On lattice polytopes, convex matroid optimization, and degree sequences of hypergraphs

Antoine Deza, Paris Sud

based on joint works with:  Asaf Levin, Technion
George Manoussakis, Ben Gurion
Shmuel Onn, Technion
Linear Optimization?

Given an $n$-dimensional vector $b$ and an $n \times d$ matrix $A$, find, in any, a $d$-dimensional vector $x$ such that:

$$Ax = b$$

$$x \geq 0$$

linear algebra  linear optimization
Linear Optimization?

Given an $n$-dimensional vector $b$ and an $n \times d$ matrix $A$ find, in any, a $d$-dimensional vector $x$ such that:

$$Ax = b \quad \quad Ax \leq b$$

linear algebra \hspace{2cm} linear optimization

*Can linear optimization be solved in strongly polynomial time?* is listed by Smale (Fields Medal 1966) as one of the top mathematical problems for the XXI century

*Strongly* polynomial : algorithm *independent* from the *input data length* and polynomial in $n$ and $d$. 
**Lattice polytopes with large diameter**

**Lattice** $(d,k)$-polytope: convex hull of points drawn from $\{0,1,\ldots,k\}^d$

**Diameter** $\delta(P)$ of polytope $P$: smallest number such that any two vertices of $P$ can be connected by a path with at most $\delta(P)$ edges

$\delta(d,k)$: largest diameter over all lattice $(d,k)$-polytopes

Ex. $\delta(3,3) = 6$ and is achieved by a **truncated cube**
**Lattice polytopes with large diameter**

**lattice** \((d,k)\)-polytope: convex hull of points drawn from \(\{0,1,\ldots,k\}^d\)

**diameter** \(\delta(P)\) of polytope \(P\): smallest number such that any two vertices of \(P\) can be connected by a path with at most \(\delta(P)\) edges

\(\delta(d,k)\): largest diameter over all lattice \((d,k)\)-polytopes

- \(\delta(P)\) : lower bound for the worst case number of iterations required by pivoting methods (simplex) to optimize a linear function over \(P\)

- *Hirsch conjecture*: \(\delta(P) \leq n - d\) \((n\ \text{number of inequalities})\) was disproved [Santos 2012]
Lattice polytopes with large diameter

\( \delta(d,k) \): largest \textbf{diameter} of a convex hull of points drawn from \( \{0,1,\ldots,k\}^d \)

\textbf{upper bounds:}

\[
\delta(d,1) \leq d \\
\delta(2,k) = O(k^{2/3}) \\
\delta(2,k) = 6(k/2\pi)^{2/3} + O(k^{1/3} \log k) \\
\delta(d,k) \leq kd \\
\delta(d,k) \leq kd - \lceil d/2 \rceil \quad \text{for } k \geq 2 \\
\delta(d,k) \leq kd - \lceil 2d/3 \rceil - (k - 3) \quad \text{for } k \geq 3
\]

[Naddef 1989]  
[Balog-Bárány 1991]  
[Thiele 1991]  
[Acketa-Žunić 1995]  
[Kleinschmid-Onn 1992]  
[Del Pia-Michini 2016]  
[Deza-Pournin 2018]
Lattice polytopes with large diameter

\( \delta(d,k) \): largest diameter of a convex hull of points drawn from \( \{0,1,\ldots,k\}^d \)

lower bounds:

\[
\delta(d,1) \geq d \quad \text{[Naddef 1989]}
\]
\[
\delta(d,2) \geq \lceil \frac{3d}{2} \rceil \quad \text{[Del Pia-Michini 2016]}
\]
\[
\delta(d,k) = \Omega(k^{2/3}d) \quad \text{[Del Pia-Michini 2016]}
\]
\[
\delta(d,k) \geq \lfloor \frac{(k+1)d}{2} \rfloor \quad \text{for } k < 2d \quad \text{[Deza-Manoussakis-Onn 2018]}
\]
### Lattice polytopes with large diameter

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\( \delta(d,1) = d \) [Naddef 1989]
### Lattice polytopes with large diameter

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- \(\delta(d,1) = d\) [Naddef 1989]
- \(\delta(2,k)\) : close form [Thiele 1991] [Acketa-Žunić 1995]
Lattice polytopes with large diameter

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$\delta(d,1) = d$

$\delta(2,k) :$ close form

$\delta(d,2) = \lceil 3d/2 \rceil$

[Thiele 1991] [Acketa-Žunić 1995]

[Del Pia-Michini 2016]
### Lattice polytopes with large diameter

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\( \delta(d,1) = d \)

\( \delta(2,k) : \) close form

\( \delta(d,2) = \lfloor 3d/2 \rfloor \)

\( \delta(4,3) = 8, \delta(3,4) = 7, \delta(3,5) = 9 \)

[Deza-Pournin 2018], [Chadder-Deza 2017]

[Naddef 1989]

[Thiele 1991] [Acketa-Žunić 1995]

[Del Pia-Michini 2016]
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- $\delta(d,1) = d$
- $\delta(2,k)$: close form
- $\delta(d,2) = \lfloor 3d/2 \rfloor$
- $\delta(4,3)=8$, $\delta(3,4)=7$, $\delta(3,5)=9$
- $\delta(5,3)=10$, $\delta(3,6)=10$

References:
- Naddef 1989
- Thiele 1991
- Acketa-Žunić 1995
- Del Pia-Michini 2016
- Deza-Pournin 2018
- Chadder-Deza 2017
- Deza-Deza-Guan-Pournin 2018
Lattice polytopes with large diameter

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Conjecture [Deza-Manoussakis-Onn 2018] $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$

and $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of primitive lattice vectors. The conjecture holds for all known entries of $\delta(d,k)$.
**Lattice polygons with many vertices**

Q. What is $\delta(2,k)$ : largest diameter of a polygon which vertices are drawn form the $k \times k$ grid?

A polygon can be associated to a set of vectors (edges) *summing up to zero*, and *without a pair of positively multiple vectors*

$$\delta(2,3) = 4$$ is achieved by the 8 vectors : $(\pm 1,0)$, $(0,\pm 1)$, $(\pm 1,\pm 1)$
Lattice polygons with many vertices

\[ \delta(2,2) = 2 \text{ ; vectors: } (\pm 1,0), (0, \pm 1) \]
Lattice polygons with many vertices

\[ \delta(2,2) = 2 \text{ ; vectors } : (\pm 1,0), (0,\pm 1) \]

\[ ||x||_1 \leq 1 \]
Lattice polygons with many vertices

\[ \delta(2,2) = 2 \; ; \text{vectors:} \; (\pm1,0), \; (0,\pm1) \]

\[ \delta(2,3) = 4 \; ; \text{vectors:} \; (\pm1,0), \; (0,\pm1), \; (\pm1,\pm1) \]

\[ ||x||_1 \leq 2 \]
$\delta(2,2) = 2$; vectors: $(\pm1,0), (0,\pm1)$

$\delta(2,3) = 4$; vectors: $(\pm1,0), (0,\pm1), (\pm1,\pm1)$

$\delta(2,9) = 8$; vectors: $(\pm1,0), (0,\pm1), (\pm1,\pm1), (\pm1,\pm2), (\pm2,\pm1)$

$Lattice polygons with many vertices$
Lattice polygons with many vertices

\[ \delta(2, k) = 2 \sum_{i=1}^{p} \varphi(i) \text{ for } k = \sum_{i=1}^{p} i\varphi(i) \]

\( \varphi(p) : \text{Euler totient function} \) counting positive integers less or equal to \( p \) relatively prime with \( p \)

\( \varphi(1) = \varphi(2) = 1, \varphi(3) = \varphi(4) = 2, \ldots \)
\[
\delta(2,k) = 2 \sum_{i=1}^{p} \phi(i) \quad \text{for} \quad k = \sum_{i=1}^{p} i \phi(i)
\]

\[
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**Lattice polygons**

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$H_1(2,p)$: Minkowski sum generated by $\{x \in \mathbb{Z}^2 : ||x||_1 \leq p, \gcd(x)=1, x \succeq 0\}$

$H_1(2,p)$ has diameter $\delta(2,k) = 2 \sum_{i=1}^{p} \varphi(i)$ for $k = \sum_{i=1}^{p} i\varphi(i)$

Ex. $H_1(2,2)$ generated by $(1,0), (0,1), (1,1), (1,-1)$ (fits, up to translation, in 3x3 grid)

$x \succeq 0$: first nonzero coordinate of $x$ is nonnegative
**Primitive zonotopes**
(generalization of the permutahedron of type $B_d$)

\[ H_q(d,p) : \text{Minkowski} \ (x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, \ x \succeq 0) \]

\[ Z_q(d,p) : \text{Zonotope} \ (x \in \mathbb{Z}^d : ||x||_q \leq p, \gcd(x)=1, \ x \succeq 0) \]

\[ x \succeq 0 : \text{first nonzero coordinate of } x \text{ is nonnegative} \]

Given a set $G$ of $m$ vectors (generators)

**Minkowski** ($G$) : convex hull of the $2^m$ sums of the $m$ vectors in $G$

**Zonotope** ($G$) : convex hull of the $2^m$ signed sums of the $m$ vectors in $G$

up to translation $Z(G)$ is the image of $H(G)$ by an homothety of factor 2

- **Primitive zonotopes**: zonotopes generated by *short integer* vectors which are *pairwise linearly independent*
**Primitive zonotopes**

*(generalization of the permutahedron of type $B_d$)*

$H_q(d,p) : \text{Minkowski} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, \ x \succeq 0)$

$Z_q(d,p) : \text{Zonotope} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, \ x \succeq 0)$

$x \succeq 0 : \text{first nonzero coordinate of } x \text{ is nonnegative}$

- $H_q(d, 1) : [0, 1]^d \text{ cube for } q \neq \infty$
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

$H_q(d,p) : \text{Minkowski} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, \ x \succeq 0)$

$Z_q(d,p) : \text{Zonotope} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, \ x \succeq 0)$

$x \succeq 0 : \text{first nonzero coordinate of } x \text{ is nonnegative}$

- $Z_1(d,2) : \text{permutahedron of type } B_d$
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

\[ H_q(d,p) : \text{Minkowski} \ (x \in \mathbb{Z}^d : ||x||_q \leq p, \ \gcd(x)=1, \ x \succeq 0) \]

\[ Z_q(d,p) : \text{Zonotope} \ (x \in \mathbb{Z}^d : ||x||_q \leq p, \ \gcd(x)=1, \ x \succeq 0) \]

\[ x \succeq 0 : \text{first nonzero coordinate of } x \text{ is nonnegative} \]

- $H_1(3,2) :$ truncated cuboctahedron
  *(great rhombicuboctahedron)*
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

$H_q(d,p) : \text{Minkowski} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \ \gcd(x)=1, \ x \succeq 0)$

$Z_q(d,p) : \text{Zonotope} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \ \gcd(x)=1, \ x \succeq 0)$

$x \succeq 0 : \text{first nonzero coordinate of } x \text{ is nonnegative}$

- $H_\infty(3,1) : \text{truncated small rhombicuboctahedron}$
**Primitive zonotopes**  
*(generalization of the permutahedron of type $B_d$)*

$H_q(d,p) : \text{Minkowski } (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$Z_q(d,p) : \text{Zonotope } (x \in \mathbb{Z}^d : \|x\|_q \leq p, \gcd(x)=1, x \succeq 0)$

$x \succeq 0 : \text{first nonzero coordinate of } x \text{ is nonnegative}$

$H^+ / Z^+ : \text{positive primitive lattice polytope } x \in \mathbb{Z}^d_+$

- $H_1(d,2)^+ : \text{Minkowski sum of the permutahedron with the } \{0,1\}^d, \text{i.e., graphical zonotope obtained by the } d\text{-clique with a loop at each node}$

  - *graphical zonotope $Z_G$: Minkowski sum of segments $[e_i,e_j]$ for all edges $\{i,j\}$ of a given graph $G$*
**Primitive zonotopes**
*(generalization of the permutahedron of type $B_d$)*

\[ H_q(d,p) : \text{Minkowski} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \ \gcd(x)=1, \ x \succeq 0) \]

\[ Z_q(d,p) : \text{Zonotope} \ (x \in \mathbb{Z}^d : \|x\|_q \leq p, \ \gcd(x)=1, \ x \succeq 0) \]

$x \succeq 0$ : first nonzero coordinate of $x$ is nonnegative

$H^+ / Z^+$: **positive** primitive lattice polytope $x \in \mathbb{Z}^d_+$

- For $k < 2d$, Minkowski sum of a subset of the generators of $H_1(d,2)$ is, up to translation, a lattice $(d,k)$-polytope with diameter \( \lfloor (k+1)d/2 \rfloor \)
Conjecture [Deza-Manoussakis-Onn 2018]  \( \delta(d,k) \leq \lceil (k+1)d/2 \rceil \)

and \( \delta(d,k) \) is achieved, up to translation, by a *Minkowski sum of primitive lattice vectors*. The conjecture holds for all known entries of \( \delta(d,k) \)
### Lattice polytopes with large diameter

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<th>( \delta(d,k) )</th>
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> Conjecture [Deza-Manoussakis-Onn 2018] \( \delta(d,k) \leq \lfloor (k+1)d/2 \rfloor \)

and \( \delta(d,k) \) is achieved, up to translation, by a Minkowski sum of primitive lattice vectors. The conjecture holds for all known entries of \( \delta(d,k) \)
Given a lattice \((d,k)\)-polytope \(P\), two vertices \(u\) and \(v\) such that \(\delta(P) = d(u,v)\), then \(d(u,v) \leq \delta(d-1,k) + k\) and \(d(u,v) < \delta(d-1,k) + k\) unless:

- \(u+v = (k,k,\ldots,k)\),
- any edge of \(P\) with \(u\) or \(v\) as vertex is \([-1,0,1]\)-valued,
- any intersection of \(P\) with a facet of the cube \([0,k]^d\) is a \((d-1)\)-dimensional face of \(P\) of diameter \(\delta(d-1,k)\).

Those conditions, combined with enumeration up to symmetry, drastically reduce the search space for lattice \((d,k)\)-polytopes such that \(\delta(P) = \delta(d-1,k) + k\).

Computationally ruling out \(\delta(d,k) = \delta(d-1,k) + k\) and using \(\delta(d,k) \leq \lceil (k+1)d/2 \rceil\) for \(k < 2d\) yields: \(\delta(3,4) = 7\) and \(\delta(3,5) = 9\)

- \(\delta(\text{great rhombicuboctahedron}) = \delta(3,5)\)

*Additional tools needed to rule out \(\delta(d,k) = \delta(d-1,k) + k - 1\)*
A034997  Number of Generalized Retarded Functions in Quantum Field Theory.
2, 6, 32, 370, 11292, 1066044, 347326352, 419172756930 (list; graph; refs; listen; history; text; internal format)
OFFSET 1,1
COMMENTS
  a(d) is the number of parts into which d-dimensional space \((x_1, \ldots, x_d)\) is split by a set of \((2^d - 1)\) hyperplanes \(c_1 x_1 + c_2 x_2 + \ldots + c_d x_d = 0\) where \(c_j\) are 0 or +1 and we exclude the case with all \(c=0\).
  Also, \(a(d)\) is the number of independent real-time Green functions of Quantum Field Theory produced when analytically continuing from euclidean time/energy \((d+1 = \text{number of energy/time variables})\). These are also known as Generalized Retarded Functions.

The numbers up to \(d=6\) were first produced by T. S. Evans using a Pascal program, strictly as upper bounds only. M. van Eijck wrote a C program using a direct enumeration of hyperplanes which confirmed these and produced the value for \(d=7\). Kamiya et al. showed how to find these numbers and some associated polynomials using more sophisticated methods, giving results up to \(d=7\). T. S. Evans added the last number on Aug 01 2011 using an updated version of van Eijck's program, which took 7 days on a standard desktop computer.

REFERENCES
...
Number of Generalized Retarded Functions in Quantum Field Theory.

370, 11292, 1066044, 347326352, 419172756930 (list; graph; refs; listen; history; text; internal format)

1,1

a(d) is the number of parts into which d-dimensional space (x_1,...,x_d) is split by a set of (2^d - 1) hyperplanes c_1 x_1 + c_2 x_2 + ...+ c_d x_d =0 where c_j are 0 or +1 and we exclude the case with all c=0.

Also, a(d) is the number of independent real-time Green functions of Quantum Field Theory produced when analytically continuing from euclidean time/energy (d+1 = number of energy/time variables). These are also known as Generalized Retarded Functions.

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T. S. Evans, N-point finite temperature expectation values at real times, Nuclear Physics B 374 (1992) 340-370.


Table of n, a(n) for n=1..8.

Computational determination of the number of vertices of primitive zonotopes

Sloane OEI sequences
$H_\infty(d,1)^+ \text{ vertices} : A034997 = \text{number of generalized retarded functions in quantum Field theory (determined till } d = 8)$

$H_\infty(d,1) \text{ vertices} : A009997 = \text{number of regions of hyperplane arrangements with \{-1,0,1\}-valued normals in dimension } d \text{ (determined till } d = 7)$

Estimating the number of vertices of $H_\infty(d,1)^+$
[Odlyzko 1988], [Zuev 1992], [Kovijanić-Vukićević 2007]

$$d^2 \ (1-o(1)) \leq \log_2 | H_\infty(d,1)^+ | \leq d^2$$
**Convex Matroid Optimization**

The optimal solution of \( \max \{ f(Wx) : x \in S \} \) is attained at a vertex of the projection integer polytope in \( \mathbb{R}^d : \text{conv}(WS) = W\text{conv}(S) \)

\( S \) : set of feasible point in \( \mathbb{Z}^n \)  
\( W \) : integer \( d \times n \) matrix  
\( f \) : convex function from \( \mathbb{R}^d \) to \( \mathbb{R} \)

**Q.** What is the maximum number \( v(d,n) \) of vertices of \( \text{conv}(WS) \) when \( S \in \{0,1\}^n \) and \( W \) is a \( \{0,1\} \)-valued \( d \times n \) matrix?

obviously \( v(d,n) \leq |WS| = O(n^d) \)  
in particular \( v(2,n) = O(n^2) \), and \( v(2,n) = \Omega(n^{0.5}) \)
Convex Matroid Optimization

[Melamed-Onn 2014] Given matroid $S$ of order $n$ and $\{0,1,\ldots,p\}$-valued $d \times n$ matrix $W$, the maximum number $m(d,p)$ of vertices of $\text{conv}(WS)$ is independent of $n$ and $S$.

Ex: maximum number $m(2,1)$ of vertices of a planar projection $\text{conv}(WS)$ of matroid $S$ by a binary matrix $W$ is attained by the following matrix and uniform matroid of rank 3 and order 8:

$$W = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$S = U(3,8) = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$\text{conv}(WS)$
Convex Matroid Optimization

The optimal solution of $\max \{ f(Wx) : x \in S \}$ is attained at a vertex of the projection integer polytope in $\mathbb{R}^d : \text{conv}(WS) = W\text{conv}(S)$

$S$ : set of feasible point in $\mathbb{Z}^n$ (in the talk $S \in \{0,1\}^n$)
$W$ : integer $d \times n$ matrix ($W$ is mostly $\{0,1,\ldots,p\}$-valued)
$f$ : convex function from $\mathbb{R}^d$ to $\mathbb{R}$

Q. What is the maximum number $v(d,n)$ of vertices of $\text{conv}(WS)$ when $S \in \{0,1\}^n$ and $W$ is a $\{0,1\}$-valued $d \times n$ matrix?

obviously $v(d,n) \leq |WS| = O(n^d)$
in particular $v(2,n) = O(n^2)$, and $v(2,n) = \Omega(n^{0.5})$

[Melamed-Onn 2014] Given matroid $S$ of order $n$ and $\{0,1,\ldots,p\}$-valued $d \times n$ matrix $W$, the maximum number $m(d,p)$ of vertices of $\text{conv}(WS)$ is independent of $n$ and $S$
Convex Matroid Optimization

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[Deza-Manoussakis-Onn 2018] Given matroid $S$ of order $n$, $\{0,1,\ldots,p\}$-valued $d \times n$ matrix $W$, maximum number $m(d,p)$ of vertices of $\text{conv}(WS)$ is equal to the number of vertices of $H_{\infty}(d,p)$

$$m(d,p) = |H_{\infty}(d,p)|$$
Convex Matroid Optimization

[Melamed-Onn 2014] Given matroid $S$ of order $n$ and $\{0,1,\ldots,p\}$-valued $d \times n$ matrix $W$, the maximum number $m(d,p)$ of vertices of $\text{conv}(WS)$ is independent of $n$ and $S$.

[Deza-Manoussakis-Onn 2018] Given matroid $S$ of order $n$, $\{0,1,\ldots,p\}$-valued $d \times n$ matrix $W$, maximum number $m(d,p)$ of vertices of $\text{conv}(WS)$ is equal to the number of vertices of $H_\infty(d,p)$.

$$m(d,p) = | H_\infty(d,p) |$$

[Melamed-Onn 2014] \[ d^2 \leq m(d,1) \leq 2 \sum_{i=0}^{d-1} \binom{3^d - 3}{i} \]

\[ 24 \leq m(3,1) \leq 158 \]
\[ 64 \leq m(4,1) \leq 19840 \]
\[ m(2,1) = 8 \]

[Deza-Manoussakis-Onn 2017] \[ d!^2 \leq m(d,1) \leq 2 \sum_{i=0}^{d-1} \binom{3^d - 3}{i} - f(d) \]

\[ m(3,1) = 96 \]
\[ m(4,1) = 5376 \]
\[ m(2,p) = 8 \sum_{i=1}^{p} \phi(i) \]
**Primitive Zonotopes**
*(degree sequences)*

\(D_d\) : convex hull of the degree sequences of all hypergraphs on \(d\) nodes

\[D_d = H_\infty(d,1)^+\]

\(D_d(k)\) : convex hull of the degree sequences of all \(k\)-uniform hypergraphs on \(d\) nodes
**Primitive Zonotopes**  
*(degree sequences)*

$D_d$: convex hull of the degree sequences of all hypergraphs on $d$ nodes  
$$D_d = H_\infty(d,1)^+$$

$D_d(k)$: convex hull of the degree sequences of all $k$-uniform hypergraphs on $d$ nodes

Q: check whether $x \in D_d(k) \cap \mathbb{Z}^d$ is the degree sequence of a $k$-uniform hypergraph. Necessary condition: sum of the coordinates of $x$ is multiple of $k$.

[Erdős-Gallai 1960]: for $k = 2$ (graphs) necessary condition is sufficient

[Liu 2013] exhibited counterexamples (holes) for $k = 3$ (Klivans-Reiner Q.)
**Primitive Zonotopes**
*(degree sequences)*

\[ D_d : \text{convex hull of the degree sequences of all hypergraphs on } d \text{ nodes} \]

\[ D_d = H_\infty(d,1)+ \]

\[ D_d(k) : \text{convex hull of the degree sequences of all } k\text{-uniform hypergraphs on } d \text{ nodes} \]

Q: check whether \( x \in D_d(k) \cap \mathbb{Z}^d \) is the degree sequence of a \( k\)-uniform hypergraph. Necessary condition: sum of the coordinates of \( x \) is multiple of \( k \).

[Erdős-Gallai 1960]: for \( k = 2 \) (graphs) necessary condition is sufficient

[Liu 2013] exhibited counterexamples (holes) for \( k = 3 \) (Klivans-Reiner Q.)

- Answer to Colbourn-Kocay-Stinson Q. (1986)
  Deciding whether a given integer sequence is the degree sequence of a 3-hypergraph is NP-complete [Deza-Levin-Meesum-Onn 2017]
\textbf{Primitive zonotopes, convex matroid optimization, and degree sequences of hypergraphs}

$\delta(d,k)$: largest diameter over all lattice $(d,k)$-polytopes

\begin{itemize}
  \item \textbf{Conjecture}: $\delta(d,k) \leq \lfloor (k+1)d/2 \rfloor$ and $\delta(d,k)$ is achieved, up to translation, by a Minkowski sum of primitive lattice vectors (holds for all known $\delta(d,k)$).
  \item $\Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor$ for $k < 2d$
  \item $m(d,p) = |H_\infty(d,p)|$ (convex matroid optimization complexity)
  \item determination of $\delta(3,k)$ and of $\delta(d,3)$? ($\delta(d,3) = 2d$?)
  \item complexity issues, e.g. decide whether a given point is a vertex of $Z_\infty(d,1)$
  \item Answer to [Colbourn-Kocay-Stinson 1986] question: Deciding whether a given integer sequence is the degree sequence of a 3-hypergraph is NP-complete [Deza-Levin-Meesum-Onn 2017]
\end{itemize}
**Primitive zonotopes, convex matroid optimization, and degree sequences of hypergraphs**

\( \delta(d,k) \): largest diameter over all lattice \((d,k)\)-polytopes

- **Conjecture**: \( \delta(d,k) \leq \lfloor (k+1)d/2 \rfloor \) and \( \delta(d,k) \) is achieved, up to translation, by a Minkowski sum of primitive lattice vectors (holds for all known \( \delta(d,k) \))

\[ \Rightarrow \delta(d,k) = \lfloor (k+1)d/2 \rfloor \text{ for } k < 2d \]

- \( m(d,p) = |H_\infty(d,p)| \) (convex matroid optimization complexity)

- determination of \( \delta(3,k) \) and of \( \delta(d,3) \)? (\( \delta(d,3) = 2d \)?)

- complexity issues, e.g. decide whether a given point is a vertex of \( Z_\infty(d,1) \)

- Answer to [Colbourn-Kocay-Stinson 1986] question:
  Deciding whether a given integer sequence is the degree sequence of a 3-hypergraph is NP-complete [Deza-Levin-Meesum-Onn 2017]

✓ thank you