O(N) Random Tensor Models

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Based on arXiv:1512.06718 [math-ph], in collaboration with Adrian Tanasa.

Motivations and purpose of this talk

Random Tensor Models have been revived in recent years, as a programme towards the definition of **random geometries** and/or **quantum gravity** theories in dimension $d \geq 3$.
[Gurau '09 '10...]

Key insights from combinatorics have unlocked many new developments in increasingly complicated settings : i.i.d. random tensor models → tensorial field theories → group field theories... [Gurau, Rivasseau, Bonzom, Tanasa, Lionni, Benedetti...]

Objectives:

- Introduce a new class of i.i.d. random tensor models, based on a $\mathrm{O}(N)$ **invariant**, which generalize $\mathrm{U}(N)$ **invariant** and **multi-orientable** ones [Tanasa...].
- Give an illustration of the role of **combinatorics** in this simple setting:
  - perturbative expansion indexed by **colored graphs**, and organized according to a combinatorial quantity called degree;
  - critical properties of the models from **analytic combinatorics**.

O(N) invariant tensor models

- **Random variable**: real tensor $T_{i_1 i_2 i_3}$, $1 \leq i_k \leq N \in \mathbb{N}^*$

- **Probability measure** defined by

$$\mathrm{d}\mu_N(T) = \frac{1}{\mathcal{Z}_N} \exp\left(-N^{3/2} S_N(T)\right) \mathrm{d}T \,,$$

  where
  - $\mathrm{d}T$ is the Lebesgue measure on N^3;
  - the action S_N is polynomial in T;
  - $\mathcal{Z}_N$ is a normalization factor, known as the **partition function**.

- We require the action to be invariant under $\mathrm{O}(N)^{\otimes 3}$:

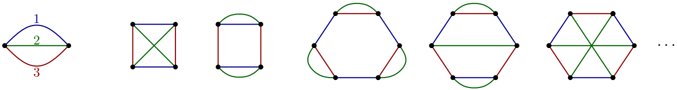
$$T_{i_1 i_2 i_3} \to \sum_{j_1, j_2, j_3} O^{(1)}_{i_1 j_1} O^{(2)}_{i_2 j_2} O^{(3)}_{i_3 j_3} \, T_{j_1 j_2 j_3}$$

  $\Rightarrow S_N(T)$ is a sum of products of **trace invariants**, which are indexed by (connected) **colored graphs**.

Colored graphs

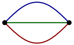
Definition: A ***k*-colored graph** is a k-regular edge-colored graph. The color of an edge is a label $\ell \in \{1, \ldots, k\}$

Examples. We will be interested in 3-colored graphs, also called **bubbles**

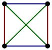


Remark. Multiple edges, as well as non-bipartite diagrams are allowed.

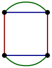
Definition. A bicolored cycle is called a **face**. We call F_b the number of faces of the bubble b. The notion of face allows to interpret bubbles as representing **2d manifolds**, but possibly non-orientable ones.



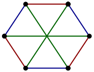
$F_b = 3$
sphere



$F_b = 3$
projective plane



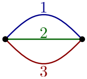
$F_b = 4$
sphere



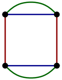
$F_b = 3$
torus

O(N) trace invariants

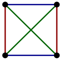
- **Trace invariants** are labelled by the bubbles:



$$\mathrm{Tr}_b(T) = \sum_{i_1, i_2, i_3} T_{i_1 i_2 i_3} T_{i_1 i_2 i_3}$$



$$\mathrm{Tr}_b(T) = \sum_{i_1, \dots, i_6} T_{i_6 i_2 i_3} T_{i_1 i_2 i_3} \times T_{i_6 i_4 i_5} T_{i_1 i_4 i_5}$$



$$\mathrm{Tr}_b(T) = \sum_{i_1, \dots, i_6} T_{i_6 i_2 i_3} T_{i_1 i_4 i_3} \times T_{i_6 i_4 i_5} T_{i_1 i_2 i_5}$$

- The action is in general a sum of not necessarily connected invariants, but we assume connectedness.

$$S_N(T) = \frac{1}{2} \mathrm{Tr}_{\ominus}(T) + \sum_{b \in \mathcal{B}} t_b \, N^{-\rho(b)} \, \mathrm{Tr}_b(T) \,,$$

where $\mathcal{B}$ is a **finite set of connected bubbles** with number of nodes $N_b > 2$.

Perturbative expansion

One can perform a formal expansion of the full measure in terms of the coupling constants t_b.

- Decompose the measure into a Gaussian part plus perturbations:

$$\begin{aligned}\mathcal{Z}_N &= \int \mathrm{d}T \exp\left(-\frac{N^{3/2}}{2}\mathrm{Tr}_{\ominus}(T)\right) \exp\left(-\sum_{b\in\mathcal{B}} t_b\, N^{3/2-\rho(b)}\,\mathrm{Tr}_b(T)\right) \\ &= \sum_{\{n_b\}} \int \mathrm{d}T \exp\left(-\frac{N^{3/2}}{2}\mathrm{Tr}_{\ominus}(T)\right) \prod_{b\in\mathcal{B}} \frac{\left(-t_b N^{3/2-\rho(b)}\right)^{n_b}}{n_b!} \left(\mathrm{Tr}_b(T)\right)^{n_b}\end{aligned}$$

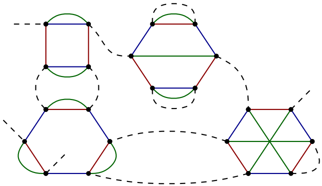
- Use **Wick's theorem** which allows to compute the moment of the Gaussian measure $\Rightarrow$ sum over **Feynman diagrams** $\mathcal{G}$:

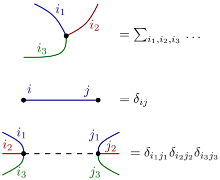
$$\mathcal{Z}_N = \sum_{\mathcal{G}} \left(\prod_{b\in\mathcal{B}} (-t_b)^{n_b(\mathcal{G})}\right) \mathcal{A}_{\mathcal{G}} \,,$$

  where

  - Feynman diagrams are **4-colored graphs**;
  - up to a combinatorial factor (that we ignore for now), the **amplitude** $\mathcal{A}_{\mathcal{G}}$ is contraction of tensor indices following the pattern of $\mathcal{G}$.

- The Feynman diagrams are 4-colored graphs, and are weighted by **amplitudes** $\mathcal{A}_{\mathcal{G}}$





- In a Feynman diagram, a **face of color ℓ** is a *cycle formed by dashed lines and color-ℓ edges*. Each face contributes with a sum of the form:

$$\sum_{i_1,\ldots,i_k} \delta_{i_1,i_2}\delta_{i_2 i_3}\ldots\delta_{i_{k-1},i_k}\delta_{i_k,i_1} = \sum_{i_1=1}^{N} \delta_{i_1,i_1} = N$$

- We have moreover: a factor $N^{3/2-\rho(b)}$ per bubble of type b; and a factor $N^{3/2}$ per dashed line.

$$\Rightarrow \mathcal{A}_{\mathcal{G}} \propto N^{3-\omega(\mathcal{G})} \qquad \text{with} \qquad \omega = 3 + \frac{3}{2}L - \sum_b \left(\frac{3}{2} - \rho\right) n_b - F\,.$$

Purpose

We know that the Feynman diagrams are weighted by $\mathcal{A}_{\mathcal{G}} \propto N^{3-\omega(\mathcal{G})}$, where

$$\omega = 3 + \frac{3}{2}L - \sum_b \left(\frac{3}{2} - \rho\right) n_b - F$$

is called the **degree**.

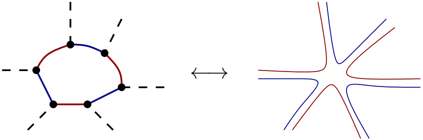
We now look for a **definition of ρ** such that:

- ω is bounded from below;
- the family of leading order diagrams (in N) is infinite.

To this effect, one needs to **count the number of faces** in a given Feynman diagram.

Definition. Given a graph $\mathcal{G}$, its **jacket of color ℓ** is obtained by deleting all edges of color ℓ.

Jackets define **ribbon graphs** and therefore carry a canonical notion of (non-orientable) **genus** $k \in \mathbb{N}$.



$$2F(\mathcal{G}) = \sum_{\ell=1}^{3} \sum_{\substack{\text{connected} \\ \text{components } i}} f(J_\ell^{(i)}) \underset{\text{(Euler)}}{=} \sum_{\ell=1}^{3} \sum_{\substack{\text{connected} \\ \text{components } i}} (2 + e - v - k)\,(J_\ell^{(i)})$$

Degree and genera of jackets

- The degree ω can be reexpressed in term of the genera of the jackets and other combinatorial quantities:

$$\omega(\mathcal{G}) = \frac{1}{2} \sum_{\ell\,;\,i} k(J_\ell^{(i)}) + \sum_{b \in \mathcal{B}} n_b \left(\rho(b) + \frac{F_b - 3}{2} \right) - \sum_{\ell} (|J_\ell| - 1)$$

- One can furthermore prove the inequality

$$\sum_{b \in \mathcal{B}} n_b \, (F_b - 3) \geq \sum_{\ell} (|J_\ell| - 1) \,,$$

which is furthermore saturated. Hence we define

$$\rho(b) := \frac{F_b - 3}{2} \,.$$

Proposition: The degree ω may be expressed as

$$\omega = \underbrace{\frac{1}{2} \sum_{\ell\,;\,i} k(J_\ell^{(i)})}_{\in \mathbb{N}/2} + \underbrace{\sum_{b \in \mathcal{B}} n_b \, (F_b - 3) - \sum_{\ell} (|J_\ell| - 1)}_{\in \mathbb{N}} \in \frac{\mathbb{N}}{2}$$

Large N expansion

- According to the previous discussion, the partition function can be organized in powers of N:

$$\mathcal{Z}_N = \sum_{\omega \in \frac{\mathbb{N}}{2}} N^{3-\omega} \, Z_\omega(t_b)\,.$$

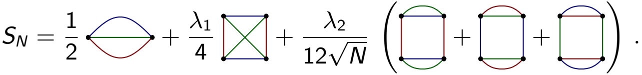
- **Leading order** ($\omega = 0$) characterized by:

$$\begin{cases} \text{All jackets are planar } (k = 0)\,, \\ \sum_{b \in \mathcal{B}} n_b \, (F_b - 3) = \sum_{\ell} (|J_\ell| - 1) \;. \end{cases}$$

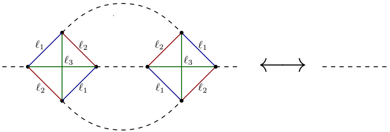
- **Next-to-leading order** ($\omega = 1/2$) characterized by:

$$\begin{cases} \exists! \; (\ell, \, i) \text{ s.t.} \quad k(J_\ell^{(i)}) = 1/2\,, \\ \text{All other jackets are planar } (k = 0)\,, \\ \sum_{b \in \mathcal{B}} n_b \, (F_b - 3) = \sum_{\ell} (|J_\ell| - 1) \;. \end{cases}$$

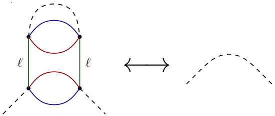
- The most general quartic action also invariant under color permutations is of the form



- The so-called melonic moves can be shown to conserve the degree:

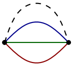


(a) Melonic move of type I.



(b) Melonic move of type II.

- Hence one can generate an infinite family of leading order graphs – the **melonic graphs** –, by melonic insertions into a degree-0 graph such as

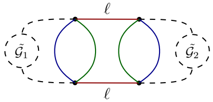


Question: Are there other degree 0 graphs apart from melonic ones ?

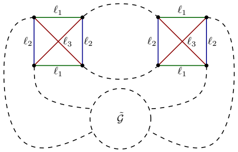
Proposition: If $\omega(\mathcal{G}) = 0$ then $\mathcal{G}$ is melonic.

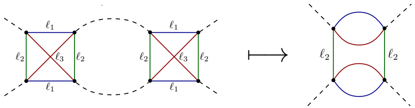
Idea of proof. Induction on $p = \lfloor n_{\boxtimes}/2 + n_{\square} \rfloor$. Works because of the following facts:

1. If $n_{\square}(\mathcal{G}) \neq 0$ then $\mathcal{G} =$



2. If $n_{\square}(\mathcal{G}) = 0$ then $\mathcal{G} =$



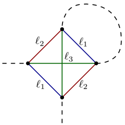
3. 

conserves the degree.

More details about second observation: one needs to show that $F_2 \geq 1$ (where F_k is the number of faces of length k).

- First:

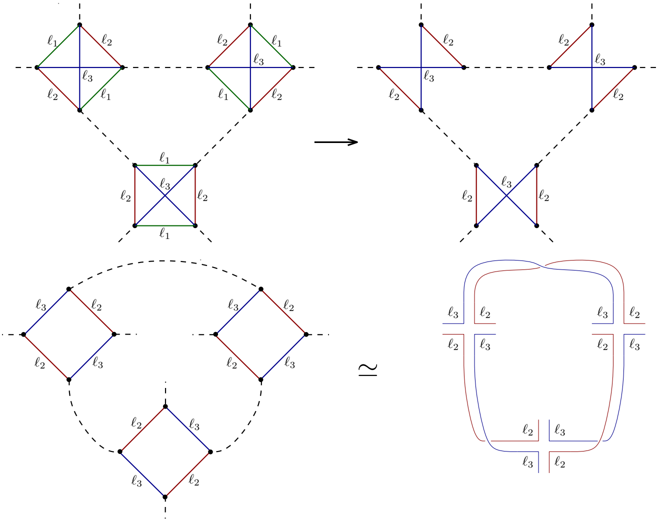
$$\begin{cases} \sum_{p \geq 1} F_p = F \underset{\omega=0}{=} 3 + \frac{3}{4} L \\ \sum_{p \geq 1} p\, F_p = 3L \end{cases} \Rightarrow \sum_p (4-p) F_p = 12 \quad \Rightarrow \quad F_1 + 2F_2 + 3F_3 > 0$$

- Second: $F_1 \geq 1 \quad \Rightarrow$  $\Rightarrow \quad \omega \geq 1/2$; hence $F_1 = 0$.

- Third: $F_3 \geq 1 \quad \Rightarrow \quad \exists$ non-orientable jacket $\quad \Rightarrow \quad \omega \geq 1/2$; hence $F_3 = 0$.

Characterization of degree 0 graphs

Graphical proof of existence of **non-orientable jacket** when $F_3 \geq 1$:



- Physically, one is interested in correlation functions, such as the **2-point function**:

$$\frac{N^{3/2}}{\mathcal{Z}_N}\int [\mathrm{d}T]\, T_{i_1 i_2 i_3}\, T_{j_1 j_2 j_3}\, \exp\left(-S_N(T)\right) = \left(G_{LO} + N^{-1/2}\, G_{NLO} + \dots\right) \prod_{\ell=1}^{3} \delta_{i_\ell j_\ell}\,.$$

- G_{LO} can be evaluated as a sum over 2-point melonic graphs. More precisely, defining

$$g := {\lambda_1}^2\,, \qquad \mu := \frac{-\lambda_2}{\lambda_1}$$

one obtains

$$G_{LO}(g,\mu) = \sum_{p,q\in\mathbb{N}} C_{p,q}\, g^{p+q}\, \mu^q$$

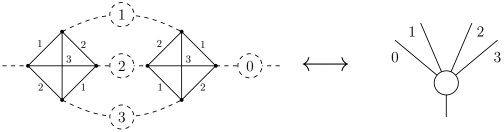
where $C_{p,q}$ is the number of 2-point melonic graphs with p melons of type I and q melons of type II.

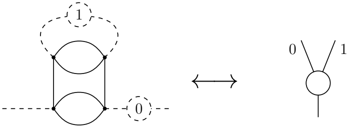
- g parametrizes the total number of melons. From a physics point of view, one is therefore interested in the behaviours of G_{LO} on the boundary of its domain of convergence:

$$|g| \to g_c(\mu) > 0\,.$$

Explicit evaluation of $C_{p,q}$

- Melonic 2-point graphs can be mapped to **rooted binary–quarternary plane trees**:





- By **Cayley's theorem**, this provides the explicit evaluation:

$$C_{p,q} = \frac{[4p + 2q]!}{p!q!(3p + q + 1)!}$$

- This is nice but not very helpful for our purpose, since:

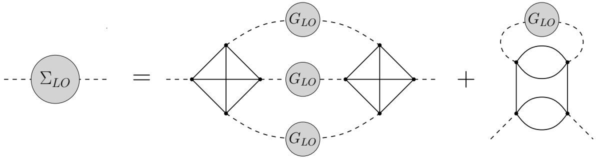
$$G_{LO}(g, \mu) = \sum_{n \in \mathbb{N}} \left(\sum_{q=0}^{n} \mu^q \frac{(4n - 2q)!}{q!(n - q)!(3n - 2q + 1)!} \right) g^n$$

Algebraic equation for G_{LO}

- In view of the tree structure of melonic graphs, their generating function verifies:

$$G_{LO} = 1 + g\, {G_{LO}}^2 \left({G_{LO}}^2 + \mu \right) .$$

- Graphical derivation:



where $G_{L0} = \frac{1}{1-\Sigma_0}$ and Σ_{LO} is the **one particle irreducible** 2-point function. Hence

$$\Sigma_{LO} = {\lambda_1}^2\, {G_{LO}}^3 - \lambda_2\, G_{LO} = g\, {G_{LO}}^3 + g\, \mu\, G_{LO}$$

- To deduce the critical behaviour of G_{LO}, we rely instead on the structure of its **analytic singularities**.

Critical behaviour from singularity analysis

For any $\mu \geq 0$, define the quantity

$$g_c(\mu) = \frac{G_c(\mu) - 1}{G_c(\mu)^2 \left(G_c(\mu)^2 + \mu \right)} ,$$

where $G_c(\mu)$ is the unique real solution of the polynomial equation

$$-3\,x^3 + 4\,x^2 - \mu\,x + 2\mu = 0 \,.$$

Proposition: G_{LO} has radius of convergence $g_c(|\mu|)$. Moreover, for any $\mu \geq 0$, there exists a constant $K(\mu) > 0$ such that:

$$G_{LO}(g, \mu) \underset{g \to g_c(\mu)^-}{=} G_c(\mu) - K(\mu) \sqrt{1 - \frac{g}{g_c(\mu)}} \left(1 + \mathcal{O}(1 - \frac{g}{g_c(\mu)}) \right) \,.$$

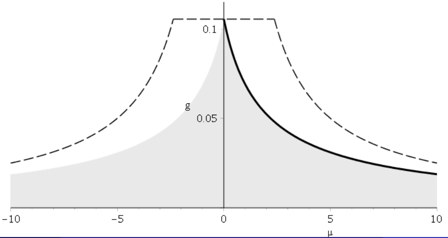
Idea of proof.

- $\mu \geq 0 \Rightarrow$ singularity at $g =$ radius of convergence (Pringsheim's theorem).
- Look for points where $g = \frac{G_{LO}-1}{{G_{LO}}^2(G_{LO}^2+\mu)} =: \Psi(G_{LO}-1)$ fails to be locally invertible, that is $\Psi'(G_{LO}-1) = 0$. This defines $G_c(\mu)$ and $g_c(\mu)$.
- Check that $\Psi''(G_c(\mu)-1) \neq 0$ and that therefore

$$g_c(\mu) - g \underset{g \sim g_c(\mu)}{\approx} -\frac{\Psi''(G_c(\mu)-1)}{2} \left(G_{LO}(g,\mu) - G_c(\mu)\right)^2 .$$

□

Moreover, one can check that their are no other real singularities:



Critical exponents and asymptotic estimates

- In physics, the critical behaviour itself is the main objective. It allows in particular to compute **critical exponents** e.g. for the free energy

$$F_N := \frac{1}{N^3} \ln \mathcal{Z}_N = F_{LO} + N^{-1/2} F_{NLO} + \dots$$
$$\underset{g \sim g_c(\mu)}{\approx} K_1(\mu) \left(1 - \frac{g}{g_c(\mu)}\right)^{3/2} + K_2(\mu) \left(1 - \frac{g}{g_c(\mu)}\right)^{1/2} + \dots$$

- From a combinatorial perspective, one may go one step further and deduce an estimation of the coefficient $\alpha_n(\mu)$ of G_{LO} in the large n limit [Flajolet, Sedgewick]

$$\alpha_n(\mu) \underset{n \to +\infty}{\sim} \frac{K(\mu)\, g_c(\mu)^{-n}}{2\sqrt{\pi}\, n^{3/2}} \,.$$

Application. Taking $\mu = 3$, one finds an estimate of the **number $\mathcal{M}_n$ of melonic 2-point graphs with n elementary melons**:

$$\mathcal{M}_n \underset{n \to +\infty}{\sim} \frac{\chi \, \beta^n}{n^{3/2}} \,,$$

with

$$\chi \approx 0.111 \qquad \text{and} \qquad \beta \approx 14.8 \,.$$

Conclusion

We have initiated the study of $\mathrm{O}(N)$ rather than $\mathrm{U}(N)$ invariant tensor models.

- Existence of a **large N expansion** for arbitrary number of interactions, labelled by not necessarily bipartite bubbles.
- Characterization of **leading and next-to-leading order graphs** $\Rightarrow$ colored graphs with **tree-like** structure as for $\mathrm{U}(N)$ invariant models.
- Hence, not surprisingly, one obtains the same type of **square-root critical behaviour** as for $\mathrm{U}(N)$ invariant models.

Outlook

- $\mathrm{O}(N)$ models with tensors of **higher rank**.
- Application to the renormalization of **multi-orientable tensorial field theories** (initial motivation for this work).

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- $\mathrm{O}(N)$ models with tensors of **higher rank**.
- Application to the renormalization of **multi-orientable tensorial field theories** (initial motivation for this work).
- Application of same combinatorial and analytic methods to **more involved tensorial theories**, which we are so far unable to compute the critical exponents of e.g. so-called **Boulatov model** (which is related to 3d quantum gravity). Main difficulty: the amplitudes depend on more involved combinatorial quantities than the mere number of faces.
- Can we **get out of the tree–like regime in tensorial theories** ? and therefore define more interesting random spaces ?

Thank you for your attention