A Computer-Algebra-Based Formal Proof of the Irrationality of $\zeta(3)$

Frédéric Chyzak

Joint work with A. Mahboubi, T. Sibut-Pinote, and E. Tassi

November 17, 2015

Apéry's Theorem (1978/1979): The Number $\zeta(3) = \sum_{m=1}^{\infty} \frac{1}{m^3}$ is Irrational

Sketch of proof, as in (van der Poorten, 1979)

- Define:

$$c_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2, \quad z_n = \sum_{m=1}^{n} \frac{1}{m^3}, \quad u_{n,k} = z_n + \sum_{m=1}^{k} \frac{(-1)^{m+1}}{2m^3 \binom{n}{m}\binom{n+m}{m}},$$

$$v_{n,k} = c_{n,k} u_{n,k}, \quad a_n = \sum_{k=0}^{n} c_{n,k}, \quad b_n = \sum_{k=0}^{n} v_{n,k}.$$

- Prove: (a_n) and (b_n) satisfy the same 2nd-order recurrence, so that

$$0 < \zeta(3) - b_n/a_n = \mathcal{O}\big(a_n^{-2}\big), \qquad a_n = \Theta\big(n^{-3/2}(\sqrt{2}+1)^{4n}\big).$$

- Define $\ell_n = \operatorname{lcm}(1, \dots, n)$ and prove $2\ell_n^3 a_n \in \mathbb{N}$, $2\ell_n^3 b_n \in \mathbb{Z}$.

- Notice $\ell_n = \mathcal{O}(e^n)$ and $e^3(\sqrt{2}+1)^{-4} \simeq 0.59$ to conclude:

$$0 < 2\ell_n^3\,(a_n\zeta(3) - b_n) = \mathcal{O}\big(n^{3/2} e^{3n} (\sqrt{2}+1)^{-4n}\big) \implies \zeta(3) \notin \mathbb{Q}.$$

Apéry's Theorem (1978/1979): The Number $\zeta(3) = \sum_{m=1}^{\infty} \frac{1}{m^3}$ is Irrational

Summary of ingredients of the proof

- Genius to invent the sequences (a_n) and (b_n)
- Elementary number theory
- Deriving same second-order recurrence for (a_n) and (b_n)
- Asymptotic estimates

Apéry's Theorem (1978/1979): The Number $\zeta(3) = \sum_{m=1}^{\infty} \frac{1}{m^3}$ is Irrational

Summary of ingredients of the proof

- Genius to invent the sequences (a_n) and (b_n)
- Elementary number theory
- Deriving same second-order recurrence for (a_n) and (b_n)
- Asymptotic estimates

Focus of the talk on proving the recurrence:

- this part is amenable to computer-algebra methods
- typical use of "creative telescoping" for summation

Beukers' Alternative Proof

(Beukers, 1979)

Observe

$$I_n = \ell_n^3 \int_0^1 \int_0^1 \int_0^1 \frac{L_n(x)\, L_n(y)}{1 - u\,(1 - xy)} dx\, dy\, du \in \mathbb{Z} + \mathbb{Z}\,\zeta(3),$$

where

$$L_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} x^n (1-x)^n \qquad \text{(Legendre orthogonal polynomials)}\,.$$

Integrations by parts and easy bounding yield

$$0 < I_n \leq 2\zeta(3)\, 3^{3n} (\sqrt{2} + 1)^{-4n}.$$

Observing $3^3(\sqrt{2}+1)^{-4} \simeq 0.79$ implies irrationality.

Beukers' Alternative Proof

(Beukers, 1979)

Observe

$$I_n = \ell_n^3 \int_0^1 \int_0^1 \int_0^1 \frac{L_n(x)\, L_n(y)}{1 - u\,(1 - xy)} dx\, dy\, du \in \mathbb{Z} + \mathbb{Z}\,\zeta(3),$$

where

$$L_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} x^n (1 - x)^n \qquad \text{(Legendre orthogonal polynomials)} .$$

Integrations by parts and easy bounding yield

$$0 < I_n \leq 2\zeta(3)\, 3^{3n} (\sqrt{2} + 1)^{-4n}.$$

Observing $3^3(\sqrt{2} + 1)^{-4} \simeq 0.79$ implies irrationality.

Mathematically more elegant, but would not illustrate CA/FP interaction.

Apéry's Recurrence for (a_n) and (b_n)

Second-order recurrence (Apéry, 1978/1979)

$$(n+1)^3 s_{n+1} - (34n^3 + 51n^2 + 27n + 5)\, s_n + n^3 s_{n-1} = 0$$

Cohen and Zagier's "Creative Telescoping" (van der Poorten, 1979)

"[They] cleverly construct

$$q_{n,k} = 4\,(2n+1)\,\left(k\,(2k+1) - (2n+1)^2\right) c_{n,k}$$

with the motive that

$$(n+1)^3 c_{n+1,k} - (34n^3 + 51n^2 + 27n + 5) c_{n,k} + n^3 c_{n-1,k} = \left[q_{n,j}\right]_{j=k-1}^{j=k}."$$

After summation over k from 0 to $n+1$:

$$(n+1)^3 a_{n+1} - (34n^3 + 51n^2 + 27n + 5)\, a_n + n^3 a_{n-1} = \underbrace{\left[q_{n,j}\right]_{j=-1}^{j=n+1}}_{0-0=0}.$$

Apéry's Recurrence for (a_n) and (b_n)

Second-order recurrence (Apéry, 1978/1979)

$$(n+1)^3 s_{n+1} - (34n^3 + 51n^2 + 27n + 5)\, s_n + n^3 s_{n-1} = 0$$

Cohen and Zagier's "Creative Telescoping" (van der Poorten, 1979)

"[They] cleverly construct

$$Q = 4\,(2n+1)\,\big(k\,(2k+1) - (2n+1)^2\big)$$

with the motive that

$$\Big((n+1)^3 S_n - (34n^3 + 51n^2 + 27n + 5) + n^3 S_n^{-1}\Big) \cdot c = (1 - S_k^{-1})\,(Q \cdot c)\,."$$

After summation over k from 0 to $n+1$:

$$\Big((n+1)^3 S_n - (34n^3 + 51n^2 + 27n + 5) + n^3 S_n^{-1}\Big) \cdot a = \underbrace{[Q \cdot c]_{j=-1}^{j=n+1}}_{0-0=0}\,.$$

Apéry's Recurrence for (a_n) and (b_n)

Second-order recurrence (Apéry, 1978/1979)

$$(n+1)^3 s_{n+1} - (34n^3 + 51n^2 + 27n + 5)\, s_n + n^3 s_{n-1} = 0$$

Cohen and Zagier's "Creative Telescoping" (van der Poorten, 1979)

"[They] cleverly construct

$$P = (n+1)^3 S_n - (34n^3 + 51n^2 + 27n + 5) + n^3 S_n^{-1}$$

and

$$Q = 4\,(2n+1)\,\left(k\,(2k+1) - (2n+1)^2\right)$$

with the motive that

$$P \cdot c = (1 - S_k^{-1})\,(Q \cdot c)\,."$$

After summation over k from 0 to $n+1$:

$$P \cdot a = [Q \cdot c]_{j=-1}^{j=n+1}.$$

Apéry's Recurrence for (a_n) and (b_n)

Cohen and Zagier's "Creative Telescoping" (van der Poorten, 1979)

"[They] cleverly construct

$$P = (n+1)^3 S_n - (34n^3 + 51n^2 + 27n + 5) + n^3 S_n^{-1}$$

and

$$Q = 4\,(2n+1)\,\bigl(k\,(2k+1) - (2n+1)^2\bigr)$$

with the motive that

$$P \cdot c = (1 - S_k^{-1})\,(Q \cdot c)\,."$$

After summation over k from 0 to $n+1$:

$$P \cdot a = [Q \cdot c]_{j=-1}^{j=n+1}.$$

Skew-polynomial algebras:

$$S_n n = (n+1) S_n, \quad S_k k = (k+1) S_k \quad \text{in} \quad \mathbb{Q}(n,k)\langle S_n, S_k \rangle$$

My Motivations to Reconsider CA from a FP Viewpoint

I do: study computer-algebra algorithms on special functions.

Can an algorithmically-generated encyclopedia be authoritative?

E.g., Dynamic Dictionary of Mathematical Functions (DDMF).

My Motivations to Reconsider CA from a FP Viewpoint

I do: study computer-algebra algorithms on special functions.

Can an algorithmically-generated encyclopedia be authoritative?

E.g., Dynamic Dictionary of Mathematical Functions (DDMF).

Doubts with the litterature related to special-functions algorithms

- some key papers are too informal to assess their correctness / I've lost proofs written too tersely in my own papers
- formal power series vs fractions vs functions? / diagonals, positive parts: Cauchy theorem vs algebraic residues?
- hypergeometric sequence vs hypergeometric term? / holonomic vs rationally holonomic vs D-finite vs ∂-finite vs P-recursive?

My Motivations to Reconsider CA from a FP Viewpoint

I do: study computer-algebra algorithms on special functions.

Can an algorithmically-generated encyclopedia be authoritative?

E.g., Dynamic Dictionary of Mathematical Functions (DDMF).

Doubts with the litterature related to special-functions algorithms

- some key papers are too informal to assess their correctness / I've lost proofs written too tersely in my own papers
- formal power series vs fractions vs functions? / diagonals, positive parts: Cauchy theorem vs algebraic residues?
- hypergeometric sequence vs hypergeometric term? / holonomic vs rationally holonomic vs D-finite vs ∂-finite vs P-recursive?

I want: banish underqualified phrasings and prevent shifts in meaning.
I don't want: reproduce informal interaction with the computer.

Example: Densities of short uniform random walks (Borwein, Straub, Wan, Zudilin, 2012).

Turning our attention to negative integers, we have for $k \geqslant 0$ an integer:

$$W_3(-2k-1) = \frac{4}{\pi^3}\left(\frac{2^k k!}{(2k)!}\right)^2 \int_0^\infty t^{2k} K_0(t)^3 \mathrm{d}t, \tag{78}$$

because the two sides satisfy the same recursion ([BBBG08, (8)]), and agree when $k = 0, 1$ ([BBBG08, (47) and (48)]).

From (78), we experimentally determined a single hypergeometric for $W_3(s)$ at negative odd integers:

Lemma 2. *For $k \geqslant 0$ an integer,*

$$W_3(-2k-1) = \frac{\sqrt{3}\binom{2k}{k}^2}{2^{4k+1}3^{2k}}\,{}_3F_2\left(\left.\begin{matrix}\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\\ k+1, k+1\end{matrix}\right|\frac{1}{4}\right).$$

Proof. It is easy to check that both sides agree at $k = 0, 1$. Therefore we need only to show that they satisfy the same recursion. The recursion for the left-hand side implies a contiguous relation for the right-hand side, which can be verified by extracting the summand and applying Gosper's algorithm ([PWZ06]). □

Summation by Computer Algebra *Is* Used in Proofs

Example: Bounding error in high-precision computation of Euler's constant (Brent, Johansson, 2013).

The "lower" sum L is precisely $\sum_{k=0}^{m/2-1} b_k x^{-2k}$. Replacing k by $2k$ in (21) (as the odd terms vanish by symmetry), we have to prove

$$\sum_{j=0}^{2k} \frac{(-1)^j [(2j)!]^2 [(4k-2j)!]^2}{(j!)^3 [(2k-j)!]^3 32^{2k}} = \frac{[(2k)!]^3}{(k!)^4 8^{2k}}. \tag{23}$$

This can be done algorithmically using the creative telescoping approach of Wilf and Zeilberger. For example, the implementation in the Mathematica package `HolonomicFunctions` by Koutschan [6] can be used. The command

```
a = ((2j)!)^2 / ((j!)^3 32^j);
CreativeTelescoping[(-1)^j a (a /. j -> 2k-j),
    {S[j]-1}, S[k]]
```

outputs the recurrence equation

$$(8+8k)b_{k+1} - \left(1+6k+12k^2+8k^3\right) b_k = 0$$

matching the right-hand side of (23), together with a telescoping certificate. Since the summand in (23) vanishes for $j < 0$ and $j > 2k$, no boundary conditions enter into the telescoping relation, and checking the initial value ($k = 0$) suffices to prove the identity.[1]

[1] Curiously, the built-in `Sum` function in Mathematica 9.0.1 computes a closed form for the sum (23), but returns an answer that is wrong by a factor 2 if the factor $[(4k-2j)!]^2$ in the summand is input as $[(2(2k-j))!]^2$.

Algorithmic theory for Special Functions and Combinatorial Sequences initiated by Zeilberger (1982, 1990, 1991)

- Replace named sequences by linear systems of recurrences (+ initial conditions to identify the right solutions)
- Develop algorithms on the level of systems for $+$, $\times$, $\sum$

Implementations exist for Maple, Mathematica, Maxima, etc.

Great success:

- fast evaluation formulae: π, the Catalan constant, ζ-values, β-values
- enumerative combinatorics: heap-ordered trees, q-analogue of totally symmetric plane partitions; positive 3D rook walks; small-step walks
- partition theory: Rogers-Ramanujan and Göllnitz-type identities
- knot theory: colored Jones functions
- mathematical physics: computation of Feynman diagrams

Computer-algebra algorithms apply to Apéry's sums!

- Zeilberger's calculation ($\leq$ 1992) for (a_n)
- Zudilin's alternate proof (1992) by two calls to Zeilberger's algorithm
- Apéry's original calculations using Zeilberger's and Chyzak's algorithms: Salvy's Maple worksheet (2003),
  `http://algo.inria.fr/libraries/autocomb/Apery2-html/apery.html`
- Using difference-field extensions (Schneider, 2007)

Computer-Aided Proofs of Apéry's Theorem

Computer-algebra algorithms apply to Apéry's sums!

- Zeilberger's calculation ($\leq$ 1992) for (a_n)
- Zudilin's alternate proof (1992) by two calls to Zeilberger's algorithm
- Apéry's original calculations using Zeilberger's and Chyzak's algorithms: Salvy's Maple worksheet (2003),
  `http://algo.inria.fr/libraries/autocomb/Apery2-html/apery.html`
- Using difference-field extensions (Schneider, 2007)

Our formalization follows the Apéry/van der Poorten/Salvy path.

A Convoluted Proof of Cassini's Identity $F_n F_{n+2} = F_{n+1}^2 + (-1)^n$

- Fibonacci numbers: $F_{n+2} = F_{n+1} + F_n, \quad F_0 = F_1 = 1.$
- Define (σ_n) by: $\sigma_{n+1} = -\sigma_n, \quad \sigma_0 = 1.$

A Convoluted Proof of Cassini's Identity $F_n F_{n+2} = F_{n+1}^2 + (-1)^n$

- Fibonacci numbers: $F_{n+2} = F_{n+1} + F_n, \quad F_0 = F_1 = 1.$
- Define (σ_n) by: $\sigma_{n+1} = -\sigma_n, \quad \sigma_0 = 1.$
- Introduce $u_n := F_{n+1}^2 + \sigma_n$ and compute the normal forms:

$$\begin{aligned} u_n &= F_{n+1}^2 + \sigma_n, \\ u_{n+1} &= F_n^2 + 2F_nF_{n+1} + F_{n+1}^2 - \sigma_n, \\ u_{n+2} &= F_n^2 + 4F_nF_{n+1} + 4F_{n+1}^2 + \sigma_n, \\ u_{n+3} &= 4F_n^2 + 12F_nF_{n+1} + 9F_{n+1}^2 - \sigma_n. \end{aligned}$$

- Solving a linear system yields: $u_{n+3} - 2u_{n+2} - 2u_{n+1} + u_n = 0.$

A Convoluted Proof of Cassini's Identity $F_n F_{n+2} = F_{n+1}^2 + (-1)^n$

- Fibonacci numbers: $F_{n+2} = F_{n+1} + F_n, \quad F_0 = F_1 = 1.$
- Define (σ_n) by: $\sigma_{n+1} = -\sigma_n, \quad \sigma_0 = 1.$
- Introduce $u_n := F_{n+1}^2 + \sigma_n$ and compute the normal forms:

$$\begin{aligned} u_n &= F_{n+1}^2 + \sigma_n, \\ u_{n+1} &= F_n^2 + 2F_nF_{n+1} + F_{n+1}^2 - \sigma_n, \\ u_{n+2} &= F_n^2 + 4F_nF_{n+1} + 4F_{n+1}^2 + \sigma_n, \\ u_{n+3} &= 4F_n^2 + 12F_nF_{n+1} + 9F_{n+1}^2 - \sigma_n. \end{aligned}$$

- Solving a linear system yields: $u_{n+3} - 2u_{n+2} - 2u_{n+1} + u_n = 0.$
- Same process for $v_n := F_n F_{n+2}$ delivers the same recurrence.
- Now, checking initial conditions and an induction ends the proof:

$$u_0 = v_0 = 2, \qquad u_1 = v_1 = 3, \qquad u_2 = v_2 = 10.$$

A Generalization: ∂-Finite Sequences (Chyzak, Salvy, 1998)

$$(t_{n,k}) \text{ is } \partial\text{-finite}$$
$$\Updownarrow$$
$$\text{the shifts } (t_{n+i,k+j}) \text{ span a finite-dimensional } \mathbb{Q}(n,k)\text{-vector space}$$

⇒ linear functional equations with rational-function coefficients.

A Generalization: ∂-Finite Sequences (Chyzak, Salvy, 1998)

$(t_{n,k})$ is ∂-finite

$\Updownarrow$

the shifts $(t_{n+i,k+j})$ span a finite-dimensional $\mathbb{Q}(n,k)$-vector space

$\Rightarrow$ linear functional equations with rational-function coefficients.

Examples: Fibonacci numbers; binomial coefficients

$$\binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \qquad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k};$$

orthogonal polynomials, Bessel functions.

A Generalization: ∂-Finite Sequences (Chyzak, Salvy, 1998)

$(t_{n,k})$ is ∂-finite

$\Updownarrow$

the shifts $(t_{n+i,k+j})$ span a finite-dimensional $\mathbb{Q}(n,k)$-vector space

$\Rightarrow$ linear functional equations with rational-function coefficients.

Examples: Fibonacci numbers; binomial coefficients

$$\binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \qquad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k};$$

orthogonal polynomials, Bessel functions.

Closures under $+$, $\times$, shifts

- Annihilating ideal $\to$ skew Gröbner basis $\to$ normal forms in finite dim.
- Iterative algorithm to search for linear dependencies

$\rightsquigarrow$ simplification and zero test of ∂-finite polynomial expressions.

A Generalization: ∂-Finite Sequences (Chyzak, Salvy, 1998)

$(t_{n,k})$ is ∂-finite

$\Updownarrow$

the shifts $(t_{n+i,k+j})$ span a finite-dimensional $\mathbb{Q}(n,k)$-vector space

$\Rightarrow$ linear functional equations with rational-function coefficients.

Examples: Fibonacci numbers; binomial coefficients

$$\operatorname{ann}\binom{n}{k} = \left\{ L_1 \left(S_n - \frac{n+1}{n+1-k} \right) + L_2 \left(S_k - \frac{n-k}{k+1} \right) : L_1, L_1 \in \mathbb{Q}(n,k)\langle S_n, S_k \rangle \right\};$$

orthogonal polynomials, Bessel functions.

Closures under $+$, $\times$, shifts

- Annihilating ideal $\rightarrow$ skew Gröbner basis $\rightarrow$ normal forms in finite dim.
- Iterative algorithm to search for linear dependencies

$\rightsquigarrow$ simplification and zero test of ∂-finite polynomial expressions.

A Convoluted Proof of $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

- Define $F_n := \sum_{k=0}^{n} \binom{n}{k}$.
- Prove

$$\binom{n+1}{k} - 2\binom{n}{k} = \left[\frac{-j\binom{n}{j}}{n+1-j} \right]_{j=k}^{j=k+1}$$

as a consequence of

$$\binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \quad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k}.$$

- Sum from $k = -1$ to $k = n+1$ to get $F_{n+1} - 2F_n = 0$.
- Now, observing $F_0 = 1$ yields the result.

Algorithms for Summing "Holonomic" ∂-Finite Sequences

Zeilberger's algorithm (1991)
INPUT: a hypergeometric term $f_{n,k}$, that is, first-order recurrences.
OUTPUT: rational functions $p_0(n), \ldots, p_r(n)$, $Q(n,k)$ with minimal r, such that $p_r(n)\, f_{n+r,k} + \cdots + p_0(n)\, f_{n,k} = Q(n,k+1)\, f_{n,k+1} - Q(n,k)\, f_{n,k}$.

Zeilberger's algorithm (1991)
INPUT: a hypergeometric term $f_{n,k}$, that is, first-order recurrences.
OUTPUT: rational functions $p_0(n), \ldots, p_r(n)$, $Q(n,k)$ with minimal r, such that $p_r(n)\, f_{n+r,k} + \cdots + p_0(n)\, f_{n,k} = Q(n,k+1)\, f_{n,k+1} - Q(n,k)\, f_{n,k}$.

Chyzak's algorithm (2000)
INPUT: a ∂-finite term u w.r.t. $A = \mathbb{Q}(n,k)\langle S_n, S_k \rangle$, a Gröbner basis G of $\operatorname{ann} u$.
OUTPUT: $P \in \mathbb{Q}(n)\langle S_n \rangle$ of minimal possible order, $Q \in A$ reduced mod. G and such that $P \cdot u = (S_k - 1)Q \cdot u$.

Algorithms for Summing "Holonomic" ∂-Finite Sequences

Zeilberger's algorithm (1991)
INPUT: a hypergeometric term $f_{n,k}$, that is, first-order recurrences.
OUTPUT: rational functions $p_0(n), \dots, p_r(n), Q(n,k)$ with minimal r, such that $p_r(n)\, f_{n+r,k} + \cdots + p_0(n)\, f_{n,k} = Q(n,k+1)\, f_{n,k+1} - Q(n,k)\, f_{n,k}$.

Chyzak's algorithm (2000)

INPUT: $\begin{cases} \text{a } \partial\text{-finite term } u \text{ w.r.t. } A = \mathbb{Q}(n,k)\langle S_n, S_k\rangle, \\ \text{a Gröbner basis } G \text{ of } \operatorname{ann} u. \end{cases}$

OUTPUT: $\begin{cases} P \in \mathbb{Q}(n)\langle S_n\rangle \text{ of minimal possible order}, \\ Q \in A \text{ reduced mod. } G \text{ and such that } P \cdot u = (S_k - 1)Q \cdot u. \end{cases}$

Example: we can get the same 2nd-order operator P for both sides of

$$\underbrace{\sum_{r=0}^{\infty}}_{\text{by C}} \underbrace{\sum_{s=0}^{\infty}}_{\text{by Z}} (-1)^{n+r+s} \binom{n}{r} \binom{n}{s} \binom{n+r}{r} \binom{n+s}{s} \binom{2n-(r+s)}{n} = \underbrace{\sum_{k=0}^{\infty}}_{\text{by Z}} \binom{n}{k}^4.$$

A Skeptic's Approach to Combining FP and CA

"Proving" an algorithm

- would prove all its results satisfy the specifications
- but it is too much work in our context

Instead, use an external computer-algebra tool as an oracle

- be as skeptical of the computer algebra as of the human
- approach of choice when checking is simpler than discovering

Inspired by (Harrison, Théry, 1997)

A Program to Derive Recurrences for Apéry's Sums

Concrete sequences ...

step	explicit form	operation	input(s)
1	$c_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$	simplification	
2	$a_n = \sum_{k=1}^{n} c_{n,k}$	creative telescoping	$c_{n,k}$
3	$d_{n,m} = \frac{(-1)^{m+1}}{2m^3 \binom{n}{m} \binom{n+m}{m}}$	simplification	
4	$s_{n,k} = \sum_{m=1}^{k} d_{n,m}$	creative telescoping	$d_{n,m}$
5	$z_n = \sum_{m=1}^{n} \frac{1}{m^3}$	simplification	
6	$u_{n,k} = z_n + s_{n,k}$	addition	z_n and $s_{n,k}$
7	$v_{n,k} = c_{n,k} u_{n,k}$	product	$c_{n,k}$ and $u_{n,k}$
8	$b_n = \sum_{k=1}^{n} v_{n,k}$	creative telescoping	$v_{n,k}$

... replaced with abstract analogues: *any* solution of a given GB

step	explicit form	operation	input GB(s)	output GB
1	$c_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$	simplification		C
2	$a_n = \sum_{k=1}^n c_{n,k}$	creative telescoping	C	A
3	$d_{n,m} = \frac{(-1)^{m+1}}{2m^3\binom{n}{m}\binom{n+m}{m}}$	simplification		D
4	$s_{n,k} = \sum_{m=1}^k d_{n,m}$	creative telescoping	D	S
5	$z_n = \sum_{m=1}^n \frac{1}{m^3}$	simplification		Z
6	$u_{n,k} = z_n + s_{n,k}$	addition	Z and S	U
7	$v_{n,k} = c_{n,k} u_{n,k}$	product	C and U	V
8	$b_n = \sum_{k=1}^n v_{n,k}$	creative telescoping	V	B

How Can a Candidate Recurrence be Checked?

Because
$$\binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \quad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k},$$
it follows:
$$\binom{n+1}{k} - 2\binom{n}{k} + \left[\frac{j\binom{n}{j}}{n+1-j}\right]_{j=k}^{j=k+1} =$$
$$\binom{n+1}{k} - 2\binom{n}{k} + \frac{(k+1)\binom{n}{k+1}}{n-k} - \frac{k\binom{n}{k}}{n+1-k} =$$
$$\underbrace{\left(\frac{n+1}{n+1-k} - 2 + \frac{k+1}{n-k}\frac{n-k}{k+1} - \frac{k}{n+1-k}\right)}_{=0}\binom{n}{k} = 0.$$

How Can a Candidate Recurrence be Checked?

Because the annihilating (left) ideal I of $\binom{n}{k}$ is generated by the GB

$$g_1 := S_n - \frac{n+1}{n+1-k}, \quad g_2 := S_k - \frac{n-k}{k+1},$$

it follows:

$$S_n - 2 + (S_k - 1)\frac{k}{n+1-k} = \\ S_n - 2 + \frac{k+1}{n-k}S_k - \frac{k}{n+1-k} = \\ g_1 + \frac{k+1}{n-k}g_2 + \underbrace{\left(\frac{n+1}{n+1-k} - 2 + \frac{k+1}{n-k}\frac{n-k}{k+1} - \frac{k}{n+1-k}\right)}_{=0} \in I.$$

How Can a Candidate Recurrence be Checked?

Because

$$\binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \quad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k},$$

it follows:

$$\binom{n+1}{k} - 2\binom{n}{k} + \left[\frac{j\binom{n}{j}}{n+1-j}\right]_{j=k}^{j=k+1} =$$

$$\binom{n+1}{k} - 2\binom{n}{k} + \frac{(k+1)\binom{n}{k+1}}{n-k} - \frac{k\binom{n}{k}}{n+1-k} =$$

$$\underbrace{\left(\frac{n+1}{n+1-k} - 2 + \frac{k+1}{n-k}\frac{n-k}{k+1} - \frac{k}{n+1-k}\right)}_{=0}\binom{n}{k} = 0.$$

How Can a Candidate Recurrence be Checked?

Because

$$k \neq n+1 \implies \binom{n+1}{k} = \frac{n+1}{n+1-k}\binom{n}{k}, \quad k \neq -1 \implies \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k},$$

it follows:

$$\binom{n+1}{k} - 2\binom{n}{k} + \left[\frac{j\binom{n}{j}}{n+1-j}\right]_{j=k}^{j=k+1} =$$

$$\binom{n+1}{k} - 2\binom{n}{k} + \frac{(k+1)\binom{n}{k+1}}{n-k} - \frac{k\binom{n}{k}}{n+1-k} =$$

$$\underbrace{\left(\frac{n+1}{n+1-k} - 2 + \frac{k+1}{n-k}\frac{n-k}{k+1} - \frac{k}{n+1-k}\right)}_{=0}\binom{n}{k} = 0$$

if $k \neq n+1$, $k \neq n$, and $k \neq -1$.

Explanation:

- Recurrences are valid out of an algebraic set Δ.
- Closures under $+$, $\times$, S_i are sound, but out of an unknown Δ.
- Meaning of summation is dubious if summation range intersects Δ.

Hope:

- Easy: Discover the recurrences by a Maple session by algorithms.
- Uneasy: Guard each of them by a proviso, but how to get it?

The Algebraic Disease and a Potential Cure

Explanation:

- Recurrences are valid out of an algebraic set Δ.
- Closures under $+$, $\times$, S_i are sound, but out of an unknown Δ.
- Meaning of summation is dubious if summation range intersects Δ.

Hope:

- Easy: Discover the recurrences by a Maple session by algorithms.
- Uneasy: Guard each of them by a proviso, but how to get it?

Remark:

- To the best of my knowledge, correctness of summation algorithms is adressed only for very limited situations (Abramov, Petkovšek, 2007).

Structure of Our Coq Files

Data of guarded recurrences for each abstracted composite sequence

- human-discovered and -written provisos for each of the recurrences
- Maple-generated coefficients of the recurrences, pretty-printed to Coq
- recurrences written in terms of the proviso name and coefficient names:
  - hypergeometric sequences ($c_{n,k}$, $d_{n,m}$) and indefinite sum (z_n): a GB directly obtained from the explicit form
  - composite under $+$ or $\times$ ($u_{n,k}$ and $v_{n,k}$): a GB directly obtained via algorithmic closure
  - composite under creative telescoping (a_n, $s_{n,k}$, b_n): first, recurrences of the form $P \cdot f = (S_k - 1)Q \cdot f$; then, conversion of the P into a GB

Structure of Our Coq Files

Data of guarded recurrences for each abstracted composite sequence

- human-discovered and -written provisos for each of the recurrences
- Maple-generated coefficients of the recurrences, pretty-printed to Coq
- recurrences written in terms of the proviso name and coefficient names

Proofs of recurrences for each abstracted sequence

- load guarded recurrences for arguments (assumed) and for the composite (being proved)
- assume arguments satisfying relevant recurrences; define the composite as a function of the arguments
- state and prove lemmas (recurrences) for the composite, e.g.:

  Lemma: $\forall c \in \mathbb{Q}^{\mathbb{Z}^2}$, $\forall u \in \mathbb{Q}^{\mathbb{Z}^2}$, $\forall v \in \mathbb{Q}^{\mathbb{Z}^2}$, if c solves C and u solves U and $v = c \times u$, then v solves V.

Structure of Our Coq Files

Data of guarded recurrences for each abstracted composite sequence

- human-discovered and -written provisos for each of the recurrences
- Maple-generated coefficients of the recurrences, pretty-printed to Coq
- recurrences written in terms of the proviso name and coefficient names

Proofs of recurrences for each abstracted sequence

- load guarded recurrences for arguments (assumed) and for the composite (being proved)
- assume arguments satisfying relevant recurrences; define the composite as a function of the arguments
- state and prove lemmas (recurrences) for the composite

Proofs of recurrences for the concrete sequences

- ad-hoc means for initial sequences ($c_{n,k}$, $d_{n,m}$, z_n)
- recurrences for other sequences follows immediately by instantiation
- finally, reduction of fourth-order recurrence for (b_n) to order 2

A Lemma for Creative Telescoping?

$$p_0(n)u_{n,k} + p_1(n)u_{n+1,k} + \cdots + p_r(n)u_{n+r,k} = Q(n,k+1)u_{n,k+1} - Q(n,k)u_{n,k}$$

$$U_n := \sum_{k=\alpha}^{n+\beta} u_{n,k}$$

$$\begin{array}{rcl}
p_0(n)u_{n,n+\beta} + p_1(n)u_{n+1,n+\beta} + \cdots + p_r(n)u_{n+r,n+\beta} &=& Q(n,n+\beta+1)u_{n,n+\beta+1} - Q(n,n+\beta)u_{n,n+\beta} \\
\vdots &=& \vdots \\
p_0(n)u_{n,\alpha+1} + p_1(n)u_{n+1,\alpha+1} + \cdots + p_r(n)u_{n+r,\alpha+1} &=& Q(n,\alpha+2)u_{n,\alpha+2} - Q(n,\alpha+1)u_{n,\alpha+1} \\
p_0(n)u_{n,\alpha} + p_1(n)u_{n+1,\alpha} + \cdots + p_r(n)u_{n+r,\alpha} &=& Q(n,\alpha+1)u_{n,\alpha+1} - Q(n,\alpha)u_{n,\alpha} \\
\hline
\end{array}$$

A Lemma for Creative Telescoping?

$$p_0(n)u_{n,k} + p_1(n)u_{n+1,k} + \cdots + p_r(n)u_{n+r,k} = Q(n,k+1)u_{n,k+1} - Q(n,k)u_{n,k}$$

$$U_n := \sum_{k=\alpha}^{n+\beta} u_{n,k}$$

$$\begin{array}{rcl} p_0(n)u_{n,n+\beta} + p_1(n)u_{n+1,n+\beta} + \cdots + p_r(n)u_{n+r,n+\beta} &=& Q(n,n+\beta+1)u_{n,n+\beta+1} - Q(n,n+\beta)u_{n,n+\beta} \\ \vdots &=& \vdots \\ p_0(n)u_{n,\alpha+1} + p_1(n)u_{n+1,\alpha+1} + \cdots + p_r(n)u_{n+r,\alpha+1} &=& Q(n,\alpha+2)u_{n,\alpha+2} - Q(n,\alpha+1)u_{n,\alpha+1} \\ p_0(n)u_{n,\alpha} + p_1(n)u_{n+1,\alpha} + \cdots + p_r(n)u_{n+r,\alpha} &=& Q(n,\alpha+1)u_{n,\alpha+1} - Q(n,\alpha)u_{n,\alpha} \\ \hline p_0(n)U_n && \end{array}$$

A Lemma for Creative Telescoping?

$$p_0(n)u_{n,k} + p_1(n)u_{n+1,k} + \cdots + p_r(n)u_{n+r,k} = Q(n,k+1)u_{n,k+1} - Q(n,k)u_{n,k}$$

$$U_n := \sum_{k=\alpha}^{n+\beta} u_{n,k}$$

$$\begin{array}{rcl}
p_1(n)u_{n+1,n+\beta+1} & = & p_1(n)u_{n+1,n+\beta+1} \\
p_0(n)u_{n,n+\beta} + p_1(n)u_{n+1,n+\beta} + \cdots + p_r(n)u_{n+r,n+\beta} & = & Q(n,n+\beta+1)u_{n,n+\beta+1} - Q(n,n+\beta)u_{n,n+\beta} \\
\vdots & = & \vdots \\
p_0(n)u_{n,\alpha+1} + p_1(n)u_{n+1,\alpha+1} + \cdots + p_r(n)u_{n+r,\alpha+1} & = & Q(n,\alpha+2)u_{n,\alpha+2} - Q(n,\alpha+1)u_{n,\alpha+1} \\
p_0(n)u_{n,\alpha} + p_1(n)u_{n+1,\alpha} + \cdots + p_r(n)u_{n+r,\alpha} & = & Q(n,\alpha+1)u_{n,\alpha+1} - Q(n,\alpha)u_{n,\alpha} \\
\hline
p_0(n)U_n + p_1(n)U_{n+1} & &
\end{array}$$

A Lemma for Creative Telescoping?

$$p_0(n)u_{n,k} + p_1(n)u_{n+1,k} + \cdots + p_r(n)u_{n+r,k} = Q(n,k+1)u_{n,k+1} - Q(n,k)u_{n,k}$$

$$U_n := \sum_{k=\alpha}^{n+\beta} u_{n,k}$$

$$\begin{array}{rcl}
p_r(n)u_{n+r,n+\beta+r} & = & p_r(n)u_{n+r,n+\beta+r} \\
\vdots & = & \vdots \\
p_1(n)u_{n+1,n+\beta+1} + \cdots + p_r(n)u_{n+r,n+\beta+1} & = & p_1(n)u_{n+1,n+\beta+1} + \cdots + p_r(n)u_{n+r,n+\beta+1} \\
p_0(n)u_{n,n+\beta} + p_1(n)u_{n+1,n+\beta} + \cdots + p_r(n)u_{n+r,n+\beta} & = & Q(n,n+\beta+1)u_{n,n+\beta+1} - Q(n,n+\beta)u_{n,n+\beta} \\
\vdots & = & \vdots \\
p_0(n)u_{n,\alpha+1} + p_1(n)u_{n+1,\alpha+1} + \cdots + p_r(n)u_{n+r,\alpha+1} & = & Q(n,\alpha+2)u_{n,\alpha+2} - Q(n,\alpha+1)u_{n,\alpha+1} \\
p_0(n)u_{n,\alpha} + p_1(n)u_{n+1,\alpha} + \cdots + p_r(n)u_{n+r,\alpha} & = & Q(n,\alpha+1)u_{n,\alpha+1} - Q(n,\alpha)u_{n,\alpha} \\
\hline
p_0(n)U_n + p_1(n)U_{n+1} + \cdots + p_r(n)U_{n+r} & &
\end{array}$$

A Lemma for Creative Telescoping?

$$p_0(n)u_{n,k} + p_1(n)u_{n+1,k} + \cdots + p_r(n)u_{n+r,k} = Q(n,k+1)u_{n,k+1} - Q(n,k)u_{n,k}$$

$$U_n := \sum_{k=\alpha}^{n+\beta} u_{n,k}$$

$$\begin{aligned}
p_r(n)u_{n+r,n+\beta+r} &= p_r(n)u_{n+r,n+\beta+r} \\
\vdots &= \vdots \\
p_1(n)u_{n+1,n+\beta+1} + \cdots + p_r(n)u_{n+r,n+\beta+1} &= p_1(n)u_{n+1,n+\beta+1} + \cdots + p_r(n)u_{n+r,n+\beta+1} \\
p_0(n)u_{n,n+\beta} + p_1(n)u_{n+1,n+\beta} + \cdots + p_r(n)u_{n+r,n+\beta} &= Q(n,n+\beta+1)u_{n,n+\beta+1} - Q(n,n+\beta)u_{n,n+\beta} \\
\vdots &= \vdots \\
p_0(n)u_{n,\alpha+1} + p_1(n)u_{n+1,\alpha+1} + \cdots + p_r(n)u_{n+r,\alpha+1} &= Q(n,\alpha+2)u_{n,\alpha+2} - Q(n,\alpha+1)u_{n,\alpha+1} \\
p_0(n)u_{n,\alpha} + p_1(n)u_{n+1,\alpha} + \cdots + p_r(n)u_{n+r,\alpha} &= Q(n,\alpha+1)u_{n,\alpha+1} - Q(n,\alpha)u_{n,\alpha} \\
\hline
p_0(n)U_n + p_1(n)U_{n+1} + \cdots + p_r(n)U_{n+r} &= Q(n,n+\beta+1)u_{n,n+\beta+1} - Q(n,\alpha)u_{n,\alpha}
\end{aligned}$$

A Lemma for Creative Telescoping?

$$p_0(n)u_{n,k} + p_1(n)u_{n+1,k} + \cdots + p_r(n)u_{n+r,k} = Q(n,k+1)u_{n,k+1} - Q(n,k)u_{n,k}$$

$$U_n := \sum_{k=\alpha}^{n+\beta} u_{n,k}$$

$$\begin{array}{rcl}
p_r(n)u_{n+r,n+\beta+r} & = & p_r(n)u_{n+r,n+\beta+r} \\
\vdots & = & \vdots \\
p_1(n)u_{n+1,n+\beta+1} + \cdots + p_r(n)u_{n+r,n+\beta+1} & = & p_1(n)u_{n+1,n+\beta+1} + \cdots + p_r(n)u_{n+r,n+\beta+1} \\
p_0(n)u_{n,n+\beta} + p_1(n)u_{n+1,n+\beta} + \cdots + p_r(n)u_{n+r,n+\beta} & = & Q(n,n+\beta+1)u_{n,n+\beta+1} - Q(n,n+\beta)u_{n,n+\beta} \\
\vdots & = & \vdots \\
p_0(n)u_{n,\alpha+1} + p_1(n)u_{n+1,\alpha+1} + \cdots + p_r(n)u_{n+r,\alpha+1} & = & Q(n,\alpha+2)u_{n,\alpha+2} - Q(n,\alpha+1)u_{n,\alpha+1} \\
p_0(n)u_{n,\alpha} + p_1(n)u_{n+1,\alpha} + \cdots + p_r(n)u_{n+r,\alpha} & = & Q(n,\alpha+1)u_{n,\alpha+1} - Q(n,\alpha)u_{n,\alpha} \\
\hline
p_0(n)U_n + p_1(n)U_{n+1} + \cdots + p_r(n)U_{n+r} & = & Q(n,n+\beta+1)u_{n,n+\beta+1} - Q(n,\alpha)u_{n,\alpha} \\
 & & + \sum_{i=1}^{r}\sum_{j=1}^{i} p_i(n)u_{n+i,n+\beta+j}
\end{array}$$

A lemma instead of a case-by-case analysis

Given $(u_{n,k}) \in \mathbb{Q}^{\mathbb{Z}^2}$, define $U_n = \sum_{k=\alpha}^{n+\beta} u_{n,k}$. Given a set Δ such that

$$(n,k) \notin \Delta \Rightarrow (P \cdot u_{\bullet,k})_n = (Q \cdot u)_{n,k+1} - (Q \cdot u)_{n,k},$$

the following identity holds for any n such that $\alpha \leq n + \beta$:

$$(P \cdot U)_n = \Big((Q \cdot u)_{n,n+\beta+1} - (Q \cdot u)_{n,\alpha}\Big) + \sum_{i=1}^{r} \sum_{j=1}^{i} p_i(n) u_{n+i,n+\beta+j}$$
$$+ \sum_{\alpha \leq k \leq n+\beta \,\wedge\, (n,k) \in \Delta} (P \cdot u_{\bullet,k})_n - (Q \cdot u)_{n,k+1} + (Q \cdot u)_{n,k}.$$

A lemma instead of a case-by-case analysis

Given $(u_{n,k}) \in \mathbb{Q}^{\mathbb{Z}^2}$, define $U_n = \sum_{k=\alpha}^{n+\beta} u_{n,k}$. Given a set Δ such that

$$(n,k) \notin \Delta \Rightarrow (P \cdot u_{\bullet,k})_n = (Q \cdot u)_{n,k+1} - (Q \cdot u)_{n,k},$$

the following identity holds for any n such that $\alpha \leq n + \beta$:

$$(P \cdot U)_n = \Big((Q \cdot u)_{n,n+\beta+1} - (Q \cdot u)_{n,\alpha}\Big) + \sum_{i=1}^{r} \sum_{j=1}^{i} p_i(n) u_{n+i,n+\beta+j}$$
$$+ \sum_{\alpha \leq k \leq n+\beta \ \wedge \ (n,k) \in \Delta} (P \cdot u_{\bullet,k})_n - (Q \cdot u)_{n,k+1} + (Q \cdot u)_{n,k}.$$

In practice: Coq's u, U, P, Q are total maps, extending the mathematical objects.

Sound Creative Telescoping

A lemma instead of a case-by-case analysis

Given $(u_{n,k}) \in \mathbb{Q}^{\mathbb{Z}^2}$, define $U_n = \sum_{k=\alpha}^{n+\beta} u_{n,k}$. Given a set Δ such that

$$(n,k) \notin \Delta \Rightarrow (P \cdot u_{\bullet,k})_n = (Q \cdot u)_{n,k+1} - (Q \cdot u)_{n,k},$$

the following identity holds for any n such that $\alpha \leq n + \beta$:

$$(P \cdot U)_n = \Big((Q \cdot u)_{n,n+\beta+1} - (Q \cdot u)_{n,\alpha}\Big) + \sum_{i=1}^{r} \sum_{j=1}^{i} p_i(n) u_{n+i,n+\beta+j}$$
$$+ \sum_{\alpha \leq k \leq n+\beta \ \wedge \ (n,k) \in \Delta} (P \cdot u_{\bullet,k})_n - (Q \cdot u)_{n,k+1} + (Q \cdot u)_{n,k}.$$

In practice: Coq's u, U, P, Q are total maps, extending the mathematical objects.

Use of the lemma: normalizing the right-hand side (to 0)

- Ill-formed terms should cancel (manual inspection)
- Normalize modulo GB (several copies of stairs: $u_{n,\alpha}$, $u_{n,n+\beta}$)
- Use rational-function normalization to get 0 (Coq's `field`)

Other Parts of the Formalization (Coq + MathComp + CoqEAL)

Elementary number theory

- definition of binomials over $\mathbb{Z}^2$
- standard properties + $1 \le i \le j \le n \implies j\binom{i}{j} \mid \ell_n$

Asymptotic estimates

- of a_n:
  - implicit use of Poincaré–Perron–Kreuser theorem(s) in Apéry's proof
  - replaced with the more elementary $33^n = \mathcal{O}(a^n)$
- of ℓ_n:
  - original proof uses $\ell_n = e^{n+o(1)}$, implied by the Prime Number Theorem
  - replaced with $\ell_n = \mathcal{O}(3^n)$

Numbers: libraries used

- proof-dedicated integers and rationals of MathComp (Gonthier *et al.*)
- computation-dedicated integers and rationals of CoqEAL (Cohen, Mörtberg, Dénès)
- algebraic numbers (Cohen)
- Cauchy reals to encode $\zeta(3)$ as $(z_n)_{n\in\mathbb{N}}$ and a Cauchy-CV proof

Other Parts of the Formalization (Coq + MathComp + CoqEAL)

Elementary number theory

- definition of binomials over $\mathbb{Z}^2$
- standard properties + $1 \leq i \leq j \leq n \implies j\binom{i}{j} \mid \ell_n$

Asymptotic estimates

- of a_n:
  - implicit use of Poincaré–Perron–Kreuser theorem(s) in Apéry's proof
  - replaced with the more elementary $33^n = \mathcal{O}(a^n)$
- of ℓ_n:
  - original proof uses $\ell_n = e^{n+o(1)}$, implied by the Prime Number Theorem
  - replaced with $\ell_n = \mathcal{O}(3^n)$ [Admitted; under continued work.]

Numbers: libraries used

- proof-dedicated integers and rationals of MathComp (Gonthier *et al.*)
- computation-dedicated integers and rationals of CoqEAL (Cohen, Mörtberg, Dénès)
- algebraic numbers (Cohen)
- Cauchy reals to encode $\zeta(3)$ as $(z_n)_{n\in\mathbb{N}}$ and a Cauchy-CV proof

End Result (as of May 2014)

We have machine-checked (a stronger statement of):

$$\boxed{\text{Theorem: } \ell_n = \mathcal{O}(3^n) \implies \zeta(3) \notin \mathbb{Q}.}$$

End Result (as of May 2014)

We have machine-checked (a stronger statement of):

$$\text{Theorem: } \ell_n = \mathcal{O}(3^n) \implies \zeta(3) \notin \mathbb{Q}.$$

```
Coq < Print lcmn_asymptotic_bound.
lcmn_asymptotic_bound =
exists (K2 K3 : rat) (N : nat),
  0 < K2  /\  0 < K3  /\  K2 ^ 3 < 33%:~R  /\
  forall (n : nat),
    (N <= n)%N -> (iter_lcmn n)%:~R < K3 * K2 ^ n
      : Prop

Coq < About zeta_3_irrational.
zeta_3_irrational :
lcmn_asymptotic_bound ->
  not (exists (r : rat), (z3 = (r%:CR))%CR)
```

Subjective Conclusions on Getting to Work with Coq (+ MathComp)

An excessively difficult endeavour: a very shallow learning curve

- different methodologies over the years $\rightsquigarrow$ documentation out of sync $\rightsquigarrow$ oral transmission
- too difficult to read *through* notation + coercions + structure inference
- understanding libraries requires a knowledge of Coq's most advanced features

Subjective Conclusions on Getting to Work with Coq (+ MathComp)

An excessively difficult endeavour: a very shallow learning curve

- different methodologies over the years $\rightsquigarrow$ documentation out of sync $\rightsquigarrow$ oral transmission
- too difficult to read *through* notation + coercions + structure inference
- understanding libraries requires a knowledge of Coq's most advanced features

Formalization: opposing goals?

- mimicking the mathematical informal interaction
- flushing doubts on proofs/interpretation of mathematical objects

Conclusions and Future Work

- (Ongoing) Complete proof by formalizing bound on $\mathrm{lcm}(1,\ldots,n)$
- Test robustness of approach by more examples of sums
- Understanding why it works, so as to automate our protocol
- (Ongoing) Differential analogue: similar approach to prove the second-order ODE for the square-lattice Green function

$$\int_0^1 \int_0^1 \frac{1}{(1-xyz)\sqrt{1-x^2}\sqrt{1-y^2}}\,\mathrm{d}x\,\mathrm{d}y$$

- Dedicated data structure to keep (skew-)polynomials normalized