Polynomial Invariants on Stranded Graphs

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Outline

1. Introduction: From "Tutte to Tensor" polynomials

2. Stranded graph structures

3. Polynomial invariants on rank 3 weakly colored stranded graph

4. Conclusion: Open questions

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Tutte's graph polynomial

• Tutte polynomial is a "Universal Invariant" for polynomials defined on simple graphs $\mathcal{G}(\mathcal{V}, \mathcal{E})$

$$T_{\mathcal{G}}(x,y) = \sum_{A\subset\mathcal{G}} (x-1)^{\mathrm{r}(\mathcal{G})-\mathrm{r}(A)}(y-1)^{n(A)}\,,\;\; \mathrm{r}(A) = V - k(A)\,,\;\; n(A) = E(A) - \mathrm{r}(A) \quad (1)$$

satisfying a contraction/deletion rule for a regular edge e

$$T_{\mathcal{G}} = T_{\mathcal{G}-e} + T_{\mathcal{G}/e}\,. \quad (2)$$

$\sim$ Special edges (bridges and self-loops) play the role of boundary conditions of this rec-rel.

$$\text{For a bridge:} \quad T_{\mathcal{G}} = xT_{\mathcal{G}-e} \quad (3)$$

$$\text{For a loop:} \quad T_{\mathcal{G}} = yT_{\mathcal{G}-e}\,. \quad (4)$$

• Tutte polynomial is "universal" in the sense that any other invariant satisfying the same rec-rel must be an evaluation of this polynomial [Brylawski, '70];

Ribbon graphs . . .

Definition (Ribbon graphs, Bollobàs-Riordan, Math. Ann. '02)

• Neighborhood of graph (cellularly) embedded in a surface.
• A ribbon graph $\mathcal{G}$ is a (not necessarily orientable) surface with boundary represented as the union of two sets of closed topological discs called vertices $\mathcal{V}$ and edges $\mathcal{E}$. These sets satisfy the following:
∼ Vertices and edges intersect by disjoint line segment,
∼ each such line segment lies on the boundary of precisely one vertex and one edge,
∼ every edge contains exactly two such line segments.

• Face: a component of a boundary of $\mathcal{G}$ considered as a geometric ribbon graph, and hence as surface with boundary. As an embedded graph, a face of $\mathcal{G}$ is simply a face of the embedding.
• Edges/Loops: can be twisted or not.
• Operations $\mathcal{G} - e$ and $\mathcal{G}/e$.
• Contraction of a trivial untwisted loop e: $\mathcal{G}/e = (\mathcal{G} - e) \sqcup \{v_0\}$.

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. . . and the Bollobàs-Riordan polynomial

- The BR polynomial:

Let $\mathcal{G}$ be a ribbon graph. We define the ribbon graph polynomial of $\mathcal{G}$ to be an element of $\mathbb{Z}[x, y, z, w]$ quotiented by the ideal generated by $w^2 - w$ as:

$$R_{\mathcal{G}}(x, y, z, w) = \sum_{A \Subset \mathcal{G}} (x-1)^{\mathrm{r}(\mathcal{G}) - \mathrm{r}(A)} (y-1)^{n(A)} z^{k(A) - F(A) + n(A)} w^{o(A)}, \quad (5)$$

where $F(A)$ is the number of faces of A, $o(A) = 0$ if A is orientable and $o(A) = 1$ if not.

- Why the exponent of z ?

$$k(A) - F(A) + n(A) = 2k(A) - (|\mathcal{V}| - |\mathcal{E}_A| - F(A)) = \kappa(A) \quad (6)$$

is nothing but the genus or twice the genus (for oriented surfaces) of the subgraph A.

- Recurence rule

For a regular edge : $\quad R_{\mathcal{G}} = R_{\mathcal{G}/e} + R_{\mathcal{G}-e}\,,$ (7)

For a bridge : $\quad R_{\mathcal{G}} = x\, R_{\mathcal{G}/e}\,,$ (8)

For a trivial untwisted loop : $\quad R_{\mathcal{G}} = y\, R_{\mathcal{G}-e}$ (9)

For a trivial twisted loop : $\quad R_{\mathcal{G}} = (1 + (y-1)zw)\, R_{\mathcal{G}-e}\,.$ (10)

What about higher dimensional space?

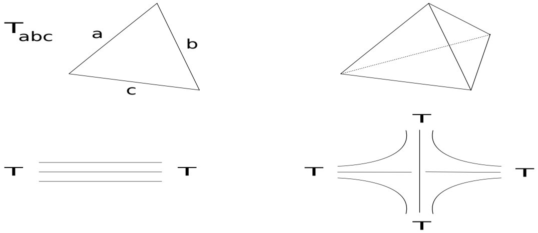
Krushkal & Renardy polynomial [2010]

• Krushkal-Renardy (KR) '10: A 2-variable polynomial $T^n_{\mathcal{G}}$ for $\mathcal{G}$ a higher dimensional simplicial or CW-complex generalizing Tutte using homology group $X^{H_{n-1}(L)-H_{n-1}(\mathcal{G})}Y^{H_n(L)}$ for $L \Subset \mathcal{G}$ subcomplex of dimension n.

Tensor graphs

- Feynman graph for Tensor Models for Quantum Gravity
- A rank d tensor $T_{p_1,\dots,p_d}$ + A Geometricial/Physical input.
- Basic building blocks $(d-1)$-simplexes & Interaction forms a D-simplex;

For e.g. in 3D:

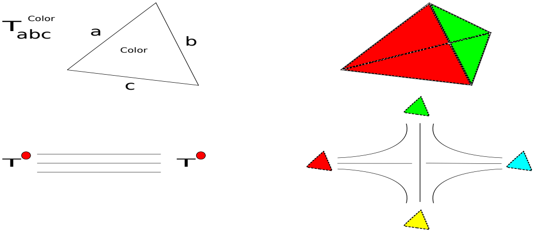


- Simplicial complex with boundaries.
- There is another vertex prescription in 3D [Tanasa, '10] the multi-orientable complex model.
- There exists for this prescription a polynomial invariant under "contraction/deletion" rules.

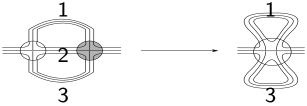
$$\mathrm{Ta}_{\mathcal{G}} = \sum_{A \in \mathcal{G}} X^{\mathrm{r}(\mathcal{G}) - \mathrm{r}(A)} Y^{n(A)} Z^{k(A) - bc(A) + n(A)} T^{2 \sum_{\mathrm{b}} g_{\mathrm{b}}} \qquad (11)$$

Colored Tensor Models

• '10 Gurau's $1/N$ expansion for colored TM [Gurau, AHP, '11]
3D:



• They admit a cellular homology and even a boundary cellular homology.
• There exists a polynomial for these colored graphs encoding even the boundary data. BUT the notion of contraction is dramatically modified (passive/active lines). $\sim$ Colored theory is not "stable" contraction:



• Gurau polynomial need a new contraction rule involving the so-called passive line.

Several questions, some answers

• Combinatorical vs Geom./Topological approaches. Combinatorics has a little advantage: you don't need to learn Hatcher's book! ☺

• ∃ Richer structures that are not seen or taking into account in the above formulation of the Krushkal and Renardy.

• Stranded Graphs ? Combinatoric approach is enough.

• The issue of contraction and deletion in a stranded/colored structures.

Today:

• Goal: Define stranded graphs and Introduce a new invariant for 3D colored simplicial complex with boundaries or colored stranded graphs.
• Method can be extended in any dimension.
• Upon reduction to simple/ribbon graphs, we reduce to Tutte/BR.

This is a compilation of works by Avohou R. Cocou, JBG, Etera Livine, M. Norbert Hounkonnou, S. Ramgoolam, R. Toriumi, arXiv:1301.1987, 1409.0398, 1310.3708, 1307.6490, 1212.5961.
Ackn: Bonzom, Gurau, Krajewski, Rivasseau, Tanasa.

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Stranded graph

Definition (Stranded vertex and edge)

• A rank D stranded vertex is a chord diagram that is a collection of $2n$ points on the unit circle (called the vertex frontier) paired by n chords, satisfying:

(a) the chords are not intersecting;

(b) the chords end points can be partitioned in sets called pre-edges with $0, 1, 2, \dots, D$ elements. These points should lie on a single arc on the frontier with no other end points on this arc;

(c) the pre-edges should form a connected collection that is, by merging all points in each pre-edge and by removing the vertex frontier, the reduced graph obtained is connected.

The coordination (also called valence or degree) of a rank $D > 0$ stranded vertex is the number of its non-empty pre-edges. By convention: (C1) we include a particular vertex made with one disc and assume that it is a stranded vertex of any rank made with a unique closed chord and (C2) a point is a rank 0 stranded vertex.

• A rank D stranded edge is a collection of segments called strands such that:

(a') the strands are not intersecting (but can cross without intersecting, i.e. cannot lie in the same plane);

(b') the end points of the strands can be partitioned in two disjoint parts called sets of end segments of the edge such that a strand cannot have its end points in the same set of end segments;

(c') the number of strands is D.

Stranded vertex and edge

- Rank D str. vertex and rank D str. edge

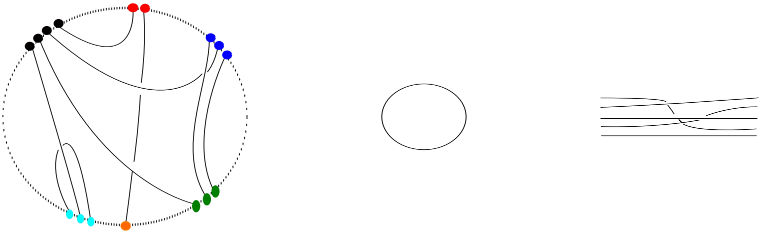

Figure: A rank 4 stranded vertex of coordination 6, with connected pre-edges (highlighted with different colors) with crossing chords; a trivial disc vertex; a rank 5 edge with non parallel strands.

- Rank $D > 0$ stranded vertices just enforce that any entering strand in the vertex should be exiting by another point at the frontier of the vertex.
- Degree or valence.

Stranded/Tensor graph

Definition (Stranded and tensor graphs)

- A rank D stranded graph $\mathcal{G}$ is a graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ which admits:

(i) rank D stranded vertices;

(ii) rank at most D stranded edges;

(iii) One vertex and one edge intersect by one set of end segments of the edge which should coincide with a pre-edge at the vertex frontier. All intersections of vertices and edges are pairwise distinct.

- A rank D tensor graph $\mathcal{G}$ is a rank D stranded graph such that:

(i') the vertices of $\mathcal{G}$ have a fixed coordination $D+1$ and their pre-edges have a fixed cardinal D. From the point of view of the pre-edges, the pattern followed by each stranded vertex is that of the complete graph K_{D+1};

(ii') the edges of $\mathcal{G}$ are of rank D.

- The notion of connectedness should be clarified.

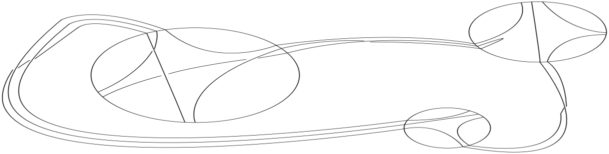

Figure: A rank 3 stranded graph.

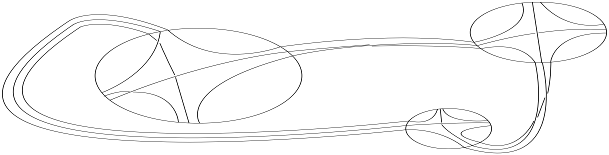

Figure: A rank 3 tensor graph with rank 3 vertices as with fixed coordination 4, pre-edges with 3 points linked by chords according to the pattern of K_4; edges are rank 3.

Lower rank reduction

- Rank 0 and 1 stranded/tensor graph are not very interesting.
- Ribbon graphs are one-to-one with particular rank 2 stranded graphs.

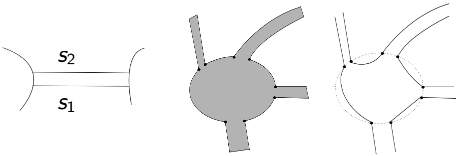

Figure: Ribbon edge and vertex of a ribbon graph as a rank 2 stranded edge and vertex of a stranded graph (the frontier vertex appears in dash).

Half-edges and cut

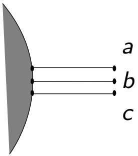

Figure: A rank 3 str half edge with its external points *a*, *b* and *c*.

- Cut of an edge

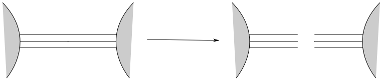

Figure: Cutting a rank 3 stranded edge.

- New category of graphs: Half-edge stranded graphs.

Contraction of a stranded edge

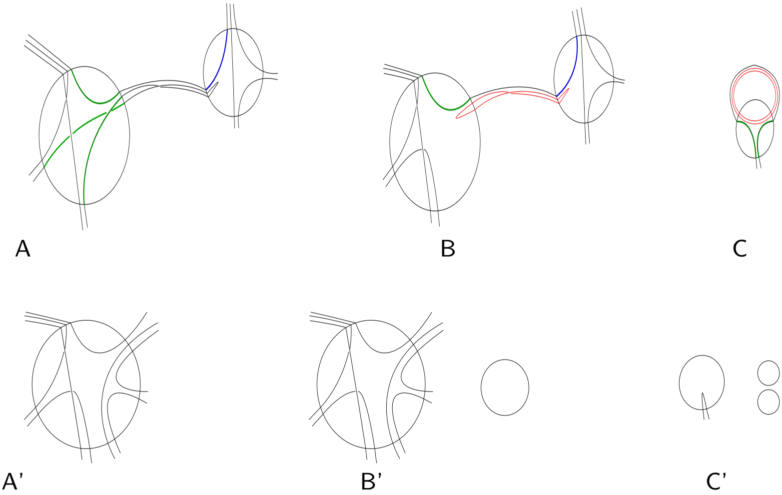

Figure: Graphs A',B' and C' obtained after edge contraction of A, B and C, respectively.

Colored tensor graphs.

Definition (Colored tensor graph)

A rank $D \geq 1$ colored tensor graph $\mathcal{G}$ is a graph such that:
- $\mathcal{G}$ is $(D+1)$ colored and bipartite;
- $\mathcal{G}$ is a rank D tensor graph.

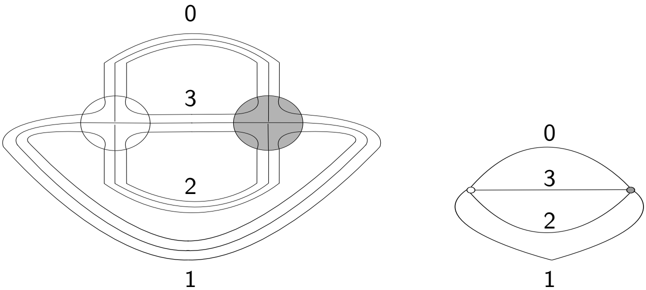

Figure: A rank 3 colored tensor graph and its compact representation.

Definition (p-bubbles, Gurau, '09)

Let $\mathcal{G}$ be a rank D colored tensor graph. A p-bubble is a connected component made with edges with p colors.

- 2-bubbles = faces of the graph. 3-bubbles = bubbles in $D = 3$;

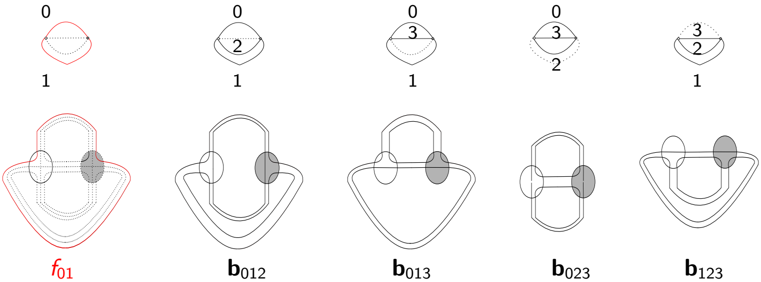

Figure: The face f_{01} and bubbles of the graph of the previous graph.

Open and boundary graphs

• Introduce colored stranded half edges: Open graphs

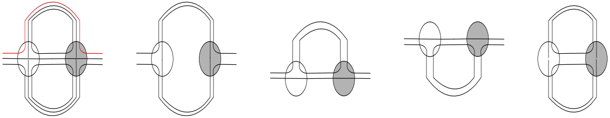

Figure: An open rank 3 colored tensor graph and its bubbles; f_{01} is an open face; $\mathbf{b}_{012}$ is an open bubble and $\mathbf{b}_{023}$ is closed.

• Boundary graph: $\partial\mathcal{G}(\mathcal{V}_\partial, \mathcal{E}_\partial)$ of a rank D HEcTG $\mathcal{G}$ is a graph encoding the boundary of the simplicial complex.

Weakly colored graphs

• Equivalence up to trivial discs: $\mathcal{G}_1 \sim \mathcal{G}_2$ if after removing their trivial discs they are isomorphic.

Definition (Rank D w-colored graph)

A rank D weakly colored or w-colored graph is the equivalence class (up to trivial discs) of a rank D half-edged stranded obtained by successive edge contractions of some rank D half-edged colored tensor graph.

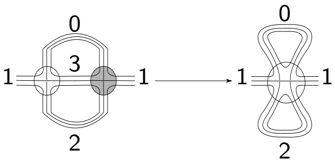

Figure: Contraction of an edge in a rank 3 HEcTG.

• Face are bi-colored, bubble tri-colored objects. This will allows to keep track of their numbers.

Hope you're still ok !

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A new invariant on rank 3 weakly-colored graphs [Avohou et al., '13]

• Consider $\mathcal{G}$ a weakly colored graph. Choose a subset of edges in $\mathcal{G}$, and define A a spanning cutting subgraph by cutting all the rest of edges in $\mathcal{G}$.

A new invariant

The following function is well defined on rank 3 W-colored graphs:

$$\mathfrak{T}_{\mathcal{G}}(X,Y,Z,S,W,Q,T) = \sum_{A\subset\mathcal{G}} (X-1)^{\mathrm{r}(\mathcal{G})-\mathrm{r}(A)} Y^{n(A)} \times$$
$$Z^{5k(A)-\left(3(V-E(A))+2(F_{\rm int}(A)-B_{\rm int}(A)-B_{\rm ext}(A))\right)} S^{C_\partial(A)} W^{F_\partial(A)} Q^{E_\partial(A)} T^{f(A)} . \qquad (12)$$

$k(A)$ its number of connected components, $F_{\rm int\,/ext}(A)$ its number of internal/external or closed/open faces, $B_{\rm int\,/ext}(A)$ its number of closed/open bubbles; $C_\partial(A)$ the number of connected component of the boundary of A, $F_\partial(A)$ the number of face of the boundary graph, $E_\partial(A) = F_{\rm ext}(A)$ the number of external faces or number of lines of the boundary graph, and $V_\partial(A) = f(A)$ the number of vertices of the boundary graph or number of half-edge of A.

• $5k(A) - (3V + 2F_{\rm int}(A))$ is independent of the representative of a w-colored stranded graphs.

Proposition

Let $\mathcal{G}$ be a representative of a w-colored graph. Then

$$\zeta(\mathcal{G}) = 3(E(\mathcal{G}) - V(\mathcal{G})) + 2[B_{int}(\mathcal{G}) + B_{ext}(\mathcal{G}) - F_{int}(\mathcal{G})] \geq -5D\,. \tag{13}$$

- Consider the set of closed and open bubbles $\mathcal{B}_{\rm int}$ and $\mathcal{B}_{\rm ext}$ of $\mathcal{G}$, with cardinal $B_{\rm int}$ and $B_{\rm ext}$, resp.
- for any $\mathbf{b}_i \in \mathcal{B}_{\rm int}$

$$2 - \kappa_{\mathbf{b}_i} = V_{\mathbf{b}_i} - E_{\mathbf{b}_i} + F_{{\rm int}\,;\mathbf{b}_i}\,, \tag{14}$$

where $\kappa_{\mathbf{b}_i}$ refers to the genus of $\mathbf{b}_i$ or twice its genus if $\mathbf{b}_i$ is oriented. Summing over all internal bubbles, we get

$$2B_{\rm int} - \sum_{\mathbf{b}_i \in \mathcal{B}_{\rm int}} \kappa_{\mathbf{b}_i} = \sum_{\mathbf{b}_i \in \mathcal{B}_{\rm int}} \Big[V_{\mathbf{b}_i} - E_{\mathbf{b}_i} + F_{{\rm int}\,;\mathbf{b}_i} \Big]\,. \tag{15}$$

Using the colors,

$$\sum_{\mathbf{b}_i \in \mathcal{B}_{\rm int}} E_{\mathbf{b}_i} + \sum_{\mathbf{b}_x \in \mathcal{B}_{\rm ext}} E_{\mathbf{b}_x} = 3E\,, \qquad \sum_{\mathbf{b}_i \in \mathcal{B}_{\rm int}} F_{{\rm int}\,;\mathbf{b}_i} + \sum_{\mathbf{b}_x \in \mathcal{B}_{\rm ext}} F_{{\rm int}\,;\mathbf{b}_x} = 2(F_{\rm int} - D)\,. \tag{16}$$

In addition, each vertex of the graph can be decomposed, at least, in three vertices (3 vertices is the minimum given by the simplest vertex of the form $\mathcal{G}_1$) which could belong to an open or closed bubble, we have

$$\sum_{\mathbf{b}_i \in \mathcal{B}_{\rm int}} V_{\mathbf{b}_i} + \sum_{\mathbf{b}_x \in \mathcal{B}_{\rm ext}} V_{\mathbf{b}_x} \geq 3(V - D)\,. \tag{17}$$

Combining (16) and (17), we re-write (15) as

$$3V - 3E + 2F_{\rm int} - 2B_{\rm int} - 5D - \sum_{\mathbf{b}_x \in \mathcal{B}_{\rm ext}} \big[V_{\mathbf{b}_x} - E_{\mathbf{b}_x} + F_{{\rm int}\,;\mathbf{b}_x} \big] \leq - \sum_{\mathbf{b}_i \in \mathcal{B}_{\rm int}} \kappa_{\mathbf{b}_i}\,. \tag{18}$$

We complete the last sum involving $\mathcal{B}_{\rm ext}$ by adding $C_\partial(\mathbf{b}_x)$ in order to get

$$\sum_{\mathbf{b}_x \in \mathcal{B}_{\rm ext}} \big[V_{\mathbf{b}_x} - E_{\mathbf{b}_x} + F_{{\rm int}\,;\mathbf{b}_x} + C_\partial(\mathbf{b}_x) \big] = \sum_{\mathbf{b}_x \in \mathcal{B}_{\rm ext}} (2 - \kappa_{\widetilde{\mathbf{b}}_x})\,, \tag{19}$$

which, substituted in (18), leads us to

$$3V - 3E + 2F_{\rm int} - 2B_{\rm int} - 2B_{\rm ext} - 5D \leq - \sum_{\mathbf{b}_i \in \mathcal{B}_{\rm int}} \kappa_{\mathbf{b}_i} - \sum_{\mathbf{b}_x \in \mathcal{B}_{\rm ext}} \big(C_\partial(\mathbf{b}_x) + \kappa_{\widetilde{\mathbf{b}}_x} \big) \tag{20}$$

from which the lemma results.

- Ccl: $5k(A) + \zeta(A) \geq 5k(A) - 5D \geq 0 \Rightarrow \mathfrak{T}_{\mathcal{G}}(X, Y, Z, S, W, Q, T)$ is a polynomial.

Contraction/Cut rule

Theorem (Contraction/cut rule for w-colored graphs)

Let $\mathcal{G}$ be a rank 3 w-colored graph. Then, for a regular edge e of any of $\mathcal{G}$, we have

$$\mathfrak{T}_{\mathcal{G}} = \mathfrak{T}_{\mathcal{G}\vee e} + \mathfrak{T}_{\mathcal{G}/e}\,, \tag{21}$$

for a bridge e, we have $\mathfrak{T}_{\mathcal{G}\vee e} = z^8 s(wq)^3 t^2 \mathfrak{T}_{\mathcal{G}/e}$ and

$$\mathfrak{T}_{\mathcal{G}} = [(x-1)z^8 s(wq)^3 t^2 + 1]\mathfrak{T}_{\mathcal{G}/e}\,; \tag{22}$$

for a trivial p-inner loop e, $p = 0, 1, 2$, we have

$$\mathfrak{T}_{\mathcal{G}} = \mathfrak{T}_{\mathcal{G}\vee e} + (y-1)z^{4p-7}\,\mathfrak{T}_{\mathcal{G}/e}\,. \tag{23}$$

- Again like the proof of the contraction/deletion of Tutte, one must prove that the subset of $\mathcal{G}$ divides into those which do not contains e (these will involve $\mathfrak{T}_{\mathcal{G}\vee e}$) and those which do (involving $\mathfrak{T}_{\mathcal{G}/e}$).
- Reduced to $T_{\mathcal{G}}$ but not "naively" to BR;
- Several reductions.

$$\mathfrak{T}_{\mathcal{G}}(x,y,z,z^{-2}s^2,s^{-1},s,s^{-1}) = \mathfrak{T}''_{\mathcal{G}}(x,y,z,s) \quad \text{Euler characterics for the boundary}$$
$$\mathfrak{T}_{\mathcal{G}}(x,y,z,z^2z^{-2},z^{-1},z,z^{-1}) = \mathfrak{T}'''_{\mathcal{G}}(x,y,z) \quad \text{Combine both in a single exponent}\,.$$

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Open questions

- Tensor graphs have a rich combinatorics.
- The coloring allows to keep track of most of the topological ingredients, like *p*-bubbles or *p*-cells.
- We have identify a rank 3 invariant on colored simplicial complex. Is it universal ?
- A Rank 2 invariant extending Bollobàs-Riordan to half-edged ribbon graphs:

$$\mathcal{R}_{\mathcal{G}}(x,y,z,s,w,t) = \sum_{A \Subset \mathcal{G}} (x-1)^{\mathrm{r}(\mathcal{G})-\mathrm{r}(A)} (y-1)^{n(A)} z^{k(A)-F_{\mathrm{int}}(A)+n(A)} s^{C_\partial(A)} w^{o(A)} t^{f(A)}, \tag{24}$$

where $C_\partial(A)$ is the number of connected component of the boundary of A.

• $\mathcal{R}_{\mathcal{G}}(x,y,z,z^{-1},w,t=1) = R_{\mathcal{G}}(x,y,z,w)$

• Higher D ?

• What do those objects count ? (Tutte and BR counting specific tree in the graphs).

Thank You for Your Attention!

The meaning of $\partial\mathcal{G}$ for half-edge ribbon graphs (HERG)

• Cellular embedding: A HERG is half-edge graph $\mathcal{G}$ "cellularly embedded" in a surface Σ with punctures in the following sense:
- Remove all half-edges from $\mathcal{G}$ we get $\mathcal{G}'$ which is then cellularly embedded in Σ such that each connected component of $\Sigma \setminus \mathcal{G}'$ is homeomorphic either to a disc or to discs with holes;
- Each of the half-edges of $\mathcal{G}$ is embedded in Σ and ends on a different puncture (but can be on the same boundary circle).

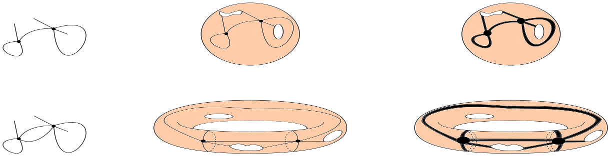

Figure: A half-edged graph, some possible cellular embeddings in punctured surfaces (a sphere top, a torus bottom) and corresponding HERGs (in bold).

- A closed face f corresponds to a component homeomorphic to a disc or to a disc with holes in $\Sigma \setminus \mathcal{G}$ such that $\partial\Sigma \cap f = \emptyset$. If $\partial\Sigma \cap f \neq \emptyset$, then it is an external face.
- An additional closed face introduced by the pinching corresponds to a component homeomorphic to a disc or to a disc with holes only after capping off some punctures in Σ. Thus pinching a HERG corresponds exactly to capping some puncture in Σ.

Multivariate form

The multivariate form associated with $\mathfrak{T}$ is defined by:

$$\widetilde{\mathfrak{T}}_{\mathcal{G}}(x, \{\beta_e\}, \{z_i\}_{i=1,2,3}, s, w, q, t) \quad (25)$$
$$= \sum_{A \Subset \mathcal{G}} x^{\mathrm{r}\,(A)} (\prod_{e \in A} \beta_e) z_1^{F_{\rm int}\,(A)} z_2^{B_{\rm int}\,(A)} z_3^{B_{\rm ext}\,(A)}\, s^{C_\partial(A)}\, w^{F_\partial(A)} q^{E_\partial(A)} t^{f(A)}\,,$$

for $\{\beta_e\}_{e \in \mathcal{E}}$ labeling the edges of the graph $\mathcal{G}$.

- Relation with Gurau polynomial but only on rank 3 stranded colored graph:

$$\widetilde{\mathfrak{T}}_{\mathcal{G}}(x, \{\beta_e\}, z_1, z_2, z_3 = 1, s, w, q, t) = G_{\mathcal{G}}(x, \{\beta_e\}, z_1, z_2, s, q, w, t)\,, \quad (26)$$