Differentially algebraic equations in physics
Youssef Abdelaziz, Jean-Marie Maillard
(Université Paris VI)
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Hamiltonian of the Ising model

\[ H = \sum_{j,k} \{ J_v \sigma_{j,k} \sigma_{j+1,k} + J_h \sigma_{j,k} \sigma_{j,k+1} \} \]

- \(J_v, J_h\): vertical and horizontal coupling constants
- The spins take the values \(\sigma_{j,k} = \pm 1\).
- The partition function: \(\exp\left(-\frac{1}{k_b T} H\right)\)
Nature of power series

- **Algebraic:** $S(x) \in \mathbb{Q}(x)$ root of a polynomial $P(t, S(t)) = 0$
- **D-finite:** $S(x) \in \mathbb{Q}(x)$ satisfying a linear differential equation with polynomial coefficients $c_r(t)S^{(t)}(t) + s + c_0(t)S(t) = 0$
- **Hypergeometric:** $S(x) = \sum_{n=0}^{\infty} s_n x^n$ s.t. $\frac{s_{n+1}}{s_n} \in \mathbb{Q}(n)$. E.g., the Gauss hypergeometric function:

$$2F_1([a, b], [c], x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!},$$

Partition function 2D square Ising model [Viswanathan, 2014]

$4F_3([1, 1, 3/2], [2, 2, 1], k^2) = k \tanh(2\beta J) - \frac{1}{2 \cosh(2\beta J)}$
Nature of power series

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$$2F_1([a, b], [c], x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{t^n}{n!}, \quad (a)_n := a(a+1) \cdots (a+n-1)$$

- E.g.: $2F_1(1, 1; 1; z) = \frac{1}{1-z}$, $2F_1(1, 1; 2; z) = -\frac{\ln(1-z)}{z}$
- Partition function 2D square Ising model [Viswanathan, 2014]

$$4F_3([1, 1, \frac{3}{2}, \frac{3}{2}], [2, 2, 2], 16k^2], \quad k = \frac{\tanh(2\beta J)}{2 \cosh(2\beta J)}$$
Magnetic susceptibility of 2D Ising model

Magnetic susceptibility → sum of two point correlation functions

\[ \chi := \beta \sum_{n=0}^{\infty} \chi^{(2n+1)} \]

ability of a material to align itself with an external imposed magnetic field

\[ \chi^{(2n+1)} \rightarrow 2n \text{ multiple integrals} \]

\[ \chi^{(3)} \] is given by the double integral:

\[ \chi^{(3)}(s) = \frac{1}{\sqrt{\pi}} \int_{0}^{2\pi} \int_{0}^{2\pi} \frac{1 + x_1 x_2 x_3}{1 - x_1 x_2 x_3} \, d\phi_1 d\phi_2 \]

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Differentially algebraic equations in physics
Magnetic susceptibility of 2D Ising model

Magnetic susceptibility → sum of two point correlation functions

\[ \chi := \beta \sum_{n=0}^{\infty} \chi(2n+1) \]

\[ \chi(2n+1) \rightarrow 2n \] multiple integrals, e.g. \( \chi(3) \) is given by the double integral:

\[ \chi(3)(s) = \frac{(1 - s)^{1/4}}{s} \frac{1}{4\pi^2} \int_{0}^{2\pi} d\phi_1 \int_{0}^{2\pi} d\phi_2 y_1 y_2 y_3 \frac{1 + x_1 x_2 x_3}{1 - x_1 x_2 x_3} F \]

\[ x_j = \frac{s}{1 + s^2 - s \cos \phi_j + \sqrt{(1 + s^2 - s \cos \phi_j)^2 - s^2}} \]

\[ y_j = \frac{s}{\sqrt{(1 + s^2 - s \cos \phi_j)^2 - s^2}}, \quad (j = 1, 2, 3) \]

\[ \phi_1 + \phi_2 + \phi_3 = 0 \]

and \( F = f_{23} \left( f_{31} + \frac{f_{23}}{2} \right) \) with \( f_{ij} = (\sin \phi_i - \sin \phi_j) \frac{x_i x_j}{1 - x_i x_j} \)
Feynman diagrams are D-finite

Feynman diagrams $\rightarrow$ first order perturbations of $n$-fold integral of the operator $S$ (scattering operator) giving the probability of such interactions:

$$S = \sum_{n=0}^{\infty} \frac{\nu^n}{n!} \left( \int \cdots \int \prod_{j=1}^{n} d^4 x_j \right) T \prod_{j=1}^{n} L(x_j)$$

$L_{\nu}(x_j) \rightarrow$ Lagrangian of interaction, $T$ the time ordered product of operators, $d^4 x_j$ four-vectors
Theorem (Kashiwara)

\[ \int \cdots \int D\text{-finite function} \ dx_1 \cdots dx_n \rightarrow D\text{-finite function} \]

(D-finite = solution of linear ODE with polynomial coefficients)
For a formal power series $F$ given by

$$F(z_1, z_2, \ldots, z_n) = \sum_{m_1=0}^{\infty} \cdots \sum_{m_n=0}^{\infty} F_{m_1, \ldots, m_n} z_1^{m_1} \cdots z_n^{m_n},$$

the diagonal of $F$ is defined as the single variable series:

$$\text{Diag}(F(z_1, z_2, \ldots, z_n)) := \sum_{m=0}^{\infty} F_{m, \ldots, m} z^m$$

**Example.** One of the many diagonals leading to Apéry numbers:

$$\text{Diag} \left( \frac{1}{(1 - z_1 - z_2)(1 - z_3 - z_4) - z_1 z_2 z_3 z_4} \right) = \sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 z^n$$
The Gauss hypergeometric function $\, _2F_1 \to \text{PHYSICS!}$, e.g. the differential operator of $\chi^{2n+1}$ factorizes into operators that annihilate $\, _2F_1$ functions. 


\[
E_4(q) = 1 + 240 \sum_{n=0}^{\infty} n^3 \frac{q^n}{1 - q^n}
\]

\[= _2F_1 \left( \left[ \frac{1}{12}, \frac{5}{12} \right], [1], \frac{1728}{j(\tau)} \right)^4\]

$q = \exp(2i\pi \tau), \ j(\tau) \to j$-invariant
Emergence of modular forms in physics through $2F_1$ functions

Modular forms emerge through covariance properties of $2F_1$:

$$2F_1\left(\alpha, \beta, \gamma, p_1(x)\right) = \mathcal{A}(x)2F_1\left(\alpha, \beta, \gamma, p_2(x)\right)$$

$\mathcal{A}(x)$, $p_1(x)$ and $p_2(x)$ are rational functions. $p_1(x)$ and $p_2(x)$ are called pullbacks, the $2F_1$ is thus called pullbacked. For instance:

$$2F_1\left(\frac{1}{12}, \frac{5}{12}, [1], \frac{1728x}{(5 + 10x + x^2)^3}\right) =$$

$$\left(\frac{5 + 10x + x^2}{3125 + 250x + x^2}\right)^{1/4} F_1\left(\frac{1}{12}, \frac{5}{12}, [1], \frac{1728x^5}{(3125 + 250x + x^2)^3}\right).$$
Emergence of modular forms in physics through $2F_1$ functions

Modular forms emerge through covariance properties of $2F_1$:

$$2F_1\left([\alpha, \beta], [\gamma], p_1(x)\right) = A(x)2F_1\left([\alpha, \beta], [\gamma], p_2(x)\right)$$

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w.l.o.g we have: $A(x)2F_1\left([\alpha, \beta], [\gamma], y(x)\right) = 2F_1\left([\alpha, \beta], [\gamma], x\right)$

$A(x)$ and $y(x)$ algebraic functions.
Modular forms as pullbacked $2F_1$ functions

- Emergence of modular forms in physics through $2F_1$ functions
- Modular forms emerge through covariance properties of $2F_1$:

$$2F_1\left([\alpha, \beta], [\gamma], p_1(x)\right) = A(x)2F_1\left([\alpha, \beta], [\gamma], p_2(x)\right)$$

$A(x)$, $p_1(x)$ and $p_2(x)$ are rational functions. $p_1(x)$ and $p_2(x)$ are called pullbacks, the $2F_1$ is thus called pullbacked. For instance:

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$$\left(\frac{5+10x+x^2}{3125+250x+x^2}\right)^{1/4}2F_1\left([\frac{1}{12}, \frac{5}{12}], [1], \frac{1728x^5}{(3125+250x+x^2)^3}\right).$$

- w.l.o.g we have: $A(x)2F_1\left([\alpha, \beta], [\gamma], y(x)\right) = 2F_1\left([\alpha, \beta], [\gamma], x\right)$

$A(x)$ and $y(x)$ algebraic functions. Modular equation $M(x, y(x)) = 0$:

$$1953125x^3y^3 - 187500x^2y^2(x+y) + 375xy(16x^2 - 4027xy + 16y^2)$$

$$-64(x+y)(x^2 + 1487xy + y^2) + 110592xy = 0$$
Schwarzian condition

Theorem (Abdelaziz–Maillard, 2016)

If we have a pullback given by:

\[ A(x) F_1([\alpha, \beta], [\gamma], x) = F_1([\alpha, \beta], [\gamma], y(x)) \]

then we have the following "Schwarzian condition":

\[ W(x) - W(y(x))y'(x)^2 + \{y(x), x\} = 0 \]

where \( W(x) := p'(x) + \frac{p(x)^2}{2} - 2q(x) \)

with \( p(x) = \frac{(\alpha + \beta + 1)x - \gamma}{x(x-1)} \)

\( q(x) = \frac{\alpha\beta}{x(x-1)} \)

NB: The Schwarzian derivative is defined by

\[ \{y(x), x\} := \frac{y'''(x)}{y'(x)} - 3 \left( \frac{y''(x)}{y'(x)} \right)^2 \]
Schwarzian condition

Theorem (Abdelaziz–Maillard, 2016)

If we have a pullback given by:

\[ A(x) \, _2F_1\left(\alpha, \beta; \gamma, x\right) = _2F_1\left(\alpha, \beta; \gamma, y(x)\right) \]

then we have the following “Schwarzian condition”:

\[ W(x) - W(y(x))y'(x)^2 + \{y(x), x\} = 0 \]

where

\[ W(x) := p'(x) + \frac{p(x)^2}{2} - 2q(x) \]

with

\[ p(x) = \frac{(\alpha + \beta + 1)x - \gamma}{x(x - 1)} \quad q(x) = \frac{\alpha\beta}{x(x - 1)} \]

NB: The Legendre derivative is defined by

\[ \{y(x), x\} := \frac{y''''(x)}{y'(x)} - 3 \left( \frac{y''(x)}{y'(x)} \right)^2 \]
Schwarzian condition

Theorem (Abdelaziz–Maillard, 2016)

If we have a pullback given by:

\[ A(x)_{2} F_{1} ([\alpha, \beta], [\gamma], x) =_{2} F_{1} ([\alpha, \beta], [\gamma], y(x)) \]

then we have the following "Schwarzian condition":

\[ W(x) - W(y(x)) y'(x)^{2} + \{ y(x), x \} = 0 \]

where

\[ W(x) := p'(x) + \frac{p(x)^{2}}{2} - 2q(x) \]

with

\[ p(x) = \frac{(\alpha + \beta + 1)x - \gamma}{x(x - 1)} \]
\[ q(x) = \frac{\alpha \beta}{x(x - 1)} \]

NB: The Schwarzian derivative is defined by

\[ \{ y(x), x \} := \frac{y''''(x)}{y'(x)} - \frac{3}{2} \left( \frac{y''(x)}{y'(x)} \right)^{2} \]
Proof of our theorem of the Schwarzian condition

We introduce

- the operator $L_2 := D_x^2 + p(x)D_x + q(x)$ annihilating $F(x) := 2F_1([\alpha, \beta], [\gamma], x)$

- the operator $L_2^{(c)} := \frac{1}{v(x)}L_2v(x)$, i.e.

$$L_2^{(c)} = D_x^2 + \left( p(x) + 2\frac{v'(x)}{v(x)} \right)D_x + q(x) + p(x)\frac{v'(x)}{v(x)} + \frac{v''(x)}{v(x)}$$

NB: $L_2^{(c)}$ annihilates $A(x)F(x)$ (with $A(x) = 1/v(x)$):

$$L_2^{(c)}\frac{1}{v}F(x) = \frac{1}{v}L_2v\frac{1}{v}F(x) = 0$$
Proof of our theorem of the Schwarzian condition

We introduce

- the operator \( L_2 := D_x^2 + p(x)D_x + q(x) \) annihilating \( F(x) := 2F_1([\alpha, \beta], [\gamma], x) \)

- the operator \( L_2^{(c)} := \frac{1}{v(x)} L_2 v(x) \), i.e.

\[
L_2^{(c)} = D_x^2 + \left( p(x) + 2 \frac{v'(x)}{v(x)} \right) D_x + q(x) + p(x) \frac{v'(x)}{v(x)} + \frac{v''(x)}{v(x)}
\]

NB: \( L_2^{(c)} \) annihilates \( A(x)F(x) \) (with \( A(x) = 1/v(x) \)):

\[
L_2^{(c)} \frac{1}{v} F(x) = \frac{1}{v} L_2 v \frac{1}{v} F(x) = 0
\]

- So, the operator annihilating \( F(y(x)) \) is

\[
L_2^{(p)} = D_x^2 + \left( p(y(x))y'(x) - \frac{y''(x)}{y'(x)} \right) D_x + q(y(x))y'(x)^2
\]

- When does the equality \( L_2^{(c)} = L_2^{(p)} \) hold?
Proof of our theorem of the Schwarzian condition

Well, identifying $L_2^{(c)} = L_2^{(p)}$ gives us two conditions:

Condition 1: \[ p(x) + 2 \frac{v'(x)}{v(x)} = p(y(x))y'(x) - \frac{y''(x)}{y'(x)} \]

Condition 2: \[ q(x) + p(x) \frac{v'(x)}{v(x)} + \frac{v''(x)}{v(x)} = q(y(x))y''(x)^2 \]

Introducing $w(x) := \exp\left(- \int p(x) \, dx\right)$, i.e. $p(x) = -\frac{w'(x)}{w(x)}$, Condition 1 rewrites

\[ -\frac{w'(x)}{w(x)} + 2 \frac{v'(x)}{v(x)} = -\frac{y''(x)}{y'(x)} + \frac{w'(y(x))}{w(y(x))} \]

Integrating the log-derivative terms we get:

\[ -\ln w(x) + 2 \ln v(x) = -\ln y'(x) - \ln w(y(x)) \]

Taking exponential gives

\[ v(x) = \sqrt{\frac{w(x)}{w(y(x))y'(x)}} \]

inserting it in Condition 2 gives the Schwarzian condition in the theorem. \qed
Assuming that the operator is globally nilpotent is equivalent to:

\[ p(x) = -\frac{w'(x)}{w(x)} \]

The following statements are a consequence of global nilpotence:

- The Wronskian is the \( n \)-th root of a rational function
- The solutions of the differential equation have rational coefficients
- The \( p \)-curvature is a nilpotent matrix mod prime
- Global nilpotence \( \rightarrow \) rational coefficients of solutions \( \rightarrow \)
  \[ p(x) = \frac{d}{dx} \ln w(x) \]
The $j$-invariant of the elliptic curve:

$$j(k) = 256 \frac{(1 - k^2 + k^4)^3}{k^4(1 - k^2)^2}$$

The Landen transformation:

$$k_L = \frac{2 \sqrt{k}}{1 + k}$$

The transform of the elliptic invariant through $k_L$:

$$j(k_L) = 16 \frac{(1 + 14k^2 + k^4)^3}{k^2(1 - k^2)^4}$$
Modular equation $M(x,y(x))=0$ and modular invariant

The $j$-invariant of the elliptic curve:

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The two corresponding Hauptmoduls (similar to a group generator):

$$x = \frac{1728}{j(k)} \quad y = \frac{1728}{j(k_L)}$$

are related through the modular equation $\tau \rightarrow 2\tau$:

$$M(x, y) = 1953125x^3y^3 - 187500x^2y^2(x + y) + 375xy(16x^2 - 4027xy + 16y^2)$$

$$-64(x + y)(x^2 + 1487xy + y^2) + 110592xy = 0$$

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Differentially algebraic equations in physics
For one $2F_1\left([a, b], [1], x\right)$ with two different pullbacks

$$
\alpha x + \cdots \\
\alpha x^2 + \cdots \\
\alpha x^3 + \cdots
$$

we obtain the isogenies series-solution “structure”
For one \( 2F_1 \left( [a, b], [1], x \right) \) with two different pullbacks

\[
\alpha x + \cdots \\
\alpha x^2 + \cdots \\
\alpha x^3 + \cdots
\]

we obtain the isogenies series-solution “structure”

This set of solutions is either:

- **Algebraic**: e.g. \( 2F_1 \left( \left[ \frac{1}{12}, \frac{5}{12} \right], [1], x \right) \), we recover “some” commutation like in the case of isogenies (as we will see below)
- **Transcendent**
Schwarzian condition and modular forms: $\tau \rightarrow 2\tau$ and beyond

The modular form:

$$A(x)\ _2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], x\right) = _2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], y(x)\right)$$

1. $A(x)$ is an algebraic function
2. $y(x)$ is an algebraic function corresponding to the modular equation corresponding to $\tau \rightarrow 2\tau$
The modular form:

\[ \mathcal{A}(x) \, {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], x\right) = {}_2F_1\left(\left[\frac{1}{12}, \frac{5}{12}\right], [1], y(x)\right) \tag{1} \]

- \( \mathcal{A}(x) \) is an algebraic function
- \( y(x) \) is an algebraic function corresponding to the modular equation corresponding to \( \tau \to 2\tau \)

\[ y(x) = \frac{1}{1728} x^2 + \frac{31}{62208} x^3 + \frac{1337}{3359232} x^4 + \frac{349115}{108839168} x^5 + \cdots \]

The Schwarzian condition is verified in this case with:

\[ W(x) = -\frac{32 x^2 - 41 x + 36}{72 x^2 (x - 1)^2} \quad p(x) = \frac{3 x - 2}{2 x (x - 1)} \quad q(x) = \frac{5}{144 x (x - 1)} \]
Schwarzian condition and modular forms: \( \tau \to 2\tau \) and beyond

The modular form:

\[
A(x)_{2F1} \left( \left[ \frac{1}{12}, \frac{5}{12} \right], [1], x \right) = 2 \ F_1 \left( \left[ \frac{1}{12}, \frac{5}{12} \right], [1], y(x) \right)
\]

- \( A(x) \) is an algebraic function
- \( y(x) \) is an algebraic function corresponding to the modular equation corresponding to \( \tau \rightarrow 2\tau \)

\[
y(x) = \frac{1}{1728} x^2 + \frac{31}{62208} x^3 + \frac{1337}{3359232} x^4 + \frac{349115}{1088391168} x^5 + \ldots
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W(x) = -\frac{32x^2 - 41x + 36}{72x^2(x - 1)^2}, \quad p(x) = \frac{3x - 2}{2x(x - 1)}, \quad q(x) = \frac{5}{144x(x - 1)}
\]

It turns out that one can write, for the modular equations corresponding to \( \tau \rightarrow N\tau \), the function in the form of (1) above. Thus the equation (1) above encapsulates all the modular equations corresponding to \( \tau \rightarrow N\tau \).
The modular equation of order three $\tau \to 3\tau$:

\[
2621440000000000x^3y^3(x + y) + 4096000000x^2y^2(27x^2 - 45946xy + 27y^2)
+ 15552000xy(x + y)(x^2 + 241433xy + y^2)
+ 729x^4 - 779997924x^3y + 1886592284694x^2y^2 - 779997924xy^3 + 729y^4
+ 2811677184xy(x + y) - 2176782336xy = 0
\]

has the series expansion starting in $x^3$ and given by:

\[
y(x) = \frac{x^3}{2985984} + \frac{31x^4}{71663616} + \frac{36221x^5}{82556485632} + \frac{29537101x^6}{71328803586048} + \ldots
\]
Modular equations of higher order

The modular equation of order three \( \tau \rightarrow 3\tau \):

\[
2621440000000000x^3y^3(x + y) + 4096000000x^2y^2(27x^2 - 45946xy + 27y^2) \\
+ 15552000xy(x + y)(x^2 + 241433xy + y^2) \\
+ 729x^4 - 779997924x^3y + 1886592284694x^2y^2 - 779997924xy^3 + 729y^4 \\
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\]

Similarly for \( \tau \rightarrow 4\tau \), we get a series starting in \( x^4 \):

\[
y(x) = \frac{x^4}{5159780352} + \frac{31x^5}{92876046336} + \frac{43909x^6}{106993205379072} + \cdots
\]
Except for this last series solution, the solution series corresponding to the isogenies \( \tau \rightarrow N\tau \) have the form \( ax^N + \cdots \).

The series solution corresponding to \( \tau \rightarrow 3\tau \) and \( \tau \rightarrow 4\tau \) are solution of the Schwarzian condition.
Except for this last series solution, the solution series corresponding to the isogenies $\tau \to N\tau$ have the form $ax^N + \cdots$

The series solution corresponding to $\tau \to 3\tau$ and $\tau \to 4\tau$ are solutions of the Schwarzian condition.

Generalizing the solution series corresponding to $\tau \to 2\tau$ we seek solution series of the Schwarzian condition of the form $ax^2 + \cdots$:

$$y_2 = ax^2 + \frac{31ax^3}{36} - \frac{a(5952a - 9511)}{13824}x^4 + \cdots$$

reducing to the solution of $\tau \to 2\tau$ when $a = 1/1728$. 

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A one-parameter family of solution-series $bx^3 + \cdots$ for the modular equation corresponding to $\tau \to 3\tau$:

$$y_3 = bx^3 + \frac{31b}{24}x^4 + \frac{36221b}{27648}x^5 + \cdots$$

reduces to a previous series having the form $x^3 + \cdots$ when $b = 1/1728^2$. 
A one-parameter family of solution-series $bx^3 + \cdots$ for the modular equation corresponding to $\tau \rightarrow 3\tau$:

$$y_3 = b x^3 + \frac{31b}{24} x^4 + \frac{36221b}{27648} x^5 + \cdots$$

reduces to a previous series having the form $x^3 + \cdots$ when $b = 1/1728^2$.

Finally the one-parameter series

$$y_4 = cx^4 + \frac{31c}{18} x^5 + \frac{43909c}{20736} x^6 + \cdots$$

reduces to a previous series of the form $x^4 + \cdots$ for $c = 1/5159780352 = 1/1728^3$
These series do not commute: $y_i(y_j(x)) \neq y_j(y_i(x))$.

Composing the solution series $y_3$ and $y_2$ with $d = ab^2$:

$$y_2(y_3(x)) = d x^6 + \frac{31 d x^7}{12} + \frac{59285 d}{13824} x^8 + \cdots$$

$$y_2(y_3(x)) = y_3(y_2(x)) \leftrightarrow ?$$
Commuting series

- These series do not commute: \( y_i(y_j(x)) \neq y_j(y_i(x)) \).
- Composing the solution series \( y_3 \) and \( y_2 \) with \( d = ab^2 \):

\[
y_2(y_3(x)) = dx^6 + \frac{31dx^7}{12} + \frac{59285d}{13824}x^8 + \cdots
\]

- \( y_2(y_3(x)) = y_3(y_2(x)) \iff ab^2 = ba^3 \)
The Schwarzian condition encapsulates the infinite number of modular equations $\tau \rightarrow N\tau$.

Strong incentive to develop more differentially algebraic tools from an algorithmic perspective: to test the non-D-finiteness of the Ising susceptibility for example!

Strong incentive to examine further the occurrence of non-linear symmetries (like the Landen transformation) in physics.
Questions: non-linear differential Galois group

- Built to generalize the differential Galois group to non-linear ODE’s and non-linear functional equations having the form $f(x + 1) = y(f(x))$.
- Having a finite non-linear differential Galois group guarantees “some integrability” and this is guaranteed by Casale’s condition:

$$\nu(y)y''(x)^2 - \nu(x) + \frac{y'''(x)}{y'(x)} - \frac{3}{2} \left( \frac{y''(x)}{y'(x)} \right)^2 = 0$$
Modular equations: definition through θ functions

With \( q = \exp(i\pi \tau) \), \( \tau = iK'/K \) the \( \theta_3 \) and \( \theta_4 \) functions are defined as follows:

\[
\theta_2 = 2q^{1/4} \prod_{n \geq 1} \left( \frac{1 - q^{4n}}{1 - q^{4n-2}} \right), \quad \theta_3 = \sum_{-\infty}^{\infty} q^n, \quad \theta_4 = \sum_{-\infty}^{\infty} (-1)^n q^n
\]

where \( K = (\pi/2)\theta_3^2(\tau) \) and \( K'(\tau) = K(-1\tau) \). We can write the identity:

\[
\theta_3(\tau)^2 + \theta_4(\tau)^2 = 2\theta_3(2\tau)^2 = \frac{2}{1 + k'}
\]

with \( \sqrt{k(\tau)} = \frac{\theta_2(\tau)}{\theta_3(\tau)} \), \( \sqrt{k'(\tau)} = \frac{\theta_4(\tau)}{\theta_3(\tau)} \) and \( l'(\tau) = k'(p\tau) \) where \( p \) is given by a positive integer, we have:

\[
\frac{1}{l'} = \frac{1}{2} \left( \sqrt{k'} + \frac{1}{\sqrt{k'}} \right)
\]

giving in the case \( p = 2 \) the modular equation that sends \( \tau \) to \( 2\tau \).
Painlevé equations

- The hypergeometric function, the Bessel function, the Airy function, the Hermite polynomials, are all “special” (appearing in problems related to physics) functions solution of linear differential equations.
- Elliptic functions are also “special” functions: they appear in physics as we shall see here, yet they are solution of simple, yet non-linear differential equations.
- Painlevé was set out to find special functions satisfying non-linear differential equations, yet have nice properties (all their singularities are poles).

Painlevé wanted to classify all differential equations of order two having the form:

\[ u_{xx} = R(x, u, u_x) \]

with \( R \) being a rational function. Painlevé found 50 equations having this form, six of these were irreducible to known functions; they are known today as the six Painlevé equations.
Magnetic susceptibility = ratio of D-finite functions?

The hypergeometric function:

\[ 2F_{1}([1/3, 1/3], [1], 27x) \]

is D-finite and verifies the following **linear** differential equation

\[ (27x^2 - x) \left( \frac{d^2}{dx^2} F(x) \right) + (45x - 1) \left( \frac{d}{dx} F(x) \right) + 3F(x). \]

Similarly the hypergeometric function given by

\[ 2F_{1}([1/2, 1/2], [1], 16x) \]

verifies the **D-finite** equation

\[ (16x^2 - x) \left( \frac{d^2}{dx^2} F(x) \right) + (32x - 1) \left( \frac{d}{dx} F(x) + 4F(x) \right). \]

**Reminder:** A function is D-finite when it is solution of a **linear** differential equation and with **rational** coefficients in \( x \).
Magnetic susceptibility = ratio of D-finite functions?

The ratio of these two D-finite functions is given by:

\[
\frac{2F_1([1/3, 1/3], [1], 27x)}{2F_1([1/2, 1/2], [1], 16x)}
\]

- While the product of two D-finite functions is always D-finite, the ratio of two D-finite functions is generally **not so** (except if the D-finite function at the denominator is an algebraic function)!
- In fact the differential equation that this ratio verifies is non-linear as we can see in the next slide.
\[-2x^2(27x - 1)(-1 + 16x)((27x - 1)(-1 + 16x)\frac{d}{dx} F(x)
\]
\[-72xF(x) - F(x))\frac{d^3}{dx^3} F(x)
\]
\[+3x^2(27x - 1)^2(-1 + 16x)^2 \left(\frac{d^2}{dx^2} F(x)\right)^2
\]
\[-2x(93312 \frac{d}{dx} F(x)x^4 - 7992 \frac{d}{dx} F(x)x^3
\]
\[-93312x^3 F(x) + 87 \frac{d}{dx} F(x)x^2 + 168x^2 F(x)
\]
\[+3 \frac{d}{dx} F(x)x + 297xF(x) - 4F(x))\frac{d^2}{dx^2} F(x)
\]
\[+(−1 + 16x)(1944x^3 − 1569x^2 + 58x − 1) \left(\frac{d}{dx} F(x)\right)^2
\]
\[+2F(x)(29376x^3 + 5580x^2 − 221x + 1)\frac{d}{dx} F(x)
\]
\[+(144x^2 − 432x + 1)F(x)^2 = 0
\]