Height representation of XOR-Ising loops via bipartite dimers

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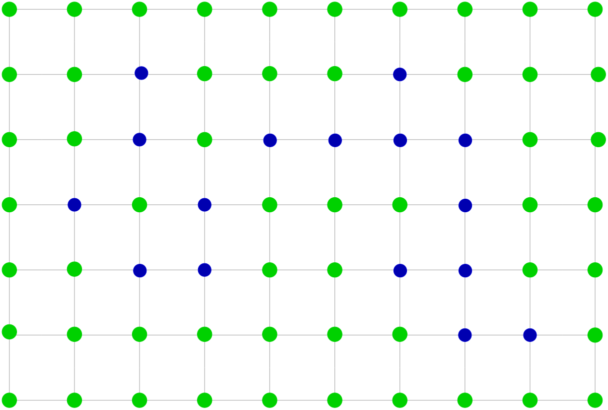
The Ising model and the XOR-Ising model

The Ising model

- Let $G = (V, E)$ be a finite graph embedded in the plane
- **spin configuration** $\sigma : V \longrightarrow \{-1, +1\}$
- σ assigns to every vertex x a spin $\sigma_x \in \{-, +\}$



$+1/-1$ are represented by green/blue dots.

The Ising model

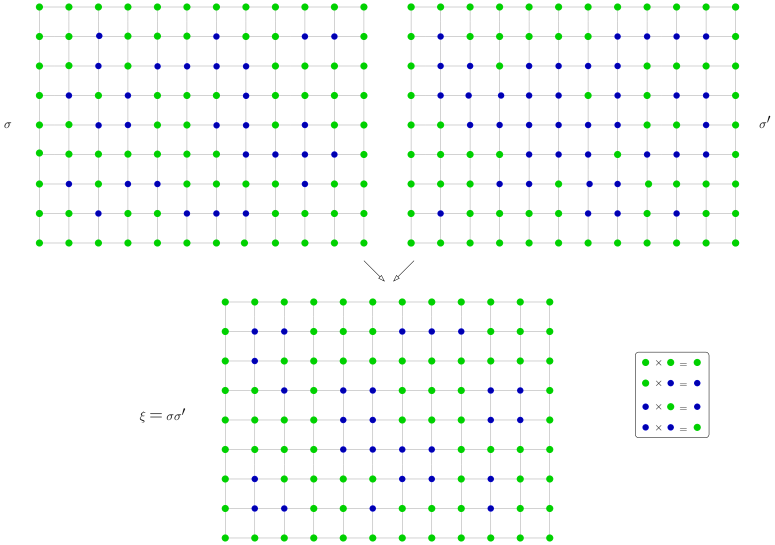
- Edges of G are assigned positive **coupling constants**: $J = (J_e)_{e \in \mathsf{E}}$.
- **Ising Boltzmann measure**:

$$\forall \sigma \in \{-1,1\}^{\mathsf{V}}, \quad \mathbb{P}_{\text{Ising}}(\sigma) = \frac{1}{Z_{\text{Ising}}(\mathsf{G}, J)} \exp\left(\sum_{e=xy \in \mathsf{E}} J_{xy} \sigma_x \sigma_y \right),$$

where $Z_{\text{Ising}}(\mathsf{G}, J) = \sum_{\sigma \in \{-1,1\}^{\mathsf{V}}} \exp\left(\sum_{e=xy \in \mathsf{E}} J_{xy} \sigma_x \sigma_y \right)$ is the **Ising partition function**.

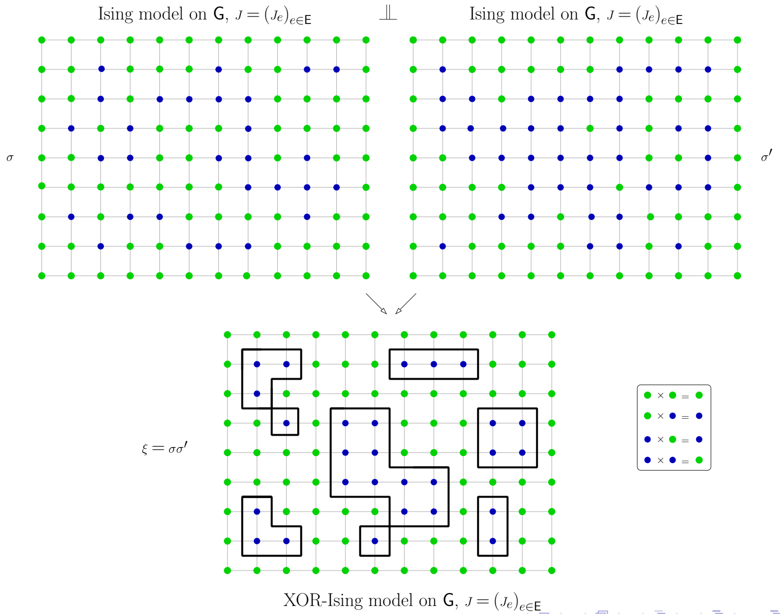
The XOR-Ising model

Ising model on G, $J = (J_e)_{e \in \mathsf{E}}$ $\perp\!\!\!\perp$ Ising model on G, $J = (J_e)_{e \in \mathsf{E}}$



XOR-Ising model on G, $J = (J_e)_{e \in \mathsf{E}}$

The XOR-Ising model



Conjecture for the XOR-Ising model

Conjecture (Wilson (11), Ikhlef–Picco–Santachiara)

The scaling limit of polygon configurations separating ± 1 *clusters of the critical XOR-Ising model are contour lines of the Gaussian free field, with the heights of the contours spaced* $\sqrt{2}$ *times as far apart as they are for [...] the double dimer model on the square lattice.*

Result

Theorem (B–dT)

- *Polygon configurations of the XOR-Ising model have the same law as a family of contours in a bipartite dimer model.*
- *This family of contours are the level lines of a restriction of the height function of this bipartite dimer model.*

Remark

Proved when the graph G *is embedded in a surface of genus* g, *or when* G *is planar, infinite.*

- When the XOR-Ising model is **critical**, so is the bipartite dimer model.
- Using results of [dT] on the convergence of the height function, this gives partial proof of Wilson's conjecture.

Contour expansion of the Ising partition function [Kramers & Wannier]

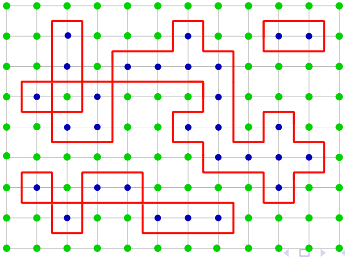
Low temperature expansion

- **Polygon configuration**: subset of edges s.t. each vertex is incident to an even number of edges.
- Write $$e^{J_e \sigma_x \sigma_y} = e^{J_e}(\delta_{\{\sigma_x = \sigma_y\}} + e^{-2J_e}\delta_{\{\sigma_x \neq \sigma_y\}}).$$

  The partition function is then equal to **(LTE)**:

$$Z_{\text{Ising}}(\mathsf{G}, J) = \sum_{\sigma \in \{-1,1\}^{\mathsf{V}}} \prod_{e=xy \in \mathsf{E}} e^{J_e \sigma_x \sigma_y} = \mathcal{C} \sum_{\mathsf{P}^* \in \mathcal{P}(\mathsf{G}^*)} \prod_{e^* \in \mathsf{P}^*} e^{-2J_e}.$$

- Geometric interp: polygon config. separate clusters of ± 1 spins.



High temperature expansion

- Write, $$e^{J_e \sigma_x \sigma_y} = \cosh(J_e)(1 + \sigma_x \sigma_y \tanh(J_e)).$$

  The partition function is then equal to **(HTE)**:

  $$Z_{\text{Ising}}(\mathsf{G}, J) = \sum_{\sigma \in \{-1,1\}^{\mathsf{V}}} \prod_{e = xy \in \mathsf{E}} e^{J_e \sigma_x \sigma_y} = \mathcal{C}' \sum_{\mathsf{P} \in \mathcal{P}(\mathsf{G})} \prod_{e \in \mathsf{P}} \tanh(J_e).$$

- No geometric interpretation using spin configurations.

Mixed contour expansion for the double Ising model

The double Ising model

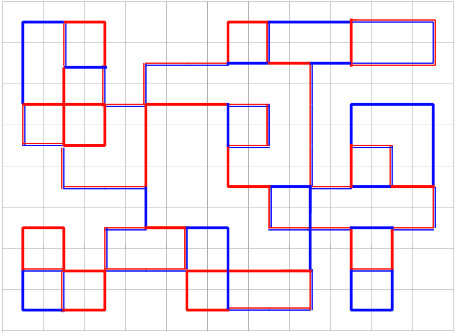
- Take 2 independent copies (red/blue) of an Ising model on G, with coupling constants J.
- Using the LTE, consider the probability measure $\mathbb{P}_{\text{2-Ising}}$: if P^*, P^* are two polygon configurations.

$$\mathbb{P}_{\text{2-Ising}}(\mathsf{P}^*, \mathsf{P}^*) = \frac{\mathcal{C}^2 \Big(\prod_{e^* \in \mathsf{P}^*} e^{-2J_e} \Big) \Big(\prod_{e^* \in \mathsf{P}^*} e^{-2J_e} \Big)}{Z_{\text{2-Ising}}(\mathsf{G}, J)},$$

where $Z_{\text{2-Ising}}(\mathsf{G}, J) = Z_{\text{Ising}}(\mathsf{G}, J)^2$.

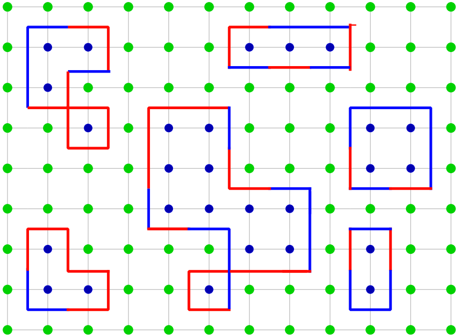
The double Ising model

- Let P^*, P^* be two polygon configurations.
- Consider the superimposition $P^* \cup P^*$.



- Define two new edge configurations:
  - $\mathrm{Mono}(P^*, P^*)$: monochromatic edges.
  - $\mathrm{Bi}(P^*, P^*)$: bichromatic edges.

Monochromatic edges



Monochromatic edge configuration of $\color{red}{\mathsf{P}^*} \cup \color{blue}{\mathsf{P}^*}$

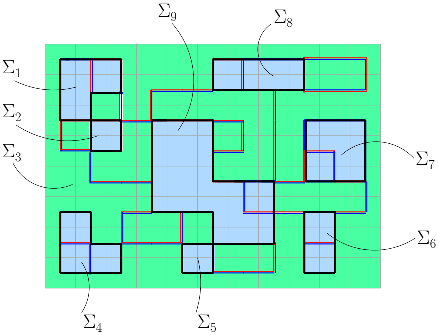
Lemma

$\mathrm{Mono}(\color{red}{\mathsf{P}^*}, \color{blue}{\mathsf{P}^*})$ *is the polygon configuration separating* ± 1 *clusters of the corresponding XOR-Ising spin configuration.*

Goal: understand the law of monochromatic edge configurations.

Bichromatic edge configurations

- Let $(\mathsf{P}^*, \mathsf{P}^*)$ be two polygon configurations.
- $\mathrm{Mono}(\mathsf{P}^*, \mathsf{P}^*)$ splits the surface into connected comp. $(\Sigma_i)_i$.



Lemma

For every i, the restriction of $\mathrm{Bi}(\mathsf{P}^*, \mathsf{P}^*)$ to Σ_i is the LTE of an Ising configuration on G_{Σ_i}, with coupling constants $(2J_e)$.

Probability of monochromatic configurations

Lemma

Let P^* *be a polygon configuration, separating the surface into* n *connected components. For every* i*, let* P_i^* *be a polygon configuration of* $\mathsf{G}^*_{\Sigma_i}$*.*

Then, there are 2^n *pairs of polygon configurations* $(\mathsf{P}^*, \mathsf{P}^*)$ *having* P^* *as monochromatic edges, and* $\mathsf{P}_1^*, \cdots, \mathsf{P}_n^*$ *as bichromatic edges.*

Denote by $W(\mathsf{P}^*)$ the contribution to $Z_{\text{2-Ising}}(\mathsf{G}, J)$ of the pairs of polygon configurations $(\mathsf{P}^*, \mathsf{P}^*)$ such that $\mathrm{Mono}(\mathsf{P}^*, \mathsf{P}^*) = \mathsf{P}^*$.

Corollary

- $W(\mathsf{P}^*) = \mathcal{C}\left(\prod_{e^* \in \mathsf{P}^*} e^{-2J_e}\right) \prod_{i=1}^n \left(2 Z_{\mathrm{LT}}(\mathsf{G}^*_{\Sigma_i}, 2J)\right)$
- $Z_{\text{2-Ising}}(\mathsf{G}, J) = \sum_{\mathsf{P}^* \in \mathcal{P}(\mathsf{G}^*)} W(\mathsf{P}^*)$

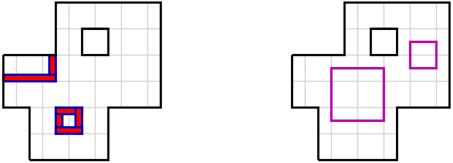
$$\boxed{\mathbb{P}_{\text{2-Ising}}(\mathrm{Mono} = \mathsf{P}^*) = \frac{W(\mathsf{P}^*)}{Z_{\text{2-Ising}}(\mathsf{G}, J)}.}$$

Mixed contour expansion

$$W(\mathsf{P}^*) = \mathcal{C}\left(\prod_{e^* \in \mathsf{P}^*} e^{-2J_e}\right) \prod_{i=1}^{n} \left(2 Z_{\mathrm{LT}}(\mathsf{G}^*_{\Sigma_i}, 2J)\right).$$

Idea [Nienhuis]: high temperature expansion in each connected component Σ_i.

$$Z_{\mathrm{LT}}(\mathsf{G}^*_{\Sigma_i}, 2J) = \mathcal{C}(\Sigma_i) Z_{\mathrm{HT}}(\mathsf{G}_{\Sigma_i}, 2J).$$



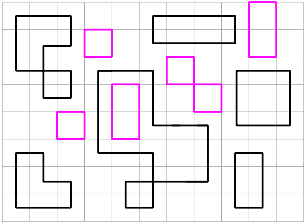
Low temp. expansion on $\mathsf{G}^*_{\Sigma_i}$ High temp. expansion on G_{Σ_i}.

Mixed contour expansion

Proposition

For every polygon configuration P^*,

$$W(\mathsf{P}^*) = \mathcal{C} \prod_{e^* \in \mathsf{P}^*} \left(\frac{2e^{-2J_e}}{1+e^{-4J_e}} \right) \sum_{\{\mathsf{P} \in \mathcal{P}(\mathsf{G}) :\, \mathsf{P}^* \cap \mathsf{P} = \emptyset\}} \prod_{e \in \mathsf{P}} \left(\frac{1-e^{-4J_e}}{1+e^{-4J_e}} \right)$$



$$\mathbb{P}_{\text{2-Ising}}(\text{Mono} = \mathsf{P}^*) = \frac{\prod_{e^* \in \mathsf{P}^*} \left(\frac{2e^{-2J_e}}{1+e^{-4J_e}} \right) \sum_{\{\mathsf{P} \in \mathcal{P}(\mathsf{G}) :\, \mathsf{P}^* \cap \mathsf{P} = \emptyset\}} \prod_{e \in \mathsf{P}} \left(\frac{1-e^{-4J_e}}{1+e^{-4J_e}} \right)}{\sum_{\mathsf{P}^* \in \mathcal{P}(\mathsf{G}^*)} \cdots}$$

Higher genus

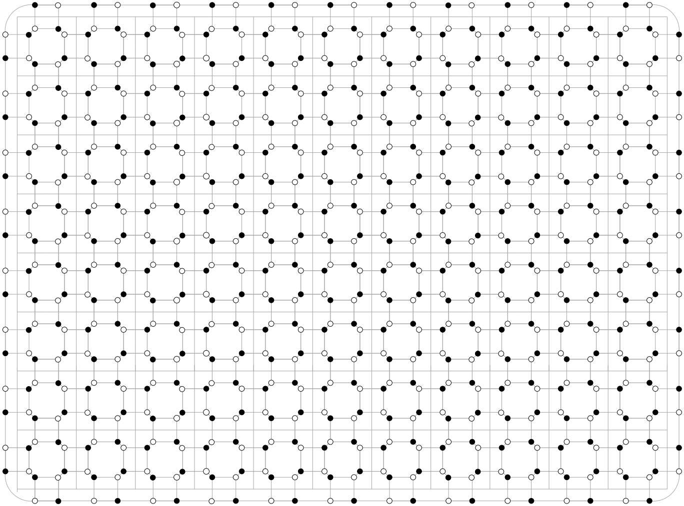
If the graph is embedded in a surface Σ of genus $g \geq 0$.

- Consider $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \simeq \{0,1\}^{2g}$.
- Family of Ising models, indexed by $\varepsilon \in \{0,1\}^{2g}$.
- The double Ising model partition function is defined as:

$$Z_{\text{2-Ising}}(\mathsf{G}, J) = \sum_{\varepsilon \in \{0,1\}^{2g}} Z^{\varepsilon}_{\text{Ising}}(\mathsf{G}, J)^2.$$

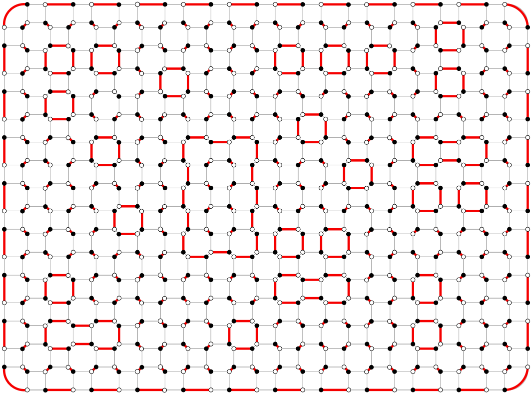
From mixed polygon configurations to dimers

The graph $G^Q = (V^Q, E^Q)$



The dimer model on G^{Q}

dimer configuration of G^{Q}: a subset of edges M such that each vertex is incident to exactly on edge of M



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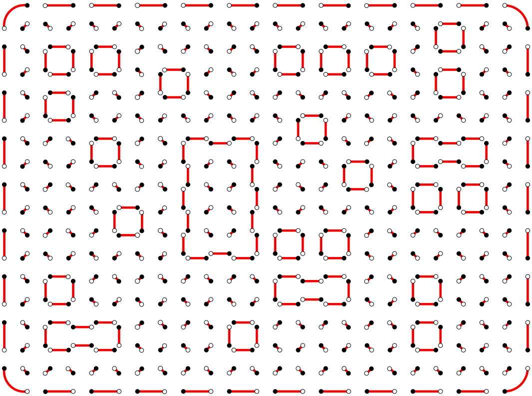

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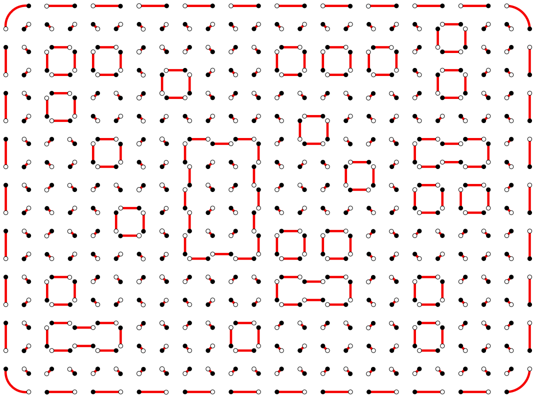
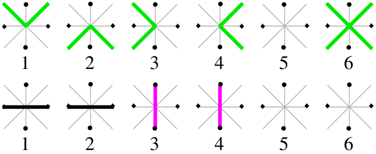


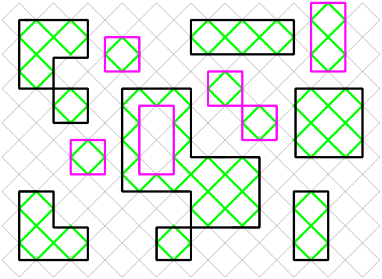
weight function ν on the edges

Dimer Boltzmann measure: $\mathbb{P}_{\text{dimer}}(\mathsf{M}) \propto \prod_{e \in \mathsf{E}^Q} \nu_e$

First step: from polygons to 6-vertex [Nienhuis]

Local mapping





Weights: $\omega_{12} = \frac{2e^{-2J_e}}{1+e^{-4J_e}}$, $\omega_{34} = \frac{1-e^{-4J_e}}{1+e^{-4J_e}}$, $\omega_{56} = 1$.

First step: from polygons to 6-vertex [Nienhuis]

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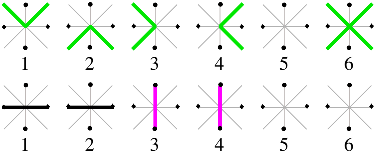

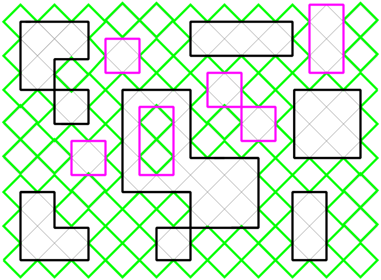

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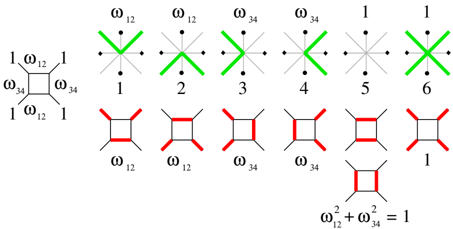
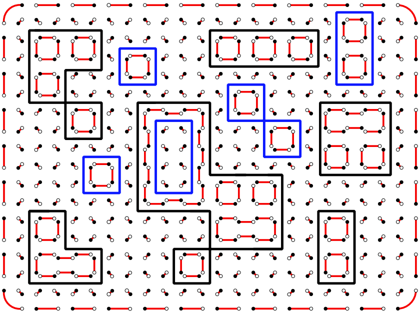
Second step: from 6V to dimers [Wu-Lin, Dubédat]

Local mapping





Conclusion

- To every dimer configuration M of G^{Q}, assign

$$\mathrm{Poly}(\mathsf{M}) = (\mathrm{Poly}_1(\mathsf{M}), \mathrm{Poly}_2(\mathsf{M})),$$

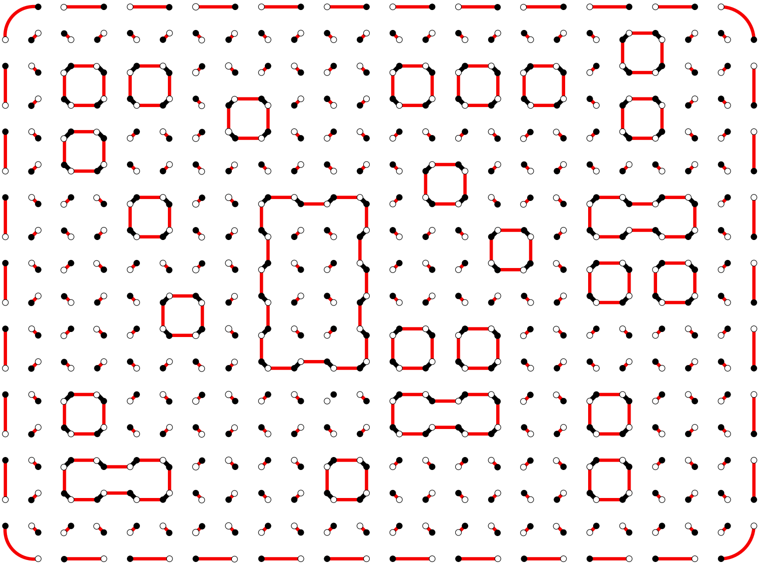
the pair of polygon configurations given by the mappings.

Theorem

For every polygon configuration P^* *of* G^*,

$$\boxed{\mathbb{P}_{\text{2-Ising}}(\mathrm{Mono} = \mathsf{P}^*) = \mathbb{P}_{\text{dimer}}(\mathrm{Poly}_1 = \mathsf{P}^*)}$$

Height function for bipartite dimers (Thurston)



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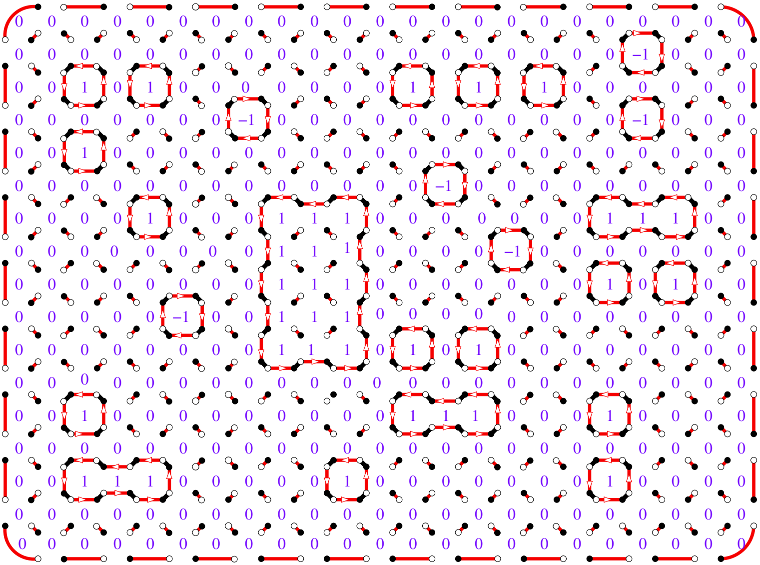

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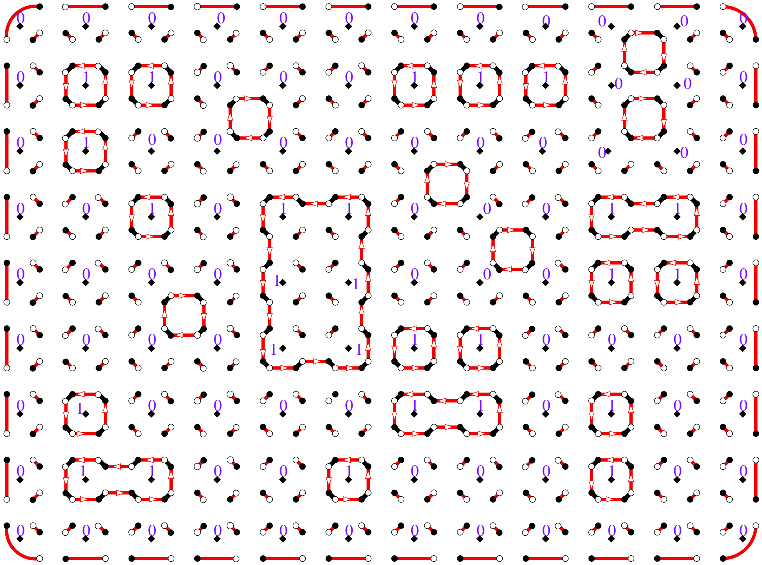

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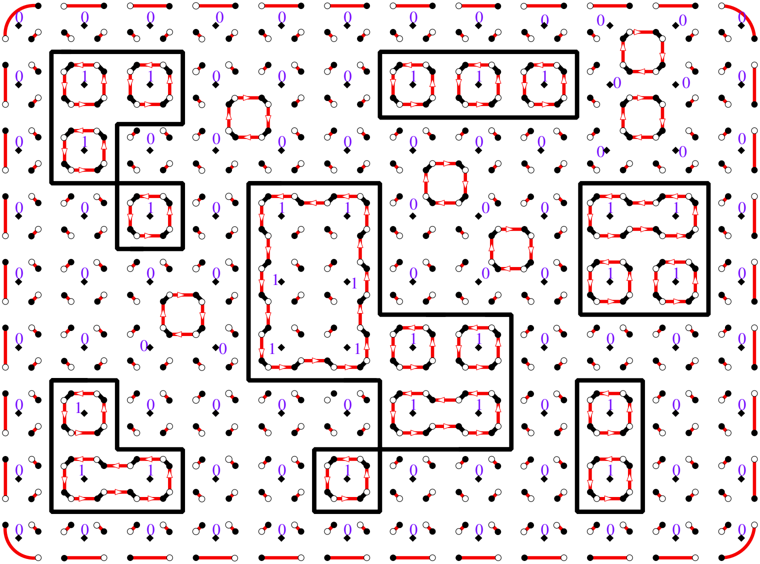
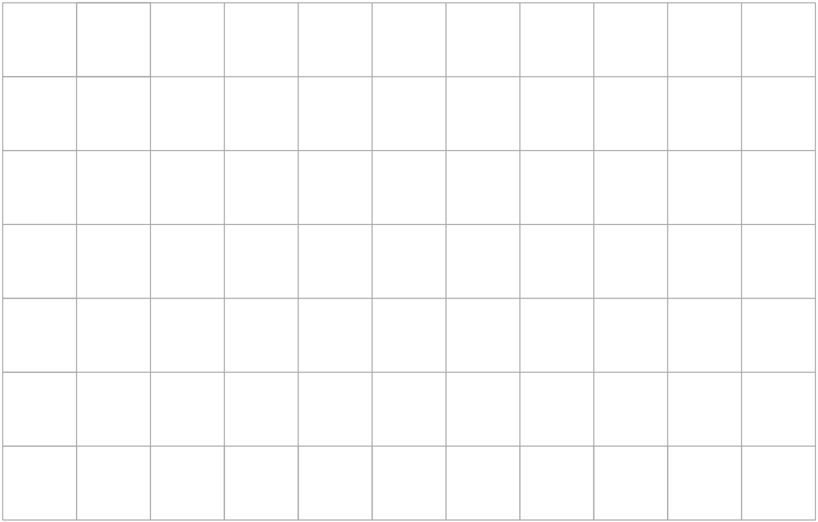


The critical XOR-Ising model on isoradial graphs

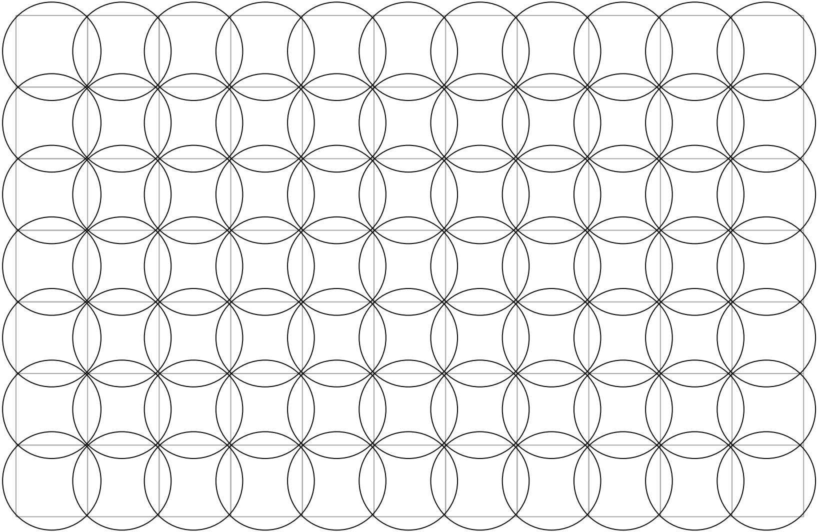
Isoradial graphs

A graph G is **isoradial** if it is planar and can be embedded in the plane in such a way that all faces are inscribed in a circle of radius 1, and that the circumcenters are in the interior of the faces. [Duffin, Mercat, Kenyon]

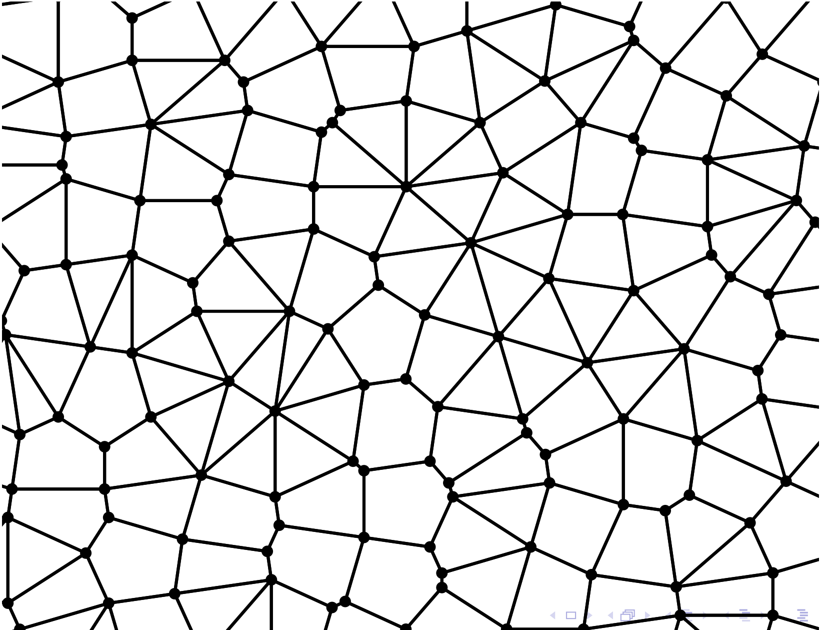
Isoradial graphs



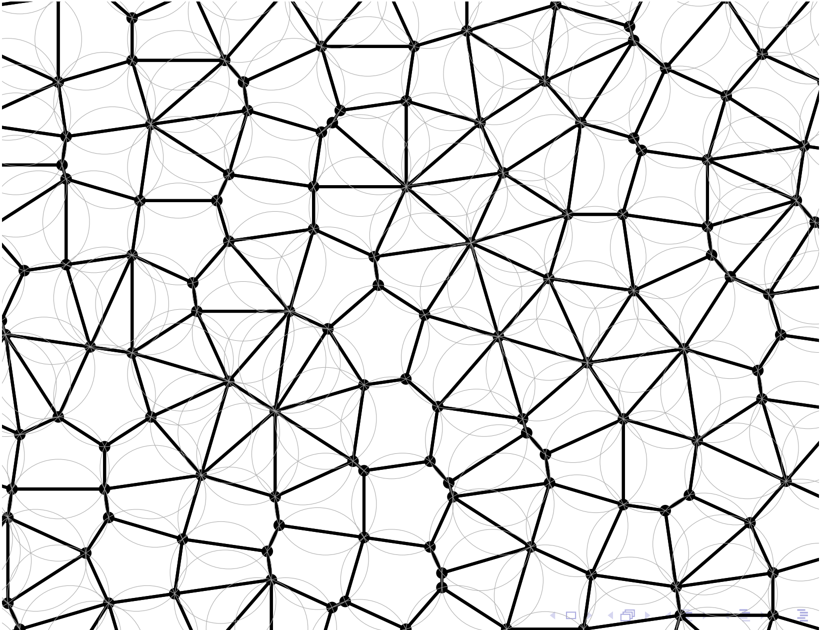
Isoradial graphs



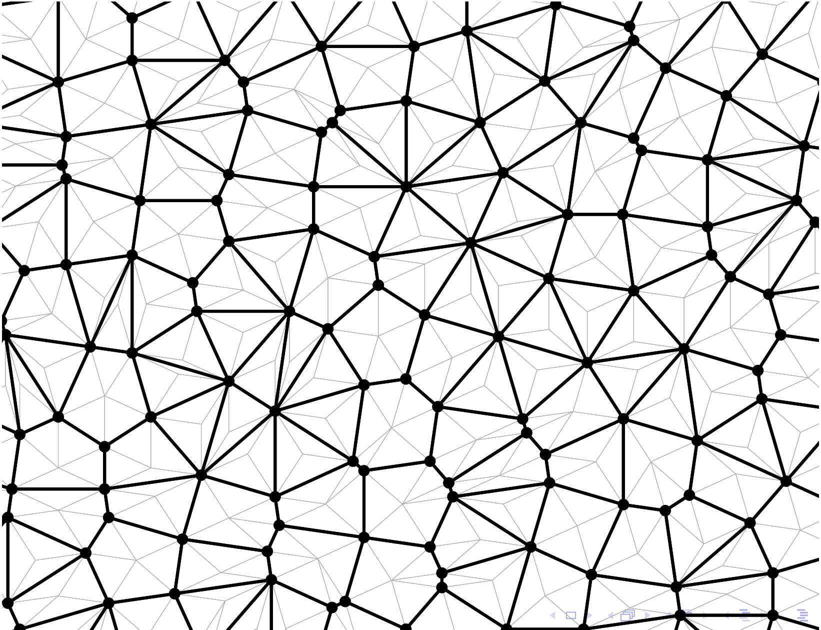
Isoradial graphs



Isoradial graphs

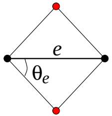


Associated rhombus graph



Critical Ising model on isoradial graphs

- To each edge e is naturally associated an angle θ_e

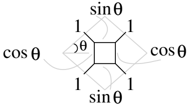


- The Ising model defined on an isoradial graph G is **critical** if the coupling constants are given by:

$$J_e = \frac{1}{2} \log \left(\frac{1 + \sin \theta_e}{\cos \theta_e} \right).$$

(Z-invariance + duality [Baxter], proof in period. case [Li, Duminil–Cimasoni])
Example: $\mathsf{G} = \mathbb{Z}^2$: $\theta_e = \frac{\pi}{4}$, $J_e = \frac{1}{2} \log(1 + \sqrt{2})$.

- The corresponding bipartite graph G^{Q} is also isoradial, and the weights are the **critical** dimer weights:



Back to Wilson's conjecture

Conjecture (Wilson)

The scaling limit of polygon configurations separating ± 1 clusters of the critical XOR-Ising model are contour lines of the Gaussian free field, with the heights of the contours spaced $\sqrt{2}$ times as far apart as they are for [...] the double dimer model on the square lattice.

Theorem (B–dT)

XOR-polygon configurations of the double Ising model on G have the same law as level lines of a restriction of the height function of the bipartite dimer model on G^{Q}, with an explicit coupling.

Theorem (dT)

The height function (as a random distribution) of the critical dimer model defined on a bipartite graph converges weakly in law to $\frac{1}{\sqrt{\pi}}$ a Gaussian free field of the plane.

Back to Wilson's conjecture

Suppose we had strong form of convergence, allowing for convergence of level lines. Then:

level lines of h^ε	$\to$	level lines of GFF	
$(k, k \in \mathbb{Z})$		$(\sqrt{\pi}k, k \in \mathbb{Z})$	
$(k + \frac{1}{2}, k \in \mathbb{Z})$		$(\frac{\sqrt{\pi}}{2}(2k+1), k \in \mathbb{Z})$	XOR loops

For the critical double dimer model. The height function is $h_1^\varepsilon - h_2^\varepsilon$, where h_1 and h_2 are independent, and each converges weakly in distribution to $\frac{1}{\sqrt{\pi}}$ a Gaussian free field. Thus, $h_1 - h_2$ converges weakly in distribution to $\frac{\sqrt{2}}{\sqrt{\pi}}$ a Gaussian free field.

level lines of $h_1^\varepsilon - h_2^\varepsilon$	$\to$	level lines of GFF	
$(k, k \in \mathbb{Z})$		$(\frac{\sqrt{\pi}}{\sqrt{2}}k, k \in \mathbb{Z})$	
$(k + \frac{1}{2}, k \in \mathbb{Z})$		$(\frac{\sqrt{\pi}}{2\sqrt{2}}(2k+1), k \in \mathbb{Z})$	d-dimer loops