Cells in the box and a hyperplane

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Fact: a line intersects at most $2n - 1$ cells (squares) of the $n \times n$ chessboard. (ger (pr3))
Question: How many cells of the $n \times \ldots \times n$ chessboard can a hyperplane intersect?

$d = 3 \quad M_n = \text{max number of cells in } \mathbb{R}^2$

Thus 1. $M_n = \frac{9}{4} n^2 + O(n)$

(higher dim later)
More precisely

\[ M_n \leq \frac{9}{4} n^2 + 2n + 1 \]

\[ M_n \geq \frac{9}{4} n^2 + n - \left\{ \begin{array}{l}
5 \quad \text{if } n \text{ is odd} \\
4 \quad \text{if } n \text{ is even}
\end{array} \right. \]

\[ M_2 = 7, \quad M_3 = 19, \quad M_4 = 35 \]

\[ 230 \leq M_{10} \leq 246 \]
$K_n = [0,n]^3$, $C(z) = \begin{array}{c} z \\ z \in \mathbb{Z}^3 \end{array}$ unit cube (cell)

$P$ a plane with equation $ax + by + z = d$

$0 < a < b < 1$

**Lower bound**

$m = \frac{3n}{2}$ (n even) = $\frac{3n-1}{2}$ (n odd)

$P = \sum \begin{array}{c} x+y+z=m+\varepsilon \end{array} (\varepsilon > 0 \, \text{small})$

interests $\frac{9n^2-8}{4}$ (n even) $\frac{9n^2-5}{4}$ (odd) cells
Proof. \( \# \{ (x, y, z) \in \mathbb{Z}^3 \mid 0 \leq x, y, z \leq a - 1 \} \) then \( x + y + z = m, m-1, m-2 \).

because \( x + y + z < m + \varepsilon \) and \( x + 1 + y + 1 + z + 1 > m + \varepsilon \).
Upper bound \( ax + by + cz = d \) is the maximizer plane \( P \).

wlog \( 0 < a < b < c = 1 \)

Claim 1. \( a + b > 1 \).

Otherwise \( a + b \leq 1 \) and

\( P \) intersects at most 2 cells in a stack.
\[ F_i = \{(x,y,i) \in \mathbb{K}_n, i \in \mathbb{Z} \} \quad \text{"floor"} \]

\[ L_i = P \cap F_i \]

\[ \overline{L}_i: \text{its projection to } F_0 = \mathbb{Q}_n \]

\[ L_i = \emptyset \quad \text{possible but} \]

Claim 1 implies that either

\[ L_i \neq \emptyset \quad \text{or} \quad L_p \neq \emptyset \quad \text{or} \quad \text{both}. \]
Assume \( L_n \neq \emptyset \). Then

\[ L_0, \ldots, L_n \neq \emptyset \quad \text{and} \quad L_0 = \ldots = L_{p-1} = \emptyset \]

\( l_i \) is the length of \(( L_i \) and of \( \overline{L_i} \))

\[ m_i = \# \text{ cells of } \overline{Q_h} \text{ hit by } \overline{L_i} \]

\[ m = \# \text{ cells of } \overline{Q_h} \text{ intersect } \overline{P} \]
Lemma  \[ \# \text{cells hit by } P = m + m_{p+1} + \ldots + m_{n-1} \]

Proof: count \( P \cap C(z) \) on the bottom face of \( C(z) \) if \( P \) hits the bottom face.

If it does not, then \( P \cap C(z) \) is counted in \( m \)
Upper bound on $M_n$

$$m \leq \text{Area } \overline{B} + m_p + m_n$$

$$\text{Area } \overline{B} = \sum_{i=p}^{n-1} h \frac{l_{i+1} + l_i}{2} + \ell_i$$

$$m_i \leq \frac{a+b}{\sqrt{a^2+b^2}} \ell_i + 1 = \frac{(a+b) h l_i + 1}{h + 1}$$

$$h = \frac{1}{\sqrt{a^2+b^2}}$$

... leads to...
Lemma. If $0 \leq a \leq b \leq 1$ then

$$(a + b + 1) \left( 1 - \frac{(a + b - 1)^2}{4ab} \right) \leq \frac{9}{4},$$

equality iff $a = b = 1$. 

stability
\[ d \geq 3 \]

\[ M^d_n = \text{max # of cells in } K^n_h = [C/n]^d \]

that a (gen. pos.) hyperplane intersects

\[ M^2_n = 2n - 1 \]

\[ M^3_n = \frac{9}{4} n^2 + O(n) \]

\[ d \geq 3 \]
Let $v \in \mathbb{R}^d$ be a unit vector with $\|v\|_2 = 1$. $P_v$ is the hyperplane orthogonal to $v$ and containing the center of $[0,1]^d$.

Define

$$V_d = \max_{\|v\|_2 = 1} \text{Vol}_{d-1}([0,1]^d \cap P_v)$$

1 \leq \text{Vol}_{d-1}([0,1]^d \cap P_v) \leq \sqrt{2} \quad \text{(K. Ball)}$$

which implies

$$\sqrt{d} \leq V_d \leq \sqrt{2d}$$

but
Thus (I. Aliëv, 2020) the maximum is attained on \( v = \frac{1}{\sqrt{d}} (1, \ldots, 1). \)

\[
V_2 = 2, \quad V_3 = \frac{9}{4}, \quad V_4 = \frac{8}{3}, \quad \ldots \quad \text{increasing}
\]

\[
V_d \rightarrow \sqrt{\frac{6d}{11}}
\]
Theorem 2.

\[ M_n^d = V_d n^{d-1} (1 + o(1)) \]

\[ M_n^d(v) = \max \# \text{ of lattice points in } K_n \text{ between two hyperplanes orthogonal to } v \text{ and at distance } \|v\|, \]

\[ S(v) \text{ is the part of } K_n \text{ between these hyperplanes \ldots} \]
Alternative definition:

\[ M^d_n = \max \{ M^d_n(v) : \|v\|_2 = 1 \} \]

Here, \( M^d_n(v) \) should be

\[ M^d_n(v) = \|v n_{\text{vol}} \cap \left([0, \frac{d}{2}] \cap P_n \right) n^{d-1} \| + o(1) \]

\[ \approx \text{vol} \ S(v) \]
because of a metamathemem:

for every $K \neq 2^n \backslash K \approx n_{0K}$

valid when $K$ is well pointed, that is, when $n_{0K}$ is large and $n_{0K}$, $ld K$ is small

**BUT:** this is not the case

$S(v)$ is a very thin slice
$K \subset \mathbb{R}^d$ convex body, $C(z) (z \in \mathbb{Z}^d)$ cell is

- inside if $C(z) \subset K$
- outside if $C(z) \cap K = \emptyset$
- empty otherwise
\[ U \cap \alpha < K < U \cap \beta \text{ inside, inside, inside or interior} \]

\[ \# \mathcal{C}(\beta) \leq \# \mathcal{K} \leq \# \mathcal{C}(\beta) \]

\[ \# \mathcal{C}(\alpha) \leq \# \mathcal{K} \cap \mathbb{Z}^d \leq \# \mathcal{C}(\beta) \text{ inside or interior} \]
Thus $A \mid \nu K - |K \cap \mathbb{Z}^d| \mid \leq \# \text{ldg cells}$

Given a basis $F=\{f_1, \ldots, f_n\}$ of $\mathbb{Z}^d$, an $F$-cell is a basic parallelootope in basis $F$.

Thus $A^* \mid \nu K - |K \cap \mathbb{Z}^d| \mid \leq \# \text{ldg } F\text{-cells}$
**surprise:**

Thus if \( B \subseteq K \cap L \) convex bodies in \( \mathbb{R}^d \), \( K \subseteq L \)

\[ \Rightarrow \# \text{bdry cells of } K \leq \# \text{bdry cells of } L \]

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**a lattice analogue of**

\[ \text{vol}_{d-1} \partial K \leq \text{vol}_{d-1} \partial L \]
Proof is easy in 2-dim in $\mathbb{R}^d$ a homotopy argument works.
Next ingredient: Given a basis $F = \{f_1, \ldots, f_n\}$ of $\mathbb{Z}^d$ an $F$-box is

$$B(\alpha, \beta, F) = \left\{ x = \sum_{i=1}^{d} x_i f_i : \alpha_i \leq x_i \leq \beta_i, \quad \forall i \right\}$$

$$B(K, F) = \min F\text{-box containing } K$$
Thus $C$ (B. Vershik '92) for every convex body $K \subset \mathbb{R}^d$ 

exists a basis $F$ of $\mathbb{Z}^d$ such that

$vol B(K, F) \ll_d vol K$

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Corollary $K \subset \mathbb{R}^d$ convex, $F$ a basis. Then

$\# \text{1-dim } F\text{-cells of } K \leq \# \text{1-dim } F\text{-cells of } B(K, F)$
Advantage: determining the
\# lying F-cells of \( b(K, f) \) is easy:

\[
2 (\beta_1 - \alpha_1) (\beta_2 - \alpha_2) + 2 (\beta_2 - \alpha_3) (\beta_3 - \alpha_1) + 2 (\beta_3 - \alpha_3) (\beta_1 - \alpha_1)
\]
(*) \[ k \cap \mathbb{Z} \] contains at least 1 independent point

\[ B(k_1 F) = B(\alpha, \beta, F) \quad \alpha_i < \beta_i \quad \gamma_i = \beta_i - \alpha_i \geq 1 \]

\[ \text{# Ady F-cells of } B(k_1 F) \approx \left( \prod_{i=1}^{d} \frac{1}{\gamma_i} \right) \left( \frac{1}{\gamma_1} + \ldots + \frac{1}{\gamma_d} \right) \]

This is not \[ B(k_1 F) \]
Theorem 3. \( K \subseteq \mathbb{R}^d \) convex, \( \exists \) a basis \( F \) s.t.

\[
\left| \text{vol } K - |K \cap \mathbb{Z}^d| \right| \ll_d \text{vol } K \left( \frac{1}{\delta_1} + \cdots + \frac{1}{\delta_d} \right)
\]

where \( \delta_1, \ldots, \delta_d \) come from the minimal box \( B(K,F) \).
$K$ convex in $\mathbb{R}^d$, $\Lambda$ a lattice in $\mathbb{R}^d$, $K$ satisfies (*) with $dt$ prob in $\Lambda \implies$

Theorem 4. Exist F of $\Lambda$ such that

$$\left| \frac{1}{\det \Lambda} \text{vol } K - |K \cap \Lambda| \right| \ll \frac{1}{\det \Lambda} \text{vol } K \left( \frac{1}{\gamma_1} + \ldots + \frac{1}{\gamma_d} \right)$$

where $\gamma_1, \ldots, \gamma_d$ come from the minimal box $B(K, F)$. 
downward on $M_d^n$ via $M_d^n(z)$ with $z \in \mathbb{R}^d$ fixed. $M_d^n(e_1) = n^{d-1} \Rightarrow M_d^n \geq n^{d-1}$.

$M_d^n(z)$ is reached on $\|z\|_2$ concentric lattice hyperplane $\perp$ to $z$, in the lattice $L < \mathbb{R}^d$ with $\det L = \|z\|_2^d$. Then 4 applies (in $\mathbb{R}^{d-1}$) with $C = [0,1]^d \cap P(\sigma,t)$ ($P(\sigma,t) = \{ x \in \mathbb{R}^d : z \cdot x = t \}$).
\[
\left| \frac{1}{\|z\|_2} \text{me } K - \left| C \varphi^q \right| \right| \leq \frac{1}{\|z\|_2} \text{vol } C \left( \frac{1}{\varphi_1} + \cdots + \frac{1}{\varphi_{d-1}} \right) \\
= O(n^{a-2})
\]

and \( \text{vol}_{d-1} C = n^{a-1} \text{vol}_{d-1} P(z) \)

\[
\Rightarrow \left| u_h^{d}(z) \right| \geq \frac{\|z\|_1}{\|z\|_2} \text{me}_{d-1} P(z) n^{a-1} \left( 1 + O\left( \frac{1}{n} \right) \right)
\]

\[
V_d(z) \rightarrow V_d
\]

\( z = (1, 1, \ldots, 1) \ldots \)
Upper bound

thin slice

target

\[ |S(v, t) \cap \mathbb{Z}^d| \leq (V_d + \varepsilon) n^{d-1} \]

maximiser

\( \varepsilon > 0 \text{ fixed} \)

\[ S(v, t) = S(v_n, t_n) = S_n \]
Consider the basis $F = \{f_1, \ldots, f_n\}$ from Thm B and the minimal box $B(S_n, F)$ ($F = F_n$).

By Thm 4

$$\left| \text{vol } S_n - |S_n \cap \mathbb{Z}^d| \right| \ll \text{vol } S_n \left( \frac{1}{\delta_1} + \ldots + \frac{1}{\delta_d} \right)$$

where $\delta_1, \ldots, \delta_d \geq 1$ are integers, $\delta_i = \delta_i(n)$.
Simple case  \( \gamma_i(n) \to \infty \quad \forall i \in [d] \)

If not, some \( \gamma_i(n) \) is \underline{bounded} along a subsequence, then the corresponding dual basis vector \( g_i(n) \) is \underline{fixed} along another subsequence. \( i=1 \text{ to } d \).

So  \( \gamma = \gamma_1(n) = \text{const} \quad \text{and} \quad g = g_1(n) = \text{const}. \)

\[ p_n = P := \text{span} \{ e(v(n)), g \} \quad \text{2-dim plane} \]
Project $K_n$ and $S_n$ to $P$

$$\Pi_n = \Pi : \mathbb{R}^d \rightarrow P$$
\[ \phi = \phi_n \text{ tends to zero} \]

\[ J_i^* = \frac{J_i}{\sqrt{2a}} \text{ deleted} \]

**Lemma** A vertical line intersects at most \(|g| + 1\) segments \(J_i\), and at most \(|g|\) segments \(J_i^*\).
\# of Lattice point in $S_n = \sum_{i=1}^{\sigma} \# \text{ Lattice point in } \pi_i^{-1}(J_i) \cap K_n$

$\sim \sum_{i=1}^{\sigma} \text{vol}_{d-1}(\pi_i^{-1}(J_i) \cap K_n)$ ....

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a technical proof.
Question:

How many lines are needed to hit all the cells of an $n \times n$ chessboard?

$n$ always suffice
Thanks!