 d_3}\, s^2 + \frac{d_1+d_2+d_3}{2d_1 d_2 d_3}\, s \\ &\quad + \frac{1}{12}\left(\frac{(d_1+d_2+d_3)^2}{d_1 d_2 d_3} + \frac{1}{d_1} + \frac{1}{d_2} + \frac{1}{d_3}\right). \end{aligned}$$

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Note: Glaisher (1908) obtained these, and all cases $m \leq 7$, by a different method.

Other authors obtained explicit polynomial parts for $\mathbf{d} = (1,2,\ldots,m)$ for small m.

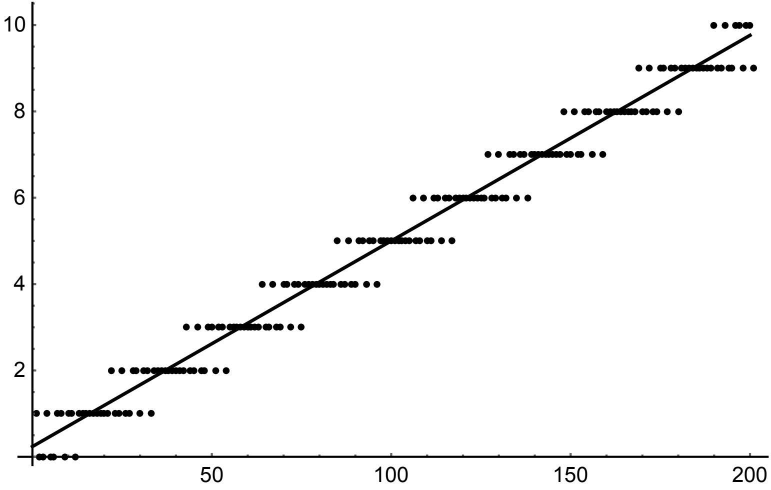

Figure 1: $W_1(s, \mathbf{d})$ (solid line) and $W(s, \mathbf{d})$ (dots) for $\mathbf{d} = (3, 5)$ and $s \leq 200$.

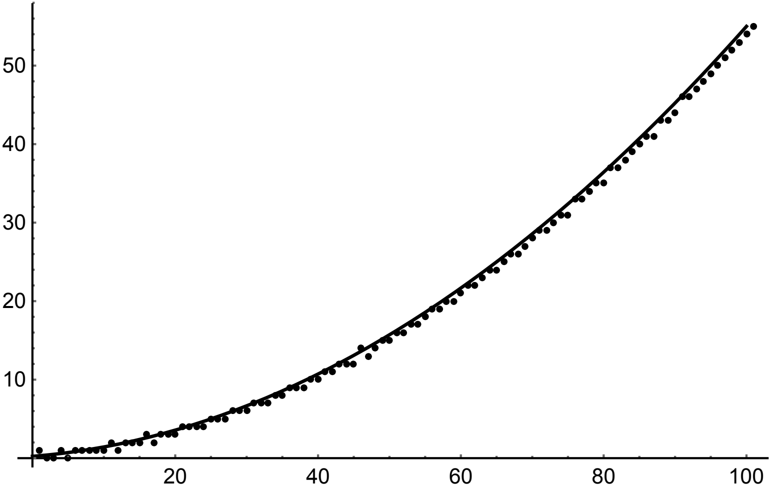

Figure 2: $W_1(s, \mathbf{d})$ (solid line) and $W(s, \mathbf{d})$ (dots) for $\mathbf{d} = (3, 5, 7)$ and $s \leq 100$.

3. Higher-order Bernoulli Polynomials

The (ordinary) Bernoulli polynomials $B_n(x)$, $n = 0, 1, 2, \ldots$ are defined by

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For an integer $k \geq 1$, the **Bernoulli polynomial of order** k is defined by

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Among numerous properties, they satisfy

$$B_{m-1}^{(m)}(x) = (x-1)(x-2)\dots(x-m+1) \qquad (m \geq 2),$$

with $B_0^{(1)}(x) = B_0(x) = 1$.

A further generalization:
For $m \geq 1$ and $\mathbf{d} = (d_1, \ldots, d_m)$ ($d_j \in \mathbb{N}$) we define the polynomials $B_n^{(m)}(x|\mathbf{d})$, $n = 0, 1, \ldots$, by

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The $B_n^{(m)}(x|\mathbf{d})$ are also known as *Bernoulli-Barnes polynomials*. (With different notation and normalization).

Main lemma: An analogue of the identity

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Let $m \in \mathbb{N}$ and $\mathbf{d} := (d_1, \dots, d_m)$, $d_j \in \mathbb{N}$. Denote $d := d_1 \dots d_m$ and $\widetilde{d}_i := d/d_i$, $1 \le i \le k$. Then

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Lemma 2 (D & Vignat)

Let $m \in \mathbb{N}$ and $\mathbf{d} := (d_1, \dots, d_m)$, $d_j \in \mathbb{N}$. Denote $d := d_1 \dots d_m$ and $\widetilde{d}_i := d/d_i$, $1 \le i \le k$. Then

$$B_{m-1}^{(m)}(x|\mathbf{d}) = \frac{1}{d^{m-1}} \sum_{\substack{0 \le \ell_1 \le \widetilde{d}_1 - 1 \\ \cdots \\ 0 \le \ell_m \le \widetilde{d}_m - 1}} \prod_{j=1}^{m-1} \left(x - jd + \ell_1 d_1 + \dots + \ell_m d_m\right). \qquad (3)$$

Note: When $\mathbf{d} = (1, \dots, 1)$, sum on the right of (3) collapses to $\ell_1 = \ldots = \ell_m = 0$; we recover (2).

Another lemma: Recall reflection formula for Bernoulli polynomials:

$$B_n(x+1) = (-1)^n B_n(-x).$$

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Higher-order analogue:

Lemma 3

Let m and $d_1, \ldots, d_m$ be as before, and $\mathbf{d} := (d_1, \ldots, d_m)$ and $\sigma := d_1 + \cdots + d_m$. Then for all $n \geq 0$,

$$B_n^{(m)}(x+\sigma|\mathbf{d}) = (-1)^n B_n^{(m)}(-x|\mathbf{d}).$$

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Can be found in Nörlund's “Differenzenrechnung" (1924).

Proof of our theorem:

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Rubinstein and Fel (2006) proved:

$$W_1(s, \mathbf{d}) = \frac{1}{(m-1)!d} B_{m-1}^{(m)}(s + \sigma | \mathbf{d}), \tag{4}$$

where, as before, $\mathbf{d} = (d_1, \ldots, d_m)$, $d = d_1 \ldots d_m$, and $\sigma = d_1 + \cdots + d_m$.

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Remark: Theorem 1 can be rewritten:

Corollary 4

Let $\mathbf{d} := (d_1, d_2, \ldots, d_m)$ *and* $d := d_1 \ldots d_m$*. Then*

$$W_1(s, \mathbf{d}) = \frac{1}{d} \sum_{\ell} \binom{m - 1 + \frac{s-\ell}{d}}{m-1}, \tag{5}$$

where the sum is taken over all ℓ *with*

$$\ell = \ell_1 d_1 + \cdots + \ell_m d_m, \quad 0 \le \ell_i \le \frac{d}{d_i} - 1, \quad i = 1, \ldots, m.$$

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When $d_1 = \cdots = d_m = 1$, (5) collapses to a single term:

$$W(s, \mathbf{d}) = W_1(s, \mathbf{d}) = \binom{m - 1 + s}{m-1}.$$

(A well-known elementary expression).

4. Symbolic Notation

We define the **Bernoulli symbol** $\mathcal{B}$ by

$$\mathcal{B}^n = B_n \qquad (n = 0, 1, \ldots),$$

where B_n is the nth Bernoulli number.

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So we can rewrite the usual definition

$$B_n(x) = \sum_{j=0}^{n} \binom{n}{j} B_j x^{n-j} \quad \text{as} \quad B_n(x) = (x + \mathcal{B})^n.$$

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$$\exp(\mathcal{B}z) = \sum_{n=0}^{\infty} \mathcal{B}^n \frac{z^n}{n!} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}.$$

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$$\exp((\mathcal{B}+1)z) = \frac{z}{e^z - 1} \cdot e^z = \frac{-z}{e^{-z} - 1} = \exp(-\mathcal{B}z),$$

and thus

$$\mathcal{B} + 1 = -\mathcal{B}.$$

The **uniform symbol** $\mathcal{U}$ is defined by

$$f(x+\mathcal{U}) = \int_0^1 f(x+u)du.$$

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Combining this with the analogous identity for $\exp(\mathcal{U}z)$,

$$1 = \frac{z}{e^z - 1} \cdot \frac{e^z - 1}{z} = \exp(z(\mathcal{B}+\mathcal{U})) = \sum_{n=0}^{\infty} (\mathcal{B}+\mathcal{U})^n \frac{z^n}{n!}.$$

Hence $\mathcal{B}$ and $\mathcal{U}$ annihilate each other, i.e.,

$$(\mathcal{B}+\mathcal{U})^n = 0 \quad \text{for all} \quad n > 0.$$

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Independence means: for any two Bernoulli symbols $\mathcal{B}_1$ and $\mathcal{B}_2$,

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Related to this, we define the **higher-order Bernoulli symbol** $\mathcal{B}^{(k)}$ by

$$\mathcal{B}^{(k)} = \mathcal{B}_1 + \cdots + \mathcal{B}_k,$$

where $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are independent Bernoulli symbols.

Application:

Recall: Bernoulli polynomial, defined by

$$e^{xz}\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n(x)\frac{z^n}{n!},$$

can be written as

$$B_n(x)=(x+\mathcal{B})^n.$$

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can be written as

$$B_n(x)=(x+\mathcal{B})^n.$$

Similarly, we can write

$$e^{xz}\prod_{i=1}^{m}\frac{d_iz}{e^{d_iz}-1}=\sum_{n=0}^{\infty}B_n^{(m)}(x|\mathbf{d})\frac{z^n}{n!}$$

symbolically as

$$B_n^{(m)}(x|\mathbf{d})=(x+d_1\mathcal{B}_1+\cdots+d_m\mathcal{B}_m)^n\,.$$

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we get, with $\sigma := d_1 + \cdots + d_m$,

$$\begin{aligned} B_n^{(m)}(x+\sigma|\mathbf{d}) &= (x + d_1(\mathcal{B}_1 + 1) + \cdots + d_m(\mathcal{B}_m + 1))^n \\ &= (x - d_1\mathcal{B}_1 - \cdots - d_m\mathcal{B}_m)^n \\ &= (-1)^n(-x + d_1\mathcal{B}_1 + \cdots + d_m\mathcal{B}_m)^n \\ &= B_n^{(m)}(-x|\mathbf{d}). \end{aligned}$$

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\end{aligned}$$

This is Lemma 3.

Lemma 2 can be obtained (and, in fact, was discovered) with similar manipulations.

5. Some Consequences of Theorem 1

Recall **Theorem 1**:

With $\mathbf{d} := (d_1, d_2, \ldots, d_m)$, $d := d_1 \ldots d_m$, and $\widetilde{d}_i := d/d_i$,

$$W_1(s, \mathbf{d}) = \frac{1}{(m-1)!d^m}$$

$$\times \sum_{\substack{0 \le \ell_1 \le \widetilde{d}_1 - 1 \\ \cdots \\ 0 \le \ell_m \le \widetilde{d}_m - 1}} \prod_{j=1}^{m-1} (s + jd - \ell_1 d_1 - \cdots - \ell_m d_m) .$$

By an easy expansion of the product in Theorem 1 we get:

Corollary 5

For $\mathbf{d} := (d_1, \dots d_m)$, $d := d_1 \dots d_m$, *and* $\sigma := d_1 + \cdots + d_m$,

$$W_1(s, \mathbf{d}) = \frac{1}{(m-1)!d}\, s^{m-1} + \frac{\sigma}{2(m-2)!d}\, s^{m-2} + \dots$$

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The leading coefficient has long been known.

The second coefficient was obtained by Rieger (1959) for $\mathbf{d} = (1, 2, \dots, m)$.

By considering the *m*-fold sum in Theorem 1 as the Riemann sum of a multiple integral, we obtain an asympotic expansion:

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Corollary 6

With $\mathbf{d}$ *and* d *as above, let* $\lambda > 0$ *and* $s \geq \lambda d$*, and let* d *grow arbitrarily large in such a way that at least two of the components* d_j*,* $1 \leq j \leq m$*, are unbounded. Then*

$$W_1(s, \mathbf{d}) \sim \frac{1}{(m-1)!d}\, s^{m-1}.$$

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$$W_1(s, \mathbf{d}) \sim \frac{1}{(m-1)!d}\, s^{m-1}.$$

In other words, $W_1(s, \mathbf{d})$ has the same asymptotic behaviour as in the case of bounded d.

Thank you – Merci