Dendric words and dendric subshifts

Paulina CECCHI B.
(Joint work with **Valérie Berthé**)

Institute de Recherche en Informatique Fondamentale
Université Paris Diderot - Paris 7

Departamento de Matemática y Ciencia de la Computación
Facultad de Ciencia. Universidad de Santiago de Chile

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Given a factor w of x, a right extension of w is a letter $a \in \mathcal{A}$ such that $aw \in \mathcal{L}(x)$. We define analogously a left extension ($wb \in \mathcal{L}(x)$) and a biextension ($awb \in \mathcal{L}(x)$) of w.

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We are interested in a family of words with linear complexity and some restrictions on the possible extensions of their factors.

Outline

- Dendric words.
- Dendric subshifts (symbolic dynamical systems).
- Exploting properties of extension graphs.
    - Balance in dendric words.
    - Invariant measures and orbit equivalence.

Dendric words

Dendric words

- Let $\mathcal{A}$ be a finite non-empty alphabet and $x \in \mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$. For $w \in \mathcal{L}(x)$, the extensions of w are the following sets,

$$L(w) = \{a \in \mathcal{A} \mid aw \in \mathcal{L}(x)\}$$
$$R(w) = \{a \in \mathcal{A} \mid wa \in \mathcal{L}(x)\}$$
$$B(w) = \{(a, b) \in \mathcal{A} \times \mathcal{A} \mid awb \in \mathcal{L}(x)\}.$$

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- The extension graph $\mathcal{E}_x(w)$ of w is the undirected bipartite graph whose set of vertices is the disjoint union of $L(w)$ and $R(w)$ and whose edges are the pairs $(a, b) \in B(w)$.

Dendric words

- Example: Consider the Fibonacci word in $\{a, b\}$

$$x = abaababaabaababaababa\cdots$$

produced by the substitution $\varphi : a \mapsto ab, b \mapsto a$ $(x = \varphi^{\omega}(a))$.

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$\mathcal{E}_x(a)$ $\mathcal{E}_x(b)$

$$\mathcal{L}_3(x) = \{aba, baa, aab, bab\}.$$

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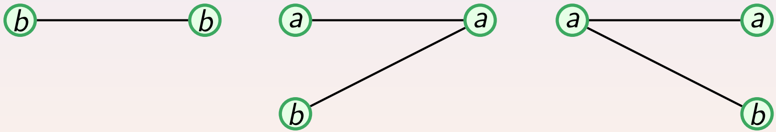
- The extension graphs of aa, ab, ba are

$\mathcal{E}_x(aa)$ $\mathcal{E}_x(ab)$ $\mathcal{E}_x(ba)$



$$\mathcal{L}_4(x) = \{abaa, baab, aaba, abab, baba\}.$$

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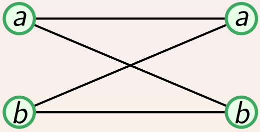
- Consider the Thue-Morse word in $\{a, b\}$ given by

$$y = abbabaabbaababba \cdots$$

produced by the Thue-Morse substitution $\sigma : a \mapsto ab, b \mapsto ba$. This word is not dendric. The extension graph of ϵ is

$$\mathcal{E}_y(\epsilon)$$



$$\mathcal{L}_2(y) = \{aa, ab, ba, bb\}.$$

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- Arnoux-Rauzy words. Consider the alphabet $\mathcal{A} = \{1, 2, \cdots, d\}$, $x \in \mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$ is an Arnoux-Rauzy word if every factor appears infinitely often; for all n,

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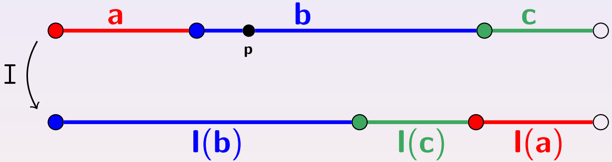
  and there exists exactly one left special factor ($L(w) \geq 2$) and one rigth special factor ($R(v) \geq 2$) of each given length.
- Sturmian words are Arnoux-Rauzy words for $d = 2$.

Dendric words: examples

- Codings of regular interval exchanges.



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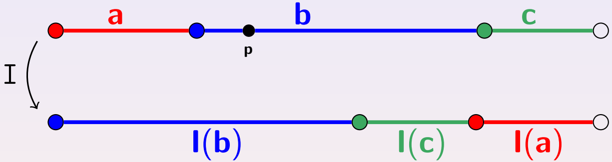

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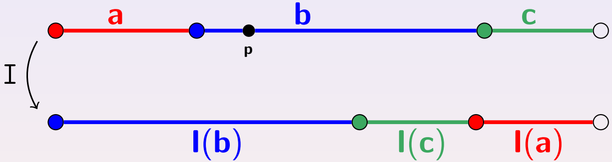

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$$x_p = \cdots abbacb \cdot bacbbacbba \cdots$$

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- For all $w \in \mathcal{L}_X$, define the cylinder w as

$$[w] = \{x \in X : x_0 \cdots x_{|w|-1} = w\}.$$

The set of all cylinders is a basis of the topology of X.

Dendric subshifts

- (X, T) is minimal if it admits no non-trivial closed and shift-invariant subset: every infinite word $x \in X$ is uniformly recurrent: every word occurring in x occurs infinitely often with bounded gaps, every $x \in X$ has the same language.

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- We focus on minimal dendric subshifts. This is the case when there is a dendric word $x = (x_n)_{n\in\mathbb{Z}}$ such that the subshift $X \subseteq \mathcal{A}^{\mathbb{Z}}$ can be obtained as

$$X = \overline{\{T^k((x_n)_{n\in\mathbb{Z}}) : k \in \mathbb{Z}\}}.$$

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- Equivalently, $X = \{y \in \mathcal{A}^{\mathbb{Z}} : \mathcal{L}(y) \subseteq \mathcal{L}(x)\}$.

Exploting properties of extension graphs.

Extension graph

Lemma ($\star$)

Let $\mathcal{T}$ be a finite tree, with a bipartition X and Y of its set of vertices, with $|X|, |Y| \geq 2$. Let E be its set of edges. For all $x \in X$, $y \in Y$, define

$$Y_x := \{y \in Y : (x, y) \in E\} \qquad X_y := \{x \in X : (x, y) \in E\}.$$

Let $(G, +)$ be an abelian group and H a subgroup of G. Suppose that there exists a function $g : X \cup Y \cup E \to G$ satisfying the following conditions:

(1) $g(X \cup Y) \subseteq H$;

(2) *for all $x \in X$, $g(x) = \sum_{y \in Y_x} g(x, y)$, and for all $y \in Y$, $g(y) = \sum_{x \in X_y} g(x, y)$.*

Then, for all $(x, y) \in E$, $g(x, y) \in H$.

Extension graph

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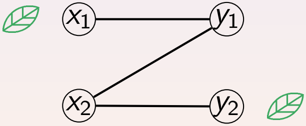
- Let $k := \max\{|X|, |Y|\}$. If $k = 2$, there is only one possibility for $\mathcal{T}$ (modulo relabeling the vertices), since $\mathcal{T}$ is connected and has no cycles, which is



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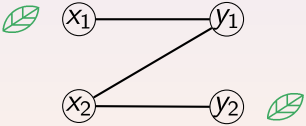
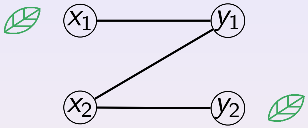


- Both $g(x_1, y_1)$ and $g(x_2, y_2)$ are in H because x_1 and y_2 are leaves.

Extension graph



- By Condition (2), one has $g(x_2) = g(x_2, y_1) + g(x_2, y_2)$, and then $g(x_2, y_1) = g(x_2) - g(x_2, y_2)$.

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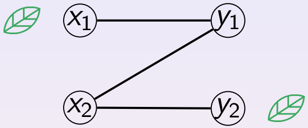

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- Since $g(x_2) \in H$ by Condition (1) and H is a group, then $g(x_2, y_1) \in H$.

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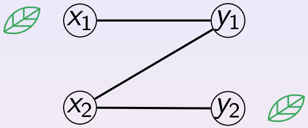

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- Since $g(x_2) \in H$ by Condition (1) and H is a group, then $g(x_2, y_1) \in H$.
- We proceed by induction for $k > 2$. □

An application: Balance.

Balance

- A word $x \in \mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$ is balanced on the factor $v \in \mathcal{L}(x)$ if there exists a constant C_v such that for every pair of factors u, w in $\mathcal{L}(x)$ with $|u| = |w|$,

$$||u|_v - |w|_v| \leq C_v.$$

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- If $x \in X$ and (X, T) is minimal, balance is a property of the language $\mathcal{L}_X$.
- Sturmian words are exactly the 1-balanced words on the letters.
- They are moreover C-balanced on factors of any length.

Frequencies

- The frequency of a factor $v \in \mathcal{L}(x)$ in $x \in \mathcal{A}^{\mathbb{Z}}$ is defined as the following limit (if it exists),

$$\lim_{n\to\infty} \frac{|x_{-n}\cdots x_n|_v}{2n+1}.$$

Proposition

The language $\mathcal{L}_X$ is balanced in the factor v if and only if v has a frequency μ_v and there exists a constant B_v such that for any factor $w \in \mathcal{L}_X$, we have $||w|_v - \mu_v|w|| \leq B_v$.

- Equivalently, v has a frequency μ_v and there exists B_v such that for all $x \in X$ and for all $n \geq 1$,

$$||x_{[0,n)}|_v - \mu_v n| \leq B_v.$$

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- Balance in factors of length $(n+1)$ implies balance in factors of length n.
- But balance behaviour can be different in factors of different lengths.
- Example: the language of the Thue-Morse word is balanced on letters and it is not balanced on the factors of length ℓ for every $\ell \geq 2$.

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Lemma

Let (X, T) be a minimal dendric subshift. Let H be the following subset of $C(X, \mathbb{Z})$:

$$H = \left\{ \sum_{a \in \mathcal{A}} \sum_{k \in K_a} \alpha(a,k) \chi_{T^k([a])} : K_a \subseteq \mathbb{Z}, |K_a| < \infty, \alpha(a,k) \in \mathbb{Z} \right\},$$

where χ_A denotes the characteristic function of the set A, for all $A \subseteq X$. Then, for all $v \in \mathcal{L}_X$, $\chi_{[v]}$ belongs to H.

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- Given any factor $v \in \mathcal{L}_X$, we want to find, for all $a \in \mathcal{A}$ a finite set $K_a \subseteq \mathbb{Z}$ and for all $k \in K_a$ an integer $\alpha(a, k)$ such that

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- Let $v = v_0 \cdots v_n$,

$$\widetilde{v} = v_1 \cdots v_n,$$

$$v' = v_0 \cdots v_{n-1},$$

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$$\begin{aligned}
\chi_{[v]}(x) &= \sum_{a \in \mathcal{A}} \sum_{k \in K_a} \alpha(a,k) \chi_{T^k([a])}(Tx) \\
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Defining $K'_a := \{k-1 : k \in K_a\}$ for all $a \in \mathcal{A}$, and $\beta(a,k) = \alpha(a,k+1)$ for all $k \in K'_a$, we conclude

$$\chi_{[v]}(x) = \sum_{a \in \mathcal{A}} \sum_{k \in K'_a} \beta(a,k) \chi_{T^k([a])},$$

and then $\chi_{[v]}$ belongs to H.

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- Note that H is a subgroup of $C(X, \mathbb{Z})$.

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- For the second condition, let $a \in L(\bar{v})$. One has

$$\chi_{[a\bar{v}]} = \sum_{b \in R(\bar{v}), (a,b) \in E(\bar{v})} \chi_{[a\bar{v}b]}(x)$$

  and thus

$$g(a) = \sum_{b \in R(\bar{v}), (a,b) \in E(\bar{v})} g(a, b).$$

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- Similarly, for $b \in R(\bar{v})$ and $x \in X$, one has

$$\chi_{T^{-1}[\bar{v}b]}(x) = \chi_{[\bar{v}b]}(Tx) = \sum_{a \in L(\bar{v}), (a,b) \in E(\bar{v})} \chi_{[a\bar{v}b]}(x).$$

We conclude that for all $b \in R(\bar{v})$,

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- Thanks to Lemma ($\star$), every $g(a,b)$ belongs to H. In particular, $g(v_0, v_n) = \chi_{[v]} \in H$.

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- Suppose (X, T) is balance on every letter. Let C be a constant of balance on the letters.
- Let $v \in \mathcal{L}_X$. Let $n \geq 3$ and let u, w be two factors of $\mathcal{L}_X$ of length $n-1$ with $n-1 > |v|$. Pick a bi-infinite word $x \in X$ such that $u = x_{[i,i+n)}$ and $w = x_{[j,j+n)}$ for some indices $i, j \in \mathbb{Z}$. We have

$$||u|_v - |w|_v| = \left| \sum_{\ell=i}^{i+n-1-|v|} \chi_{[v]}(T^\ell x) - \sum_{\ell=j}^{j+n-1-|v|} \chi_{[v]}(T^\ell x) \right|.$$

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Then, $||u|_v - |w|_v| \leq |\mathcal{A}|KC$, $K = \max_{a\in\mathcal{A}} \left\{ \sum_{k\in K_a} |\alpha(a,k)| \right\}$.

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- Let $\mathcal{A} = \{1, \cdots, d\}$. The set of Elementary Arnoux-Rauzy substitutions defined on $\mathcal{A}$ is $\{\sigma_i : i \in \mathcal{A}\}$ given by

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- Another sufficient conditions exist to guarrantee balance on letters for Arnoux-Rauzy words.

Another application: Invariant measures and Orbit equivalence.

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- A probability measure μ on the compact metric space X is T-invariant if for all Borel subset B of X,

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- Every dynamical system has invariant measures (Krylov-Bogolyubov's theorem).
- (X, T) is said to be uniquely ergodic if there exists only one invariant measure.
- For minimal systems, unique ergodicity is equivalent to having frequencies.

Invariant measures

- Let $\mathcal{M}(X, T)$ the set of $T-$invariant measures on X. The Image subgroup of (X, T) is the following subgroup of $\mathbb{R}$.

$$I(X, T) = \bigcap_{\mu \in \mathcal{M}(X,T)} \left\{ \int f d\mu : f \in C(X, \mathbb{Z}) \right\}.$$

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- The triple $(I(X,T), I(X,T)\cap\mathbb{R}^+, 1)$ is an order group with unit.
- $(I(X,T), I(X,T)\cap\mathbb{R}^+, 1)$ is total invariant for Orbit equivalence [Giordano-Putnam-Skau95].
- (X,T) and (Y,S) are orbit equivalent if there is a homeomorphism $h : X \to Y$ such that for all $x \in X$

$$h(\{T^n(x) : n \in \mathbb{Z}\}) = \{S^n(h(x)) : n \in \mathbb{Z}\},$$

that is, h sends orbits onto orbits.

Invariant measures

Theorem

Let (X, T) a uniquely ergodic dendric subshift over the alphabet $\mathcal{A}$ and μ its unique invariant measure. Then, the image subgroup of (X, T) is

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All dendric subshifts over a three-letter alphabet with the same letter frequencies are orbit equivalent.