Height representation of XOR-Ising loops via bipartite dimers

Cédric Boutillier (UPMC)
Béatrice de Tilière (UPMC)

LAGA Université Paris Nord – March 11, 2015
The Ising model and the XOR-Ising model
The Ising model

- Let $G = (V, E)$ be a finite graph embedded in the plane
- spin configuration $\sigma : V \rightarrow \{-1, +1\}$
- $\sigma$ assigns to every vertex $x$ a spin $\sigma_x \in \{-, +\}$

+1/−1 are represented by green/blue dots.
The Ising model

- Edges of $G$ are assigned positive **coupling constants**:
  $$J = (J_e)_{e \in E}.$$

- **Ising Boltzmann measure**:
  $\forall \sigma \in \{-1, 1\}^V, \quad \mathbb{P}_{\text{Ising}}(\sigma) = \frac{1}{Z_{\text{Ising}}(G, J)} \exp \left( \sum_{e=xy \in E} J_{xy} \sigma_x \sigma_y \right),$

where $Z_{\text{Ising}}(G, J) = \sum_{\sigma \in \{-1, 1\}^V} \exp \left( \sum_{e=xy \in E} J_{xy} \sigma_x \sigma_y \right)$ is the Ising partition function.
The XOR-Ising model

Ising model on $G$, $J = (J_e)_{e \in E}$

$\downarrow$

Ising model on $G$, $J = (J_e)_{e \in E}$

$\xi = \sigma \sigma'$

XOR-Ising model on $G$, $J = (J_e)_{e \in E}$
The XOR-Ising model

Ising model on $G$, $J = (J_e)_{e \in E}$  

$\sigma$  

$\sigma'$

$\xi = \sigma \sigma'$

XOR-Ising model on $G$, $J = (J_e)_{e \in E}$
Conjecture (Wilson (11), Ikhlef–Picco–Santachiara)

The scaling limit of polygon configurations separating $\pm 1$ clusters of the critical XOR-Ising model are contour lines of the Gaussian free field, with the heights of the contours spaced $\sqrt{2}$ times as far apart as they are for [...] the double dimer model on the square lattice.
Theorem (B–dT)

- Polygon configurations of the XOR-Ising model have the same law as a family of contours in a bipartite dimer model.
- This family of contours are the level lines of a restriction of the height function of this bipartite dimer model.

Remark

Proved when the graph $G$ is embedded in a surface of genus $g$, or when $G$ is planar, infinite.

- When the XOR-Ising model is critical, so is the bipartite dimer model.
- Using results of [dT] on the convergence of the height function, this gives partial proof of Wilson’s conjecture.
Contour expansion of the Ising partition function [Kramers & Wannier]
Low temperature expansion

- **Polygon configuration**: subset of edges s.t. each vertex is incident to an even number of edges.

- Write
  \[ e^{J_e \sigma_x \sigma_y} = e^{J_e \left( \delta_{\{\sigma_x = \sigma_y\}} + e^{-2J_e} \delta_{\{\sigma_x \neq \sigma_y\}} \right)} \]

The partition function is then equal to \((\text{LTE})\):

\[
Z_{\text{Ising}}(G, J) = \sum_{\sigma \in \{-1,1\}^V} \prod_{e=xy \in E} e^{J_e \sigma_x \sigma_y} = c \sum_{P^* \in \mathcal{P}(G^*)} \prod_{e^* \in P^*} e^{-2J_e}. 
\]

- Geometric interp: polygon config. separate clusters of \(\pm 1\) spins.
High temperature expansion

Write,

\[ e^{J_x \sigma_x \sigma_y} = \cosh(J_e)(1 + \sigma_x \sigma_y \tanh(J_e)) \].

The partition function is then equal to (HTE):

\[
Z_{\text{Ising}}(G, J) = \sum_{\sigma \in \{-1, 1\}^V} \prod_{e=xy \in E} e^{J_e \sigma_x \sigma_y} = C' \sum_{P \in \mathcal{P}(G)} \prod_{e \in P} \tanh(J_e).
\]

No geometric interpretation using spin configurations.
Mixed contour expansion for the double Ising model
The double Ising model

- Take 2 independent copies (red/blue) of an Ising model on $G$, with coupling constants $J$.
- Using the LTE, consider the probability measure $\mathbb{P}_{2\text{-Ising}}$: if $P^*$, $P^*$ are two polygon configurations.

$$\mathbb{P}_{2\text{-Ising}}(P^*, P^*) = \frac{c^2 \left( \prod_{e^* \in P^*} e^{-2J_e} \right) \left( \prod_{e^* \in P^*} e^{-2J_e} \right)}{Z_{2\text{-Ising}}(G, J)},$$

where $Z_{2\text{-Ising}}(G, J) = Z_{\text{Ising}}(G, J)^2$. 
The double Ising model

Let $P^*$, $P^*$ be two polygon configurations.

Consider the superimposition $P^* \cup P^*$.

Define two new edge configurations:

- $\text{Mono}(P^*, P^*)$: monochromatic edges.
- $\text{Bi}(P^*, P^*)$: bichromatic edges.
Monochromatic edges

Lemma

\( \text{Mono}(P^*, P^*) \) is the polygon configuration separating \( \pm 1 \) clusters of the corresponding XOR-Ising spin configuration.

Goal: understand the law of monochromatic edge configurations.
Bichromatic edge configurations

- Let \((P^*, P^*)\) be two polygon configurations.
- \(\text{Mono}(P^*, P^*)\) splits the surface into connected comp. \((\Sigma_i)_{i}\).

Lemma

For every \(i\), the restriction of \(\text{Bi}(P^*, P^*)\) to \(\Sigma_i\) is the LTE of an Ising configuration on \(G_{\Sigma_i}\), with coupling constants \((2J_e)\).
Lemma

Let $P^*$ be a polygon configuration, separating the surface into $n$ connected components. For every $i$, let $P_i^*$ be a polygon configuration of $G_{\Sigma_i}^*$.

Then, there are $2^n$ pairs of polygon configurations $(P^*, P^*)$ having $P^*$ as monochromatic edges, and $P_1^*, \cdots, P_n^*$ as bichromatic edges.

Denote by $W(P^*)$ the contribution to $Z_{2\text{-Ising}}(G, J)$ of the pairs of polygon configurations $(P^*, P^*)$ such that $\text{Mono}(P^*, P^*) = P^*$.

Corollary

- $W(P^*) = \mathcal{C} \left( \prod_{e^* \in P^*} e^{-2J_e} \right) \prod_{i=1}^{n} \left( 2Z_{\text{LT}}(G_{\Sigma_i}^*, 2J) \right)$

- $Z_{2\text{-Ising}}(G, J) = \sum_{P^* \in \mathcal{P}(G^*)} W(P^*)$

\[
\mathbb{P}_{2\text{-Ising}}(\text{Mono} = P^*) = \frac{W(P^*)}{Z_{2\text{-Ising}}(G, J)}.
\]
Mixed contour expansion

\[ W(P^*) = \mathcal{C} \left( \prod_{e^* \in P^*} e^{-2J_e} \right) \prod_{i=1}^{n} \left( 2Z_{LT}(G_{\Sigma_i}^*, 2J) \right). \]

**Idea** [Nienhuis]: high temperature expansion in each connected component \( \Sigma_i \).

\[ Z_{LT}(G_{\Sigma_i}^*, 2J) = \mathcal{C}(\Sigma_i) Z_{HT}(G_{\Sigma_i}, 2J). \]

Low temp. expansion on \( G_{\Sigma_i}^* \)  
High temp. expansion on \( G_{\Sigma_i} \).
Mixed contour expansion

**Proposition**

For every polygon configuration $P^*$,

$$W(P^*) = C \prod_{e^* \in P^*} \left( \frac{2e^{-2Je}}{1 + e^{-4Je}} \right) \sum_{\{P \in \mathcal{P}(G): P^* \cap P = \emptyset\}} \prod_{e \in P} \left( \frac{1 - e^{-4Je}}{1 + e^{-4Je}} \right)$$

$$\mathbb{P}_{2\text{-Ising}}(\text{Mono} = P^*) = \prod_{e^* \in P^*} \left( \frac{2e^{-2Je}}{1 + e^{-4Je}} \right) \sum_{\{P \in \mathcal{P}(G): P^* \cap P = \emptyset\}} \prod_{e \in P} \left( \frac{1 - e^{-4Je}}{1 + e^{-4Je}} \right) \sum_{P^* \in \mathcal{P}(G^*)} \ldots$$
Higher genus

If the graph is embedded in a surface $\Sigma$ of genus $g \geq 0$.

- Consider $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z}) \simeq \{0, 1\}^{2g}$.
- Family of Ising models, indexed by $\varepsilon \in \{0, 1\}^{2g}$.
- The double Ising model partition function is defined as:

$$Z_{2\text{-Ising}}(G, J) = \sum_{\varepsilon \in \{0, 1\}^{2g}} Z_{\text{Ising}}(G, J)^2.$$
From mixed polygon configurations to dimers
The graph $G^Q = (V^Q, E^Q)$
The dimer model on $G^Q$

dimer configuration of $G^Q$: a subset of edges $M$ such that each vertex is incident to exactly one edge of $M$
The dimer model on $G^Q$

dimer configuration of $G^Q$: a subset of edges $M$ such that each vertex is incident to exactly one edge of $M$

\[ P_{\text{dimer}}(M) \propto \prod_{e \in E} Q^\nu_e \]
The dimer model on $G^Q$

**dimer configuration** of $G^Q$: a subset of edges $M$ such that each vertex is incident to exactly one edge of $M$

weight function $\nu$ on the edges

**Dimer Boltzmann measure**: $\mathbb{P}_{\text{dimer}}(M) \propto \prod_{e \in E^Q} \nu_e$
First step: from polygons to 6-vertex [Nienhuis]

Local mapping

Weights: $\omega_{12} = \frac{2e^{-2Je}}{1+e^{-4Je}}$, $\omega_{34} = \frac{1-e^{-4Je}}{1+e^{-4Je}}$, $\omega_{56} = 1$. 
First step: from polygons to 6-vertex [Nienhuis]

Local mapping

Weights: $\omega_{12} = \frac{2e^{-2Je}}{1+e^{-4Je}}$, $\omega_{34} = \frac{1-e^{-4Je}}{1+e^{-4Je}}$, $\omega_{56} = 1$. 
Second step: from $6V$ to dimers [Wu-Lin, Dubédat]

Local mapping

\[ \omega_{12}^2 + \omega_{34}^2 = 1 \]
To every dimer configuration \( M \) of \( G^Q \), assign

\[
\text{Poly}(M) = (\text{Poly}_1(M), \text{Poly}_2(M)),
\]

the pair of polygon configurations given by the mappings.

**Theorem**

*For every polygon configuration \( P^* \) of \( G^* \),*

\[
\mathbb{P}_{\text{2-Ising}}(\text{Mono} = P^*) = \mathbb{P}_{\text{dimer}}(\text{Poly}_1 = P^*)
\]
Height function for bipartite dimers (Thurston)
Height function for bipartite dimers (Thurston)
Height function for bipartite dimers (Thurston)
The critical XOR-Ising model on isoradial graphs
Isoradial graphs

A graph $G$ is **isoradial** if it is planar and can be embedded in the plane in such a way that all faces are inscribed in a circle of radius 1, and that the circumcenters are in the interior of the faces. [Duffin, Mercat, Kenyon]
Isoradial graphs
Isoradial graphs
Isoradial graphs
Isoradial graphs
Associated rhombus graph
Critical Ising model on isoradial graphs

- To each edge $e$ is naturally associated an angle $\theta_e$
- The Ising model defined on an isoradial graph $G$ is **critical** if the coupling constants are given by:

$$J_e = \frac{1}{2} \log \left( \frac{1 + \sin \theta_e}{\cos \theta_e} \right).$$

($Z$-invariance + duality [Baxter], proof in period. case [Li, Duminil–Cimasoni])

**Example:** $G = \mathbb{Z}^2$: $\theta_e = \frac{\pi}{4}$, $J_e = \frac{1}{2} \log(1 + \sqrt{2})$.

- The corresponding bipartite graph $G^Q$ is also isoradial, and the weights are the **critical** dimer weights:
Conjecture (Wilson)

The scaling limit of polygon configurations separating ±1 clusters of the critical XOR-Ising model are contour lines of the Gaussian free field, with the heights of the contours spaced $\sqrt{2}$ times as far apart as they are for [...] the double dimer model on the square lattice.

Theorem (B–dT)

XOR-polygon configurations of the double Ising model on $G$ have the same law as level lines of a restriction of the height function of the bipartite dimer model on $G^Q$, with an explicit coupling.

Theorem (dT)

The height function (as a random distribution) of the critical dimer model defined on a bipartite graph converges weakly in law to $\frac{1}{\sqrt{\pi}}$ a Gaussian free field of the plane.
Suppose we had strong form of convergence, allowing for convergence of level lines. Then:

<table>
<thead>
<tr>
<th>level lines of $h_\varepsilon$</th>
<th>→</th>
<th>level lines of GFF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(k, k \in \mathbb{Z})$</td>
<td></td>
<td>$(\sqrt{\pi}k, k \in \mathbb{Z})$</td>
</tr>
<tr>
<td>$(k + \frac{1}{2}, k \in \mathbb{Z})$</td>
<td></td>
<td>$(\frac{\sqrt{\pi}}{2}(2k + 1), k \in \mathbb{Z})$</td>
</tr>
</tbody>
</table>

XOR loops

For the critical double dimer model. The height function is $h_1^\varepsilon - h_2^\varepsilon$, where $h_1$ and $h_2$ are independent, and each converges weakly in distribution to $\frac{1}{\sqrt{\pi}}$ a Gaussian free field. Thus, $h_1 - h_2$ converges weakly in distribution to $\frac{\sqrt{2}}{\sqrt{\pi}}$ a Gaussian free field.

<table>
<thead>
<tr>
<th>level lines of $h_1^\varepsilon - h_2^\varepsilon$</th>
<th>→</th>
<th>level lines of GFF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(k, k \in \mathbb{Z})$</td>
<td></td>
<td>$(\frac{\sqrt{\pi}}{\sqrt{2}}k, k \in \mathbb{Z})$</td>
</tr>
<tr>
<td>$(k + \frac{1}{2}, k \in \mathbb{Z})$</td>
<td></td>
<td>$(\frac{\sqrt{\pi}}{2\sqrt{2}}(2k + 1), k \in \mathbb{Z})$</td>
</tr>
</tbody>
</table>

d-dimer loops