Phase Transition for the mixing time of Glauber Dynamics on Regular Trees at Reconstruction: Colorings and Independent Sets.

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JOINT WORK WITH
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Coloring graphs

Given

- A graph $G = (V, E)$ on $n$ vertices and maximum degree $\Delta$
- A set of $k$ colors

A $k$-coloring of $G$ is an assignment $f : V \rightarrow \{1, \ldots, k\}$ such that

for all $(u, v) \in E$, $f(u) \neq f(v)$

Given a graph with maximum degree $\Delta$

- How to construct $k$-colorings
  - Trivial for $k > \Delta$
- How to sample (uniformly at) random $k$-colorings
  - Non-trivial even for $k > \Delta$
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Random Coloring graphs

Why are we interested?

- How to sample (uniformly at) random $k$-colorings?

- How do random $k$-colorings look?
  - Random colorings is the Gibbs distribution for (zero-temperature) anti-ferromagnetic Potts model.
  - Efficient sampler yields approximation algorithm for counting colorings, which is #P-complete.
Glauber dynamics

Let $\Omega$ denote the set of all proper $k$-colorings of $G$.

Glauber dynamics (heat bath version)

Given $X_t \in \Omega$,

- Take $v \in V$ uniformly at random (u.a.r.)
- Take $c$ u.a.r. from available colors for $v$ in $X_t$:
  $$A_{X_t} = \{c : c \notin X_t(N(v))\}.$$
- Obtain $X_{t+1} \in \Omega$ by recoloring $v$ to color $c$. 
Glauber dynamics

A natural threshold: $k \geq \Delta + 2$

For $k \geq \Delta + 2$:

- Glauber dynamics is always ergodic.
- The (unique) stationary distribution is uniform over $\Omega$, independently of initial coloring.

$t \to \infty$: distribution of $X_t \to$ uniform dist. on $\Omega$.

- Run chain long enough to get close to uniform state.
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For $k \leq \Delta + 1$:
- There are graphs where the Glauber dynamics is not ergodic.
- Some graphs are not even colorable for $k \leq \Delta$. 
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$T_{\text{mix}}$: Mixing time
Time until the chain is within total variation distance $\leq 1/4$ from uniform distribution independently of initial state.
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Theorem (Hayes and Sinclair 05)

$T_{\text{mix}} = \Omega(n \ln n)$ for general graphs.

- Intuitively, time necessary to see all vertices.

Conjecture (folklore)

In general graphs, for $k \geq \Delta + 2$ the mixing time is optimal, i.e.,

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Towards the conjecture (selection of results):

**Conjecture (folklore)**

*In general graphs, for* \( k \geq \Delta + 2 \) *the mixing time is optimal, i.e.,*

\[
T_{\text{mix}} = O(n \ln n)
\]

- **\( k > 2\Delta \):** [Jerrum ’95]
- **\( k > 11\Delta/6 \):** \((T_{\text{mix}} = O(n^2))\) [Vigoda ’99]
- **Girth and/or max degree assumptions:** [Dyer-Frieze’01], [Molloy’02], [Hayes’03], [Hayes-Vigoda’03], [Frieze-Vera’04], [Dyer-Frieze-Hayes-Vigoda’04]
- **\( \Delta \)-regular trees, any fix boundary:** \( k \geq \Delta + 3 \), [Martinelli-Sinclair-Weitz ’04]
- **Planar graphs, \( k \geq 100\Delta/\ln\Delta \):** \( T_{\text{mix}} = O^*(n^3) \) [Hayes-Vera-Vigoda ’07]
- **For \( \Delta \)-regular trees, \( k = C\Delta/\ln\Delta \):** \( T_{\text{mix}} = n^{\Theta(\min(1,1/C))} \) [Lucier-Molloy ’08], [Goldberg-Jerrum-Karpinski ’08]
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Significance of $\Delta/\ln \Delta$:

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- **What’s significance of $\Delta/\ln \Delta$**

- **What happens below $\Delta/\ln \Delta$?**

**Goal:** Get detailed picture on trees.

- Better understanding for planar and sparse random graphs.
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Outline

1. Introduction
   - Motivation
   - Glauber dynamics
   - Mixing Time

2. Colorings on the complete $\Delta$-Tree
   - Reconstruction
   - Main Result

3. Relation between reconstruction and mixing time

4. Independent Sets on the complete $\Delta$-Tree
Significance of $\Delta/\ln \Delta$:

Consider the complete tree with branching factor $\Delta$ and height $h$.

Recall

For $k = \Delta + 1$ there are colorings that “freeze” the root

- Colors of leaves determine color of root

But this is not true for “typical” colorings

Question

For which values of $k$ does a random coloring of the leaves freeze the root?
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For which values of $k$ does a random coloring of the leaves freeze the root?
Reconstruction

Generating a random coloring of the tree: Broadcasting model

1. Choose a random color for the root, call it $\sigma(r)$.
2. For each vertex $v$, given the color of its parent $\sigma(p(v))$, choose a random different color.

Reconstruction

$Reconstruction$ holds if the leaves have a non-vanishing (as $h \to \infty$) influence on the root in expectation.

$$\lim_{h \to \infty} E_{\sigma_L} \left[ \left| \frac{\mu(\tau(r)|\tau(L) = \sigma_L)}{k} - \frac{1}{k} \right| \right] > 0.$$ 

Given (random) coloring of leaves can guess color of root

Reconstruction threshold

Threshold is at $\approx \Delta/\ln \Delta$ [Sly’08, Bhatnagar-Vera-Vigoda’08]
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Connections of reconstruction to the efficiency of local algorithms on trees and tree-like graphs

- $T_{mix} = O(n \ln n)$ on the complete tree implies non-reconstruction [Berger-Kenyon-Mossel-Peres ’05]
- “Clustering of solution space” in reconstruction region for several constraint satisfaction problems, including colorings, on sparse random graphs [Achlioptas,Coja-Oghlan ’08]

We prove:
Mixing time of the Glauber dynamics for random colorings of the complete tree undergoes a phase transition. Critical point appears to coincide with the reconstruction threshold.
Main Result

Theorem

Let $k = \frac{C\Delta}{\ln \Delta}$. There exists $\Delta_0$ such that, for all $\Delta > \Delta_0$, the Glauber dynamics on the complete $\Delta$-tree on $n$ vertices satisfies:

1. For $C \geq 1$:
   \[
   \Omega(n \ln n) \leq T_{\text{mix}} \leq O\left(n^{1+o_{\Delta}(n)} \ln^2 n\right)
   \]

2. For $C < 1$:
   \[
   \Omega\left(n^{1/C+o_{\Delta}(n)}\right) \leq T_{\text{mix}} \leq O\left(n^{1/C+o_{\Delta}(n)} \ln^2 n\right)
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Ideas on lower bound for reconstruction region ($C < 1$)
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Ideas on lower bound for reconstruction region (\( C < 1 \))
Lowerbound for $C < 1$

Usually reconstruction is proven via a *reconstruction algorithm*

**Reconstruction Algorithm**

Function $A : \Omega_L \rightarrow \{0, 1\}$ (ideally efficiently computable)

- For any $\sigma$, $A(\sigma_L)$ and $\sigma(r)$ are positively correlated.
- Assume: when coloring of $L$ freezes the root, $A$ gives correct answer

Given reconstruction algorithm $A$

- Let

$$S_c = \{ \sigma \in \Omega : A(\sigma_L) = c \}$$

- $S_c \supseteq \{ \sigma \in \Omega : \sigma_L \text{ freezes } r \text{ to } c \}$.

**Intuitive key Idea:**

Under reconstruction: If initial coloring in $S_R$ it is "difficult" to get to coloring in $S_B$. 

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Phase Transition at Reconstruction

LIPN: CALIN, Apr 2010
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Conductance
Let $S \subseteq \Omega$ and $\bar{S} = \Omega \setminus S$. Define

$$\Phi_S = \frac{\sum_{\sigma \in S} \sum_{\eta \in \bar{S}} \pi(\sigma)P(\sigma, \eta)}{\pi(S)\pi(\bar{S})}$$

- Related to probability of escaping from $S$ in one step

Theorem (Lawler-Sokal '88. Sinclair-Jerrum '89)
For all $S \subseteq \Omega$

$$T_{mix} \geq \Omega(1/\Phi_S)$$

Formalized Key idea
Show $S_c$ has small conductance
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**Formalized Key idea**
Show $S_c$ has small conductance
Relation between reconstruction and conductance

Goal
Show $S_c$ has small conductance

Theorem
Under reconstruction, for any reconstruction function $A$,

$$\Phi_{S_c} = O(\mathbb{E}_\sigma [\Psi_A(\sigma)|\sigma \in S_c])$$

where

$$\Psi_A(\sigma) = \#\{v \in L : \exists d \in [k] A(\sigma^v,d) \neq A(\sigma)\}.$$

- **Sensitivity** of $A$ at $\sigma$: For how many leaves, changing color of leaf will change outcome of $A$. 
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**Goal**

Show $S_c$ has small conductance

Actually, let $S = \bigcup_{c<k/2} S_c$.

- [Goldberg, Jerrum, and Karpinski - 08] For $0 < C < 1/2$
  \[ \Phi_S = O(n^{-\frac{1}{6C}}) \]

- We prove for $C < 1$
  \[ \Phi_S = O^*(n^{-1/C}) \]
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Theorem

Let \( k = C\Delta / \ln \Delta \). There exists \( \Delta_0 \) such that, for all \( \Delta > \Delta_0 \), the Glauber dynamics on the complete \( \Delta \)-tree on \( n \) vertices satisfies:

1. For \( C \geq 1 \):

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\Omega\left(n^{1/C-o_\Delta(n)}\right) \leq T_{\text{mix}} \leq O\left(n^{1/C+o_\Delta(n)} \ln^2 n\right)
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Does a similar phenomenon hold for independent sets?

No, more interesting phenomenon occurs.
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### Hard-core model

- Graph $G = (V, E)$ with $n$ vertices and maximum degree $\Delta$.
- Independent set is a subset $S \subseteq V$ where for all $(v, w) \in E$, either $v \notin S$ and/or $w \notin S$.
- Activity (or fugacity) $\lambda > 0$.
- Hard-core distribution (i.e., Gibbs measure): $\mu(S) \sim \lambda^{|S|}$. 

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Reconstruction threshold for the hard-core model

Consider the complete tree with branching factor $\Delta$ and height $h$. Let $\omega$ be the solution to $\lambda = \omega(1 + \omega)^\Delta$.

Broadcasting model:

1. Occupy the root with probability $p = \frac{\omega}{1 + \omega}$ and leave it unoccupied with $1-p$.
2. For each vertex $v$, if the parent is unoccupied, occupy $v$ with probability $p$.

Reconstruction is said to hold if the leaves have a non-vanishing (as $h \to \infty$) influence on the root in expectation:

$$\lim_{h \to \infty} E_{\sigma_L} \left[ \mu(r \in \tau|\tau(L) = \sigma_L) - \frac{\omega}{1 + \omega} \right] > 0.$$
Reconstruction threshold and mixing of the Glauber dynamics?

- **Reconstruction threshold:** \( \omega_r \approx \frac{\ln \Delta + \ln \ln \Delta}{\Delta} \)
  
  [Bhatnagar-Sly-Tetali ’10],[Brightwell-Winkler ’04]

- Rapid mixing for free boundary: For the complete tree on \( n \) vertices, \( T_{\text{mix}} = O(n \log n) \) for all \( \lambda \)
  
  [Martinelli-Sinclair-Weitz ’04]

So, no slow down at reconstruction?

- Free boundary does not correspond to the broadcast process for the hard-core model.
  - it does for colorings.

- There exist boundary conditions with a slow down at reconstruction.
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  ▶ it does for colorings.

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Results for hard-core model:

**Theorem**

For the Glauber dynamics on the hard-core model with activity \( \lambda = \omega (1 + \omega)^\Delta \) on the complete \( \Delta \)-tree with \( n \) vertices:

1. **For all** \( \omega \leq \ln \Delta / \Delta \): \( \Omega(n) \leq T_{rel} \leq O^*(n) \).
2. **For all** \( \delta > 0 \) and \( \omega = (1 + \delta) \ln \Delta / \Delta \):
   
   1. For every boundary condition,
      
      \[ T_{rel} \leq O^*(n^{1+\delta}). \]
   2. Exists a sequence of boundary conditions with \( h \to \infty \) such that,
      
      \[ T_{rel} \geq \Omega^*(n^{1+\delta/2}). \]
Current and Future Work

- Similar analysis of other CSPs (spin systems).
  - e.g. k-SAT

- Analysis of more general graphs.
- Poisson tree closely related to sparse random graph $G(n, d/n)$.
  - For constant $d, k$, $k \geq \text{poly}(d)$, $T_{\text{mix}}(\text{Col}) = \text{poly}(n)$ whp. [Mossel-Sly ’08]
  - Open: Prove “rapid mixing” down to $d/\ln d$ colors.

- Explore more general relation between reconstruction and ”local algorithms”