Generalized Fourier Series for Solutions of Linear Differential Equations

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I Introduction

Generalized Fourier Series for Solutions of Linear Differential Equations.
Generalized Fourier Series

\[ f(x) = \sum a_n \psi_n(x) \]

Some Examples

\[
\sin(x) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n}(x)
\]

\[
\arccos(x) = \frac{1}{2\pi} T_0(x) - \sum_{n=0}^{\infty} \frac{4}{(2n+1)^2 \pi} T_{2n+1}(x)
\]

\[
\text{erf}(x) = 2 \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{1}{\sqrt{\pi} (2n+1) n!} \text{!!}_{1}^{1} F_{1} \left( \frac{n + \frac{1}{2}}{2n + 2} \right) - x
\]

More generally \((\psi_n(x))_{n \in \mathbb{N}}\) can be an orthogonal basis of a Hilbert space.
Applications: Good approximation properties.

Approximation of \( \text{arctan}(2x) \) by Taylor expansion of degree 1
Applications: Good approximation properties.

Bad approximation outside its circle of convergence

\[ \text{arctan}(2x) \]

Taylor approximation
Applications: Good approximation properties.

approximation of $\arctan(2x)$ by Chebyshev expansion of degree 1

arctan(2x)

Taylor expansion

Chebyshev expansion
Applications: Good approximation properties.

bad approximation over $\mathbb{R}$

arctan(2x)

Taylor expansion

Chebyshev expansion
Applications: Good approximation properties.

approximation of \( \arctan(2x) \) by Hermite expansion of degree 1

\[ \arctan(2x) \]

- Taylor expansion
- Chebyshev expansion
- Hermite expansion
Families of functions $\psi_n(x)$ with two special properties

**Mult by $x$ ($\mathcal{P}_x$)**

\[ \mathcal{R} \mathcal{e} \mathcal{c}_{x2} (x \psi_n(x)) = \mathcal{R} \mathcal{e} \mathcal{c}_{x1} (\psi_n(x)) \]

**Examples**

- Monomial polynomials
  \[ (M_n = x^n) \]
- All orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions
Families of functions $\psi_n(x)$ with two special properties

\[
R_{2x} (x \psi_n(x)) = R_{1x} (\psi_n(x))
\]

**Examples**

- Monomial polynomials
  
  \[
  (M_n = x^n)
  \]
  \[
  xM_n = M_{n+1}
  \]

- All orthogonal polynomials
  
  \[
  2x T_n(x) = T_{n+1}(x) + T_{n-1}(x)
  \]

- Bessel functions
  
  \[
  \frac{1}{n} (xJ_{n+1} - xJ_{n-1}) = 2J_n
  \]

- Legendre functions

- Parabolic cylinder functions
Our framework

Families of functions \( \psi_n(x) \) with two special properties

**Mult by** \( x \) (\( P_x \))

\[
\text{Rec}_{x2} (x \psi_n(x)) = \text{Rec}_{x1} (\psi_n(x))
\]

**Differentiation** (\( P_\partial \))

\[
\text{Rec}_{\partial 2} (\psi_n'(x)) = \text{Rec}_{\partial 1} (\psi_n(x))
\]

**Examples**

- Monomial polynomials
- Classical orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions
Our framework

Families of functions $\psi_n(x)$ with two special properties

**Mult by $x$** ($\mathcal{P}_x$)

$$\mathcal{R} \mathcal{e} \mathcal{c}_{x2} (x \psi_n(x)) = \mathcal{R} \mathcal{e} \mathcal{c}_{x1} (\psi_n(x))$$

**Differentiation** ($\mathcal{P}_\partial$)

$$\mathcal{R} \mathcal{e} \mathcal{c}_{\partial 2} (\psi'_n(x)) = \mathcal{R} \mathcal{e} \mathcal{c}_{\partial 1} (\psi_n(x))$$

**Examples**

- Monomial polynomials
- Classical orthogonal polynomials
- Bessel functions
- Legendre functions
- Parabolic cylinder functions

For example:

- $M'_n = nM_{n-1}$
- $\frac{1}{n+1} T'_{n+1}(x) - \frac{1}{n-1} T'_{n-1}(x) = 2T_n(x)$
- $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$
Families of functions $\psi_n(x)$ with two special properties

**Mult by $x$ ($P_x$)**

$$\text{Rec}_{x2} (x\psi_n(x)) = \text{Rec}_{x1} (\psi_n(x))$$

**Differentiation ($P_\partial$)**

$$\text{Rec}_{\partial2} (\psi_n'(x)) = \text{Rec}_{\partial1} (\psi_n(x))$$

This is our data-structure for $\psi_n(x)$
If $\psi_n(x)$ satisfies $P_x$ and $P_\partial$, for any $f(x) = \sum a_n \psi_n(x)$ solution of a linear differential equation with polynomial coefficients, the coefficients $a_n$ are cancelled by a linear recurrence relation with polynomial coefficients.
Main Idea

If $\psi_n(x)$ satisfies $(\mathcal{P}_x)$ and $(\mathcal{P}_\partial)$, for any $f(x) = \sum a_n \psi_n(x)$ solution of a linear differential equation with polynomial coefficients, the coefficients $a_n$ are cancelled by a linear recurrence relation with polynomial coefficients.

Applications:

- Efficient numerical computation of the coefficients.
- Computation of closed-form for the coefficients (when it's possible).
Previous work

- Clenshaw (1957): numerical scheme to compute the coefficients when \( \psi_n(x) = T_n(x) \) (Chebyshev series).
- Lewanowicz (1976-2004): algorithms to compute a recurrence relation when \( \psi_n \) is an orthogonal or semi-orthogonal polynomial family.
- Rebillard and Zakražsek (2006): General algorithm computing a recurrence relation when \( \psi_n \) is a family of hypergeometric polynomials.
- Benoit and Salvy (2009): Simple unified presentation and complexity analysis of the previous algorithms using Fractions of recurrence relations when \( \psi_n = T_n \). New and fast algorithm to compute the Chebyshev recurrence.
**New Results (2011)**

- Simple unified presentation of the previous algorithms using **Pairs of recurrence relations**.
- **New general algorithm** computing the recurrence relation of the coefficients for a Generalized Fourier Series when $\psi_n(x)$ satisfies $(P_x)$ and $(P_\partial)$. 
II Pairs of Recurrence Relations
Examples: Chebyshev case \( f(x) = \sum u_n T_n(x) \)

Basic rules:

\[
xf = \sum a_n T_n \quad (P_x)
\]

\[
f' = \sum b_n T_n \quad (P_{\partial})
\]

\[
a_n = \frac{u_{n-1} + u_{n+1}}{2}
\]

\[
b_{n-1} - b_{n+1} = 2nu_n.
\]
Examples: Chebyshev case \( f(x) = \sum u_n T_n(x) \)

Basic rules:

\[
xf = \sum a_n T_n \quad (\mathcal{P}_x) \\
\frac{df}{dx} = \sum b_n T_n \quad (\mathcal{P}_\partial)
\]

\[
a_n = \frac{u_{n-1} + u_{n+1}}{2} \\
b_{n-1} - b_{n+1} = 2nu_n.
\]

Combine:

\[
f' + 2xf = \sum c_n T_n \quad (\mathcal{P}_\partial + 2\mathcal{P}_x) \\
c_{n-1} - c_{n+1} = \text{Rec}_1(u_n).
\]

Application: Chebyshev series for \( \exp(-x^2) \).
Examples: Chebyshev case \( f(x) = \sum u_n T_n(x) \)

Basic rules:

\[
xf = \sum a_n T_n \quad \left( P_x \right)
\]

\[
f' = \sum b_n T_n \quad \left( P_\partial \right)
\]

\[
a_n = \frac{u_{n-1} + u_{n+1}}{2}
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\[
b_{n-1} - b_{n+1} = 2nu_n.
\]

Combine:

\[
f' + 2xf = \sum c_n T_n \quad \left( P_\partial + 2P_x \right)
\]

\[
c_{n-1} - c_{n+1} = \text{Rec}_1(u_n).
\]

Application: Chebyshev series for \( \exp(-x^2) \).

\[
(f' + 2xf)' = \sum d_n T_n \quad \left( P_\partial \right)
\]

\[
d_{n-1} - d_{n+1} = 2nc_n,
\]

\[
\text{Rec}_2(d_n) = \text{Rec}_3(u_n),
\]

\[
\text{Rec}_4(e_n) = \text{Rec}_5(u_n).
\]

Application: Chebyshev series for \( \text{erf}(x) \).
Theorem (Least Common Left Multiple (Ore 33))

Given $\text{Rec}_1$ and $\text{Rec}_2$, there exists a recurrence relation $\text{Rec}$ and a pair $(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)$ such that for all sequences $(u_n)_{n \in \mathbb{N}}$:

\[
\text{Rec} (u_n) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 (u_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_2 (u_n)
\]
Theorem (Least Common Left Multiple (Ore 33))

Given $\text{Rec}_1$ and $\text{Rec}_2$, there exists a recurrence relation $\text{Rec}$ and a pair $(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)$ such that for all sequences $(u_n)_{n \in \mathbb{N}}$:

$$\text{Rec} (u_n) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 (u_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_2 (u_n)$$

- The LCLM is the recurrence relation $\text{Rec}$ with minimal order.
Theorem (Least Common Left Multiple (Ore 33))

Given $\text{Rec}_1$ and $\text{Rec}_2$, there exists a recurrence relation $\text{Rec}$ and a pair $(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)$ such that for all sequences $(u_n)_{n \in \mathbb{N}}$:

$$\text{Rec} (u_n) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 (u_n) = \tilde{\text{Rec}}_2 \circ \text{Rec}_2 (u_n)$$

- The LCLM is the recurrence relation $\text{Rec}$ with minimal order.
- Computation: Euclidean algorithm.
Operations of addition and composition

\[ \text{Rec} = \text{lclm}(\text{Rec}_1, \text{Rec}_2) = \text{Rec}_1 \circ \text{Rec}_1 = \text{Rec}_2 \circ \text{Rec}_2 \]

Operation 1: Addition

\[
\begin{align*}
\text{Rec}_1(a_n) &= \text{Rec}_3(u_n), \quad \text{Rec}_2(b_n) = \text{Rec}_4(u_n) \\
\text{Rec}(a_n) &= \text{Rec}_1 \circ \text{Rec}_3(u_n), \quad \text{Rec}(b_n) = \text{Rec}_2 \circ \text{Rec}_4(u_n) \\
\rightarrow \text{Rec}(a_n + b_n) &= \left( \text{Rec}_1 \circ \text{Rec}_3 + \text{Rec}_2 \circ \text{Rec}_4 \right)(u_n).
\end{align*}
\]
Operations of addition and composition

\[ \text{Rec} = \text{lclm}(\text{Rec}_1, \text{Rec}_2) = \tilde{\text{Rec}}_1 \circ \text{Rec}_1 = \tilde{\text{Rec}}_2 \circ \text{Rec}_2 \]

**Operation 1: Addition**

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\begin{align*}
\text{Rec}_1(a_n) &= \text{Rec}_3(u_n), \quad \text{Rec}_2(b_n) = \text{Rec}_4(u_n) \\
\text{Rec}(a_n) &= \text{Rec}_1 \circ \text{Rec}_3(u_n), \quad \text{Rec}(b_n) = \text{Rec}_2 \circ \text{Rec}_4(u_n) \\
\implies \text{Rec}(a_n + b_n) &= \left( \text{Rec}_1 \circ \text{Rec}_3 + \text{Rec}_2 \circ \text{Rec}_4 \right)(u_n).
\end{align*}
\]

**Operation 2: Composition**

\[
\begin{align*}
\text{Rec}_1(u_n) &= \text{Rec}_3(a_n), \quad \text{Rec}_2(u_n) = \text{Rec}_4(b_n) \\
\text{Rec}(u_n) &= \text{Rec}_1 \circ \text{Rec}_1(u_n) = \text{Rec}_2 \circ \text{Rec}_2(u_n) \\
\implies \text{Rec}_1 \circ \text{Rec}_3(a_n) &= \text{Rec}_2 \circ \text{Rec}_4(b_n).
\end{align*}
\]
Main Result

Main Result : Morphism

There exists a morphism $\varphi$ such that if $f = \sum u_n \psi_n(x)$ and $g = \sum v_n \psi_n(x)$ are related by $L(f) = g$ ($L$ a linear differential operator), then:

$$\varphi(L) = (\text{Rec}_1, \text{Rec}_2) \quad \text{with} \quad \text{Rec}_1(u_n) = \text{Rec}_2(v_n)$$

In particular if $L(f) = 0$, then $\text{Rec}_1(u_n) = 0$. 

Alexandre Benoît

Generalized Fourier Series for Solutions of Linear Differential Equations.
Definition of the Morphism $\varphi$

\[ f = \sum u_n \psi_n(x) \quad g = \sum v_n \psi_n(x) \]

- If $xf = g$, then $\text{Rec}_{x2}(u_n) = \text{Rec}_{x1}(v_n)$
- If $f' = g$, then $\text{Rec}_{\partial2}(u_n) = \text{Rec}_{\partial1}(v_n)$

Example for Chebyshev series:

\[ 2xT_n(x) = T_{n+1}(x) + T_{n-1}(x) \]

\[ T'_n(x) = T_{n-1}(x) - T_{n+1}(x) \]

Example for Bessel series:

\[ \frac{1}{x} J_n(x) J_{n+1}(x) - J_{n-1}(x) J_n(x) = 2 J_n(x) \]

\[ \frac{1}{x} J'_n(x) = J_{n-1}(x) - J_{n+1}(x) \]
Definition of the Morphism $\varphi$

\[ f = \sum u_n \psi_n(x) \quad g = \sum v_n \psi_n(x) \]

\[ \text{Rec}_{x_2}(x \psi_n(x)) = \text{Rec}_{x_1}(\psi_n(x)) \]

\[ \varphi(x) \]

\[ \text{if } xf = g, \text{ then } \text{Rec}_{x_2}(u_n) = \text{Rec}_{x_1}(v_n) \]

\[ \text{Rec}_{\partial_2}(\psi_n'(x)) = \text{Rec}_{\partial_1}(\psi_n(x)) \]

\[ \varphi(\partial) \]

\[ \text{if } f' = g, \text{ then } \text{Rec}_{\partial_1}(u_n) = \text{Rec}_{\partial_2}(v_n) \]

Example for Chebyshev series:

\[ 2xT_n(x) = T_{n+1}(x) + T_{n-1}(x) \]

\[ \frac{T'_{n+1}(x)}{n+1} - \frac{T'_{n-1}(x)}{n-1} = 2T_n(x) \]

\[ u_{n+1} + u_{n-1} = 2v_n \]

\[ 2u_n = \frac{1}{n} (v_{n-1} - v_{n+1}) \]

Example for Bessel series

\[ \frac{1}{n} (xJ_{n+1} - xJ_{n-1}) = 2J_n \]

\[ 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \]

\[ 2u_n = \frac{v_{n+1}}{n+1} + \frac{v_{n-1}}{n-1} \]

\[ u_{n+1} - u_{n-1} = 2v_n \]
General Algorithm

Recall

- Definition of $\varphi(x)$ and $\varphi(\partial)$
- Algorithms to compute addition and composition between two pairs
Recall

- Definition of $\varphi(x)$ and $\varphi(\partial)$
- Algorithms to compute addition and composition between two pairs

General Algorithm

We deduce from this morphism a general Horner-like algorithm to compute the recurrence relation satisfied by the coefficients of a generalized Fourier series solution of a linear differential equation.
III Recurrences of Smaller Order
Greatest Common Left Divisor and Reduction of Order

**GCLD**

Given a pair \((\text{Rec}_1, \text{Rec}_2)\), the Euclidean algorithm computes the greatest recurrence relation \(\text{Rec}\) (GCLD) such that there exists a pair \((\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)\) with the following relations for all sequences \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\):

\[
\begin{align*}
\text{Rec} \circ \tilde{\text{Rec}}_1 (u_n) &= \text{Rec}_1 (u_n) \\
\text{Rec} \circ \tilde{\text{Rec}}_2 (v_n) &= \text{Rec}_2 (v_n)
\end{align*}
\]
Given a pair $(\text{Rec}_1, \text{Rec}_2)$, the Euclidean algorithm computes the greatest recurrence relation $\text{Rec}$ (GCLD) such that there exists a pair $(\tilde{\text{Rec}}_1, \tilde{\text{Rec}}_2)$ with the following relations for all sequences $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$:

$$\text{Rec} \circ \tilde{\text{Rec}}_1 (u_n) = \text{Rec}_1 (u_n)$$
$$\text{Rec} \circ \tilde{\text{Rec}}_2 (v_n) = \text{Rec}_2 (v_n)$$

The orders of the recurrence relations $\tilde{\text{Rec}}_i$ are at most those of $\text{Rec}_i$.
Greatest Common Left Divisor and Reduction of Order

**GCLD**

Given a pair \((\text{Rec}_1, \text{Rec}_2)\), the Euclidean algorithm computes the greatest recurrence relation \(\text{Rec} (\text{GCLD})\) such that there exists a pair \((\widetilde{\text{Rec}}_1, \widetilde{\text{Rec}}_2)\) with the following relations for all sequences \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\):

\[
\begin{align*}
\text{Rec} \circ \widetilde{\text{Rec}}_1 (u_n) &= \text{Rec}_1 (u_n) \\
\text{Rec} \circ \widetilde{\text{Rec}}_2 (v_n) &= \text{Rec}_2 (v_n)
\end{align*}
\]

The orders of the recurrence relations \(\widetilde{\text{Rec}}_i\) are at most those of \(\text{Rec}_i\).

**Remark**

In a general case, we don’t have:

\[
\text{Rec}_1 (u_n) = \text{Rec}_2 (v_n) \Rightarrow \widetilde{\text{Rec}}_1 (u_n) = \widetilde{\text{Rec}}_2 (v_n),
\]
GCD

Given a pair \((\text{Rec}_1, \text{Rec}_2)\), the Euclidean algorithm computes the greatest recurrence relation \(\text{Rec} \) (GCLD) such that there exists a pair \((\widetilde{\text{Rec}}_1, \widetilde{\text{Rec}}_2)\) with the following relations for all sequences \((u_n)_{n \in \mathbb{N}}\) and \((v_n)_{n \in \mathbb{N}}\):

\[
\begin{align*}
\text{Rec} \circ \widetilde{\text{Rec}}_1(u_n) &= \text{Rec}_1(u_n) \\
\text{Rec} \circ \widetilde{\text{Rec}}_2(v_n) &= \text{Rec}_2(v_n)
\end{align*}
\]

The orders of the recurrence relations \(\widetilde{\text{Rec}}_i\) are at most those of \(\text{Rec}_i\).

Remark

In a general case, we don’t have:

\[
\text{Rec}_1(u_n) = \text{Rec}_2(v_n) \Rightarrow \widetilde{\text{Rec}}_1(u_n) = \widetilde{\text{Rec}}_2(v_n),
\]

\[
(-1)^{n+2} - (-1)^n = (-1)^{2(n+1)} - (-1)^{2n} \not\Rightarrow (-1)^{n+1} + (-1)^n = (-1)^{2n}
\]
GLCD for reduction of order

Theorem

Given $L$ a linear differential operator, $f = \sum u_n \psi_n(x)$, $g = \sum v_n \psi_n(x)$ such that $L(f) = g$ and a pair $(\text{Rec}_1, \text{Rec}_2) = \varphi(L)$. We have

$$\tilde{\text{Rec}}_1(u_n) = \tilde{\text{Rec}}_2(v_n)$$
Theorem

\[ f = \sum u_n \psi_n(x), \quad g = \sum v_n \psi_n(x) \]

such that \( L(f) = g \) and a pair \((\text{Rec}_1, \text{Rec}_2) = \varphi(L)\). We have

\[ \tilde{\text{Rec}}_1(u_n) = \tilde{\text{Rec}}_2(v_n) \]

Application: Adaptation of the previous algorithm

At the end of the previous algorithm, add a final step:

Remove the \textit{GC LD} of the two recurrence relations of the pair.
Example of reduction for Chebyshev series

\[ \sqrt{1 - x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n + 1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x) \]

\(\sqrt{1 - x^2}\) is the solution of the differential equation:

\[ xy(x) + (1 - x^2)y'(x) = 0 \]
Example of reduction for Chebyshev series

\[ \sqrt{1 - x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n + 1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x) \]

\[ \sqrt{1 - x^2} \] is the solution of the differential equation:

\[ xy(x) + (1 - x^2)y'(x) = 0 \]

With the general algorithm we obtain the pair of recurrence relations:

Rec_1 (u_n) = (n+3)u_{n+2} - 2nu_n + (n-3)u_{n-2} and Rec_2 (v_n) = 2 (-v_{n+1} + v_{n-1}).

We deduce:

\[ (n + 3)c_{n+2} - 2nc_n + (n - 3)c_{n-2} = 0. \]
Example of reduction for Chebyshev series

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\sqrt{1 - x^2} = \sum_{n \in \mathbb{N}} \frac{4}{\pi(2n+1)} T_{2n}(x) = \sum_{n \in \mathbb{N}} c_n T_n(x)
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\(\sqrt{1 - x^2}\) is the solution of the differential equation:

\[xy(x) + (1 - x^2)y'(x) = 0\]

With the general algorithm we obtain the pair of recurrence relations:

Rec_1 (u_n) = (n+3)u_{n+2} - 2nu_n + (n-3)u_{n-2} and Rec_2 (v_n) = 2 (-v_{n+1} + v_{n-1}).

We deduce:

\((n + 3)c_{n+2} - 2nc_n + (n - 3)c_{n-2} = 0.\)

\[\tilde{\text{Rec}}_1 (u_n) = (n + 2)u_{n+1} - (n - 2)u_{n-1} \text{ and } \tilde{\text{Rec}}_2 (v_n) = 2v_n.\]

We deduce:

\((n + 2)c_{n+1} - (n - 2)c_{n-1} = 0.\)
IV Conclusion
Conclusion

Contributions:

- Use of Pairs of recurrence relations.
- New general algorithm.
- Use of the GCLD to reduce order of the recurrence.

Example

\[ \text{erf}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \frac{\Gamma(n + \frac{1}{2})}{T_2^{n+1}(x)} \]

Integration in the Dynamic Dictionary of Mathematical Functions.
Conclusion

Contributions:

- Use of Pairs of recurrence relations.
- New general algorithm.
- Use of the GCLD to reduce order of the recurrence.

Perspectives:

- Computation of the recurrence of minimal order.
- Numerical computation of the coefficients.
- Closed form for the coefficients.

Example

\[ \text{erf} (x) = \sum_{n=0}^{\infty} 2 \frac{4^{-n} (-1)^n \text{$_1$F$_1$}(n + 1/2; 2 n + 2; -1)}{\sqrt{\pi} (2 n + 1) n!} T_{2n+1} (x). \]

- Integration in the Dynamic Dictionary of Mathematical Functions.