Combinatorial state sum invariant from categories

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  - 3D: Ponzano-Regge Rep(SU(2)) , Turaev-Viro Rep($U_q(\mathfrak{su}(2))$)
  - 4D: Dijkgraaf-Witten (finite groups), Ooguri Rep(SU(2)), Crane-Yetter Rep($U_q(\mathfrak{su}(2))$)
  - Models of 4d quantum gravity: Barrett-Crane, EPRL-FK

Constructing manifold invariants:

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- Combinatorial construction of manifold invariants, TQFT's
- Models of quantum geometry:
  - Triangulation independent models of quantum geometry ?
    Issue tied to diffeomorphism symmetry
  - 'Metric' models: explicit data on the edges of the triangulation ?

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  2. Going up dimensions: algebra elements → objects in a monoidal category
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'Categorification' boosts dimensions

- Going up to 4 dimensions: monoidal 2-categories Mackaay '99

Need examples...

Introduction

Which 2-category?

New set of examples: Barrett, Mackaay '04

Representation 2-category of a 2-group

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Representation 2-category of a 2-group

- Finite dimensional representations on **2-Vect**: Barrett, Mackaay '04
- Infinite dimensional representations on 'measurable categories' **Meas**: Crane, Yetter '04; Baez, AB, Freidel, Wise '08

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- Possible relevance for models of quantum geometry: 2-Poincaré group Barrett, Mackaay '04, Crane, Sheppeard '04

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Explicit model ? ...

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2. Explicit example of a state sum model using the 2-category representation of the 'Euclidean 2-group'
   AB, Freidel '11; AB, Wise '09; AB '11
3. The model shows up in a combinatorial (state sum) reformulation of the Feynman graph amplitudes in ordinary QFT on flat Euclidean spacetime
   AB, Freidel '06

Outline

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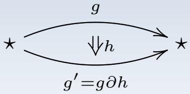

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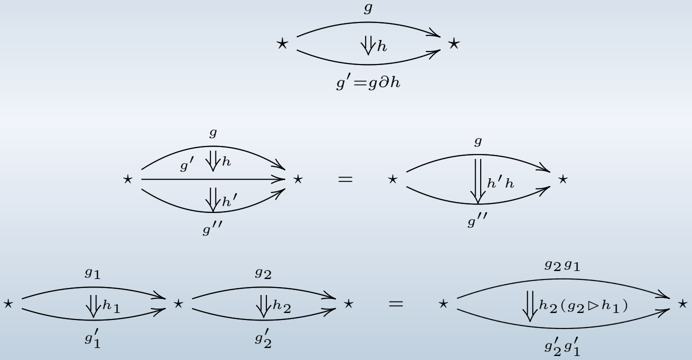

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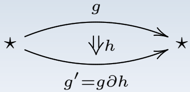
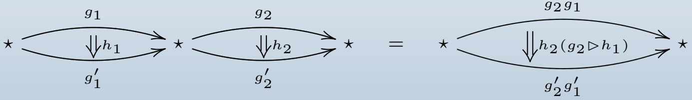
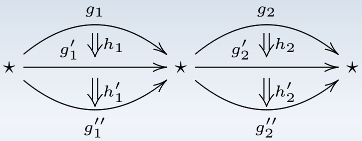




2-group data: 'crossed module' $(G, H, \triangleright, \partial)$

$$\partial(g \triangleright h) = g\partial(h)g^{-1} \qquad \partial(h) \triangleright h' = hh'h^{-1}$$



- 2-groups allows to define 'surface holonomies' on discretized surfaces.
- Algebraic structure underlying 'higher gauge theory' see Baez, Huerta '09

Example: the Poincaré 2-group

- $G = \mathrm{SO}_0(3,1)$: the group of Lorentz transformations,
- $H = \mathbb{R}^4$: the group of translations of Minkowski space,
- the obvious action of $\mathrm{SO}_0(3,1)$ on $\mathbb{R}^4$.
- $\partial = 1$ (source = target)

Groups are usually represented in the category on vector spaces.

- A representation is a functor $\rho\colon G \to \mathrm{Vect}$.

$$\rho(*) = V, \qquad V \xrightarrow{\rho(g)} V$$

- An intertwiner is a natural transformation:

$$\begin{array}{ccc} V_1 & \xrightarrow{\rho_1(g)} & V_1 \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi} \\ V_2 & \xrightarrow[\rho_2(g)]{} & V_2 \end{array}$$

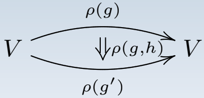
Group representations and intertwiners between these form a monoidal category.

2-group representations
From groups to 2-groups

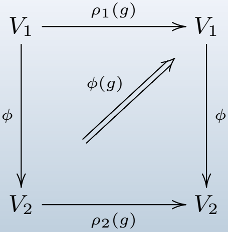
A 2-group $\mathcal{G}$ may be represented on suitable '2-vector spaces'.

- A representation is a '2- functor' $\rho\colon \mathcal{G} \to \mathbf{2Vect}$

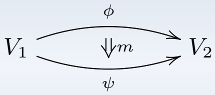
$$\rho(*) = V,$$



- An intertwiner is a 'pseudo-natural transformation'.



- Novelty: there are also 2-intertwiners between interwiners:



Representations of a given 2-group, intertwiners and 2-intertwiners between these, form a monoidal 2-category.

2-group representations

2-vector spaces: a flavor

ordinary linear algebra	higher linear algebra
$\mathbb{C}$	Vect
$+$	$\oplus$
$\times$	$\otimes$
0	$\{0\}$
1	$\mathbb{C}$

Kapranov and Voevodsky '94

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    $TV_m = \bigoplus_n T_{m,n} \otimes V_n$

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- Infinite dimensional Crane, Yetter '03; Baez, AB, Freidel, Wise '08
  - 2-Hilbert space $\mathcal{H}^X$
  - '1-map' $T\colon \mathcal{H}^X \to \mathcal{H}^Y$: field of Hilbert spaces $T_{y,x}$ + measures μ_y on X
  - '2-map' $\alpha_{y,x}\colon T_{y,x} \to T'_{y,x}$

Let's focus on the Euclidean 2-group $\mathcal{E}$:

$$G = \mathrm{SO}(4),\ H = \mathbb{R}^4,\ \triangleright,\ \partial = 1$$

A representation $\quad H^X \underset{\rho(g)}{\overset{\rho(g)}{\Downarrow \rho(g,h)}} H^X \qquad G = \mathrm{SO}(4),\ H = \mathbb{R}^4$

is determined by:

- a space X with an action of SO(4): $\quad \rho(g)\mathcal{H}_x = \mathcal{H}_{gx}$

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- an SO(4)-equivariant map $\chi\colon X \to H^* \sim \mathbb{R}^4$:

$$\rho(1,h)\colon \mathcal{H}_x \to \mathcal{H}_x, \qquad \varphi_x \mapsto e^{i\chi(x).h}\varphi_x$$

Representations of 2-groups

Euclidean 2-group: representations

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Irreducible representations labelled by $\ell \in \mathbb{R}_+$

Tensor product of two representations gives the bundle:

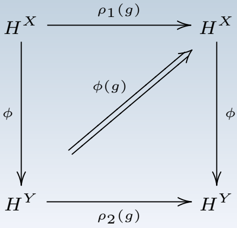
$$\begin{array}{c} X_1 \times X_2 \\ \Big\downarrow {\scriptstyle \chi_1+\chi_2} \\ H^* \end{array}$$

determined by:

- the direct product $X_1 \times X_2$ with diagonal action: $g(x_1, x_2) := (gx_1, gx_2)$
- the SO(4)-equivariant map $(x_1, x_2) \mapsto \chi_1(x_1) + \chi_2(x_2)$

Tensor product of Irreps: $x_i \in \mathbb{R}^4, |x_i| = \ell_i, \qquad (x_1, x_2) \mapsto x_1 + x_2$

An intertwiner between two reps (X, χ) and (Y, ξ):



$$\phi\mathcal{H}_y = \int^{\oplus} \mathrm{d}\mu_y(x) V_{y,x} \otimes \mathcal{H}_x, \qquad \Phi^g_{y,x} \colon V_{y,x} \to V_{g(y,x)}$$

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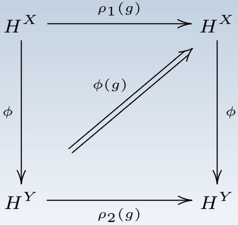

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- a SO(4)-Hilbert bundle (V_z, Φ^g_z) over the pullback $Z = \{(y,x) \in Y \times X \;:\; \chi(x) = \xi(y)\}$

$$\Phi^g_z \colon V_z \longrightarrow V_{gz}$$

$$\Phi^{gg'}_z = \Phi^g_{gz} \circ \Phi^g_z \qquad \text{(cocycle)}$$

Representations of 2-groups

Euclidean 2-group: intertwiners

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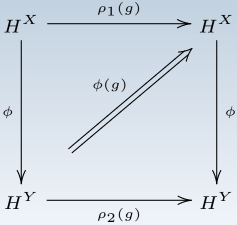

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- a SO(4)-equivariant family of measures μ_y supported on Z.

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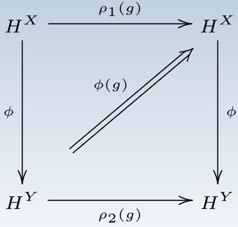

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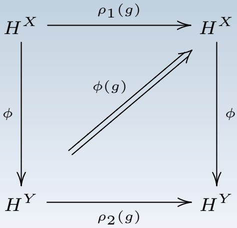

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Remark: $(V_z, \Phi_z^g)_{g \in G_z}$ representation of stabilizer $G_z \subset \mathrm{SO}(4)$ of z.

Mackey's induced representation theory

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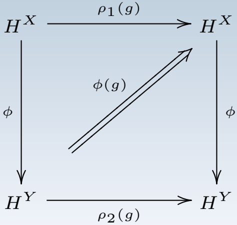

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Irreducible intertwiners: representation of stabilizer G_o irreducible

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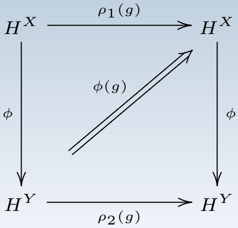

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Mackey's induced representation theory

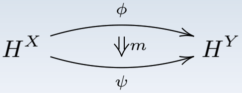
When Z is a transitive space:

Irreducible intertwiners: representation of stabilizer G_o irreducible

Irreducible intertwiners labelled by an irreducible group representation of stabilizer G_o

Two intertwiners $\phi = (V_z, \Phi_z^g), \psi = (W_z, \Psi_z^g)$ give two SO(4)-Hilbert bundles over $Z = \{(y,x) \in Y \times X \,:\, \chi(x) = \xi(y)\}$.

A 2-intertwiner:



is determined by **a map of SO(4)-Hilbert bundle**:

$$m_z \colon V_z \to W_z, \qquad \Psi_z^g \circ m_z = m_{gz} \circ \Phi_z^g$$

Two intertwiners $\phi = (V_z, \Phi_z^g), \psi = (W_z, \Psi_z^g)$ give two SO(4)-Hilbert bundles over $Z = \{(y,x) \in Y \times X \,:\, \chi(x) = \xi(y)\}$.

A 2-intertwiner:

$$H^X \underset{\psi}{\overset{\phi}{\rightrightarrows}} H^Y \qquad (\Downarrow m)$$

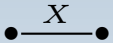
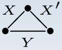
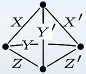
is determined by **a map of SO(4)-Hilbert bundle**:

$$m_z \colon V_z \to W_z, \qquad \Psi_z^g \circ m_z = m_{gz} \circ \Phi_z^g$$

- each m_z intertwines the representations of the stabilizer G_z of z.
- m defines an intertwiner between the two induced representations of SO(4) defined by ϕ, ψ.

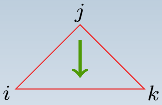
Outline

The model
Building up the categorical state sum

	irreducible representation
	irreducible intertwiner from $X \otimes X'$ to Y
	2-intertwiner 

The model

Building up the categorical state sum



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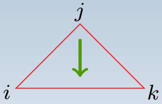
Building up the categorical state sum



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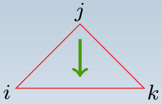

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   - Base space: $\mathcal{T}_{ijk} = \{(x_{ij}, x_{jk}, x_{ik}) \in (\mathbb{R}^4)^3 \ : \ |x_{ij}| = l_{ij}, \ x_{ij} + x_{jk} = x_{kl}\}$
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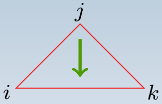

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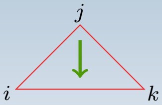
The model

Building up the categorical state sum



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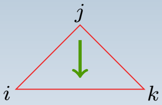

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   - unique up to equivalence $\rightarrow$ this is a normalization choice.

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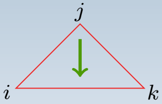

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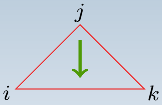

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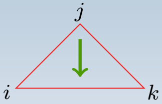

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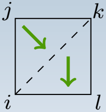
Representation of $\mathrm{SO}(4)$ on $\int^{\oplus} \mathrm{d}\mu(\triangle) V_{\triangle}$ induced by the U(1) representation s.

The model

Building up the categorical state sum

- Gluing Triangles :



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The model

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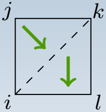

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The model
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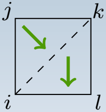

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The model

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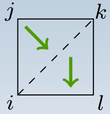

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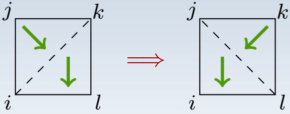
   - use $\mathrm{d}\mu = \mathrm{d}_{l_{il}} x_{il} \otimes \mathrm{d}\mu_{x_{il}}$ to get a representation of $\mathrm{SO}(4)$ on $\int^{\oplus} \mathrm{d}\mu(\boxslash) V_{\triangledown} \otimes V_{\triangle}$

- for each tetrahedron: a 2-intertwiner



$$\Phi^g_{\text{◸}}\Phi^g_{\text{◿}}\, m_{\boxtimes} = m_{g\boxtimes}\, \Phi^g_{\text{◺}}\Phi^g_{\text{◹}}$$

$$\boxtimes \in \mathrm{T}_{ijkl} = \mathcal{Q}^{ik}_{ijkl} \cap \mathcal{Q}^{jl}_{ijkl}$$

The model

Building up the categorical state sum

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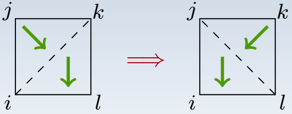

$$\Phi^{g}_{\triangledown}\Phi^{g}_{\lrtriangle}\, m_{\boxtimes} = m_{g\boxtimes}\, \Phi^{g}_{\llcorner}\Phi^{g}_{\triangledown}$$

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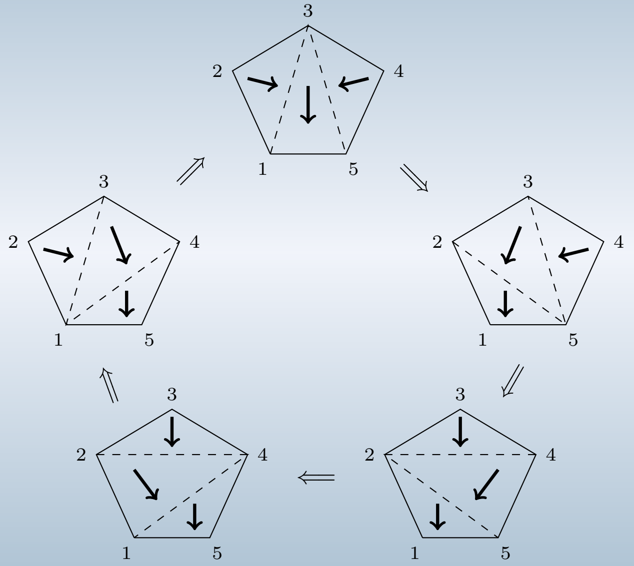
- m fully determined by the value $m_{\boxtimes^o}$ on a given 'reference tetrahedron' $\boxtimes^o \in \mathrm{T}_{ijkl}$: $\boxtimes^o$, $m_{\boxtimes^o}$ normalization choices

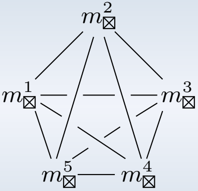
$$m_{g\boxtimes^o} = \Phi^{g}_{\triangledown}\Phi^{g}_{\lrtriangle} m_{\boxtimes^o} (\Phi^{g}_{\llcorner}\Phi^{g}_{\triangledown})^{-1}$$

The model

Simplex weight: '20j-symbol'



$$\int \prod_{(ij)\text{ext.}} \mathrm{d}^4 x_{ij} \prod_{(ij)} \delta_{l_{ij}}(|x_{ij}|)$$



The model is written formally as

$$Z_{\Delta} = \int_{\mathbb{R}_+} \prod_e l_e \mathrm{d}l_e \sum_{s_t \in \mathbb{N}} W_{\Delta}(l_e, s_t)$$

where

$$W_{\Delta}(l_e, s_t) = \prod_t \mathcal{A}_t(l_e) \prod_{\sigma} \frac{\cos S_{\sigma}(l_e, s_t)}{\mathcal{V}_{\sigma}(l_e)}$$

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- $\mathcal{A}_t(l_e)$ area triangle t with edge lengths l_e
- $\mathcal{V}_\sigma(l_e)$ volume 4-simplex with edge lengths l_e
- $S_\sigma(l_e, s_t)$ first order Regge action:

$$S_\sigma(l_e, s_t) := \sum_{t \subset \sigma} s_t \phi_t^\sigma(l_e), \qquad \phi_t^\sigma \text{ dihedral angle } t \subset \sigma$$

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$$\sum_{\epsilon_1,\epsilon_2,\epsilon_3} \sum_{s_{045}} \mathcal{A}_{045} \frac{e^{\imath\epsilon_1 S_1}}{\mathcal{V}_1} \frac{e^{\imath\epsilon_2 S_2}}{\mathcal{V}_2} \frac{e^{\imath\epsilon_3 S_3}}{\mathcal{V}_3} = \sum_{\epsilon_0,\epsilon_4,\epsilon_5} \sum_{s_{123}} \mathcal{A}_{123} \frac{e^{\imath\epsilon_0 S_0}}{\mathcal{V}_0} \frac{e^{\imath\epsilon_4 S_4}}{\mathcal{V}_4} \frac{e^{\imath\epsilon_5 S_5}}{\mathcal{V}_5}$$

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Invariance under Pachner moves

Outline

'Quantum flat space': A.B, Freidel '06

Feynman amplitudes QFT on 4d Euclidean space-time:

$$I_\Gamma = \int_{\mathbb{R}^D} \mathrm{d}^D x_1 \cdots \mathrm{d}^D x_n \prod_{(ij)\in\Gamma} G^F(\vec{x}_i - \vec{x}_j) \tag{1}$$

formulated as background free as state sums:

$$I_\Gamma = \int_{\text{gauge fixing}} \prod_{e\in\Delta} \ell_e d\ell_e \sum_{\{s_t\}} W_\Delta(s_t, \ell_e) \prod_{e\in\Gamma} G(\ell_e) \tag{2}$$

$$I_\Gamma = \int_{\mathbb{R}^D} \mathrm{d}^D x_1 \cdots \mathrm{d}^D x_n \prod_{(ij)\in\Gamma} G^F(\vec{x}_i - \vec{x}_j) \tag{3}$$

Limit of quantum gravity amplitudes?

$$\widetilde{I}_\Gamma := \int \mathcal{D}g \, I_\Gamma(g) \, e^{iS_{\mathrm{grav}}[g]} \quad \longrightarrow_{G_N \to 0} \quad I_\Gamma$$

State sum structure of I_Γ may tell us something about the structure of the quantum gravity amplitude

Outline

Conclusion
Summary

- Explicit example state sum model based on the representation 2-category of the 'Euclidean 2-group'

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- Formal invariance under Pachner moves: state sum invariant.
- Models flat space: shows up in state sum formulation of QFT Feynman amplitudes

- Conjecture: related to higher gauge theory functional integral

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- Topological invariance: generic feature of 2-group state sum models? Can one get interesting invariants out of this?
- Algebraic ways to think of deformations of field theory structure
- Basis for gravity models?