(Co)-Algebras, Shifts and a theorem by E. Abe and M. P. Schützenberger

Matthieu Deneufchâtel

Laboratoire d'Informatique de Paris Nord,
Université Paris 13

CIP,
26 October 2010

Notations

In this talk,

- Let $(\mathfrak{A}, \mu, u)$ and $(\mathfrak{C}, \Delta, \epsilon)$ denote respectively an associative algebra with unit u and a coalgebra (coassociative with counit ϵ), k the ground field.
- $\mathfrak{A}^*$ denotes the algebraic dual of the vector space $\mathfrak{A}$.
- For V, W two vector spaces, $f \in \mathrm{Hom}(V, W)$ and $v \in V$, $\langle f|v\rangle$ denotes the value of $f(v)$.
- Transpose of a linear map : we denote by ${}^t f : W^* \to V^*$ the transpose of f :

$$\langle {}^t f(w)|v\rangle = \langle w|f(v)\rangle, \forall v \in V.$$

Useful properties 1/3

- (i) Let U, V and W be vector spaces and $\phi : U \times V \to W$ a bilinear map. Then if $z \in \mathrm{Im}(\phi)$ and if

$$z = \sum_{i=1}^{n} \phi(x_i, y_i)$$

with n minimal, the families $(x_i)_{1 \leq i \leq n}$ and $(y_i)_{1 \leq i \leq n}$ are free in their respective spaces.

Application to the tensor product : Let y be an element of $V \otimes W$. Then there exists $n \in \mathbb{N}$ and two families $(a_i)_i \subset V$, $(b_i)_i \subset W$, $(a_i)_i$ being free such that

$$y = \sum_{i=0}^{n} a_i \otimes b_i$$

If n minimal then both families are free.

Proof : Assume, without loss of generality, that $x_n = \sum_{i=1}^{n-1} \alpha_i x_i$. Then

$$\begin{aligned} z &= \sum_{i=1}^{n-1} \phi(x_i, y_i) + \phi(x_n, y_n) \\ &= \sum_{i=1}^{n-1} \phi(x_i, y_i) + \phi(\sum_{j=1}^{n-1} \alpha_j x_j, y_n) \\ &= \sum_{i=1}^{n-1} \phi(x_i, y_i) + \sum_{j=1}^{n-1} \alpha_j \phi(x_j, y_n) \end{aligned}$$

since ϕ is bilinear. The same argument allows us to "factor" the x_i's in the two sums :

$$z = \sum_{i=1}^{n-1} \phi(x_i, y_i + \alpha_i y_n)$$

Therefore, there would be another decomposition of z with $n' = n - 1 < n$ terms. This is impossible since n is minimal. Therefore, $(x_i)_i$ is a free family. The same argument also applies for $(y_i)_i$.

Useful properties 2/3

- (ii) If $\mathfrak{C}$ is a coalgebra, then $\mathfrak{C}^*$ is an algebra for the following maps :
  - $\mu = {}^t\Delta$: $\langle \mu(f \otimes g) | z \rangle = \langle f \otimes g | \Delta(z) \rangle$;
    Indeed, $\mu = {}^t\Delta\rho$ with

$$\begin{aligned} \rho : \mathfrak{C}^* \otimes \mathfrak{C}^* &\to (\mathfrak{C} \otimes \mathfrak{C})^* \\ f \otimes g &\mapsto \rho(f \otimes g) : c \otimes c' \mapsto f(c)g(c') \end{aligned}$$

  - $u = \epsilon^* \phi$ where $\phi : k \to k^*$ is the canonical isomorphism.

$$\mathfrak{C} \xrightarrow{\epsilon} k$$

$$\begin{array}{ccc} k^* & \xrightarrow{{}^t\epsilon} & \mathfrak{C}^* \\ \sim\uparrow & \nearrow_{u} & \\ k & & \end{array}$$

$$\begin{aligned} k^* &\to k \\ f &\mapsto f(1_k) \end{aligned}$$

$$\begin{aligned} k &\to k^* \\ k &\mapsto (u \mapsto uk) \end{aligned}$$

Useful properties 2/3

- (iii) If $\mathfrak{A}$ is a finite dimensional algebra, then $\mathfrak{A}^*$ has a coalgebra structure for the following maps :
  - $\Delta = j_{\mathrm{nat}}^{-1} \circ {}^t\mu$ where j_{nat} is the canonical isomorphism between $\mathfrak{A}^* \otimes \mathfrak{A}^*$ and $(\mathfrak{A} \otimes \mathfrak{A})^*$:

$$j_{\mathrm{nat}}(f \otimes g)(a \otimes a') = f(a)g(a').$$

  - $\epsilon = \psi\, {}^t u$ where ψ is the canonical isomorphism that associates to a linear form its value at 1_k.

Useful properties 2/3

- (iii) If $\mathfrak{A}$ is a finite dimensional algebra, then $\mathfrak{A}^*$ has a coalgebra structure for the following maps :
  - $\Delta = j_{\mathrm{nat}}^{-1} \circ {}^t\mu$ where j_{nat} is the canonical isomorphism between $\mathfrak{A}^* \otimes \mathfrak{A}^*$ and $(\mathfrak{A} \otimes \mathfrak{A})^*$:

$$j_{\mathrm{nat}}(f \otimes g)(a \otimes a') = f(a)g(a').$$

  - $\epsilon = \psi \, {}^t u$ where ψ is the canonical isomorphism that associates to a linear form its value at 1_k.

Remark : The two following properties are equivalent :

- $\mathfrak{A}$ is of finite dimension;
- j_{nat} is an isomorphism.

Useful properties 3/3

- (iv) The correspondences $\mathfrak{C} \mapsto \mathfrak{C}^*$ and $f \mapsto {}^t f$ define a (contravariant) functor from the category $k\text{-}\mathrm{Cog}$ of all coalgebras to the category $k\text{-}\mathrm{Alg}$ of algebras.
  Proof : On the black(/white?)board.

The problem

If $\mathfrak{A}$ is an infinite dimensional algebra, $\mathfrak{A}^*$ is not in general a coalgebra. The transpose of the multiplication μ takes its values in $(\mathfrak{A} \otimes \mathfrak{A})^*$:

$$\langle {}^t\mu(f) | x \otimes y \rangle = \langle f | \mu(x \otimes y) \rangle = f(xy)$$

and j_{nat} is no more an isomorphism between $(\mathfrak{A} \otimes \mathfrak{A})^*$ and $\mathfrak{A}^* \otimes \mathfrak{A}^*$. The previous construction does not work anymore.

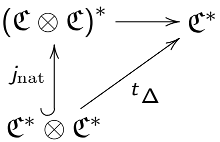
The problem

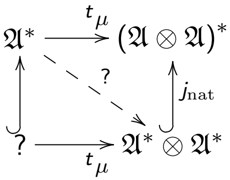
If $\mathfrak{A}$ is an infinite dimensional algebra, $\mathfrak{A}^*$ is not in general a coalgebra. The transpose of the multiplication μ takes its values in $(\mathfrak{A} \otimes \mathfrak{A})^*$:

$$\langle {}^t\mu(f) | x \otimes y \rangle = \langle f | \mu(x \otimes y) \rangle = f(xy)$$

and j_{nat} is no more an isomorphism between $(\mathfrak{A} \otimes \mathfrak{A})^*$ and $\mathfrak{A}^* \otimes \mathfrak{A}^*$. The previous construction does not work anymore.

Some diagrams :





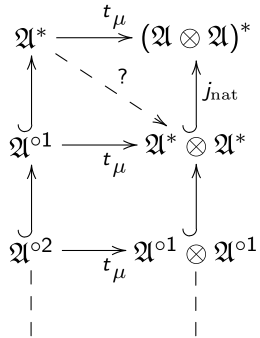
$\mathfrak{C}^*$ has an algebra structure.

$\mathfrak{A}^*$ does not have a coalgebra structure.

Definition : Finite dual (Sweedler's dual)

$$\mathfrak{A}^\circ = \left({}^t\mu\right)^{-1}\left(\mathfrak{A}^* \otimes \mathfrak{A}^*\right)$$

A priori, $\forall f \in \mathfrak{A}^\circ$, ${}^t\mu(f) \in \mathfrak{A}^\circ \otimes \mathfrak{A}^\circ$. Therefore,



and the interesting set is $\bigcap_i \mathfrak{A}^{\circ i}$. In fact, $\mathfrak{A}^{\circ 1} \otimes \mathfrak{A}^{\circ 1} = \mathfrak{A}^{\circ 2} \otimes \mathfrak{A}^{\circ 2}$.

Finite dual (Sweedler's dual)

Properties

- If $(\mathfrak{A}, \mu)$ is an algebra, $(\mathfrak{A}^\circ, {}^t\mu)$ is a coalgebra.
- If, morever, $\mathfrak{A}$ admits a unit 1, then $(\mathfrak{A}^\circ, {}^t\mu, \epsilon)$ is coalgebra with counit ϵ such that $\epsilon(f) = f(1)$.
- If $\mathfrak{A}$ is associative, then $\mathfrak{A}^\circ$ is coassociative.

$\mathfrak{A}^\circ$ is the biggest coalgebra contained in $\mathfrak{A}^*$ and induced by μ.
The mappings $\mathfrak{A} \mapsto \mathfrak{A}^\circ$ and $f \mapsto f^\circ$ define a contravariant functor from k-Alg to k-Cog (f° denotes the morphism induced by f^* on B° for $f \in \mathrm{Hom}(A, B)$).

Notations for shifts

Let $\mathfrak{A}$ be an algebra and $f \in \mathfrak{A}^*$. Then we define :

- $f_x : y \mapsto f(xy) = \langle f | xy \rangle$ (left shift);
- ${}_x f : y \mapsto f(yx)$ (right shift);
- ${}_x f_y : z \mapsto f(yzx)$.

Theorem by Abe (extended by Schützenberger's condition)

Characterization of the elements of $\mathfrak{A}^\circ$

Let $\mathfrak{A}$ be an algebra and $f \in \mathfrak{A}^*$. The following properties are equivalent :

- (i) ${}^t\mu(f) \in \mathfrak{A}^* \otimes \mathfrak{A}^*$;
- (ii) The family $(f_x)_{x\in\mathfrak{A}}$ is of finite rank;
- (iii) The family $({}_xf)_{x\in\mathfrak{A}}$ is of finite rank;
- (iv) The family $({}_xf_y)_{x,y\in\mathfrak{A}}$ is of finite rank;
- (v) $f(xy) = \sum_{i=1}^{n} f_i(x)g_i(y)$;
- (vi) $\exists\ \alpha : \mathfrak{A} \to k^{n\times n}$ and $(\lambda,\gamma) \in k^{1\times n} \times k^{n\times 1}$ such that

$$\forall x \in \mathfrak{A}, \quad f(x) = \lambda\,\alpha(x)\,\gamma;$$

- (vii) $\mathrm{Ker}(f)$ contains an ideal of finite codimension (i.e. there exists an ideal I such that $\dim(\mathrm{Ker}(f)/I) < \infty$).

Proof (elements) 1/4 :$(i) \Rightarrow (v)$ and $(v) \Rightarrow (ii), (iii)$

We assume that ${}^t\mu(f) \in \mathfrak{A}^* \otimes \mathfrak{A}^*$. Therefore ${}^t\mu(f) = \sum_{i=1}^n f_i \otimes g_i$. Now

$$\langle {}^t\mu(f) | x \otimes y \rangle = f(xy) = \sum_{i=1}^n \langle f_i \otimes g_i | x_i \otimes y_i \rangle = \sum_{i=1}^n f_i(x) g_i(y) \qquad (2)$$

which is the condition (v).
Hence we can write that

$$({}_y f) = \sum_{i=1}^n g_i(y) f_i; \qquad {}_{\mathfrak{A}}f \subset \mathrm{span}(f_i).$$

$$(f_x) = \sum_{i=1}^n f_i(x) g_i; \qquad f_{\mathfrak{A}} \subset \mathrm{span}(g_i).$$

The orbits and ${}_{\mathfrak{A}}f$ and $f_{\mathfrak{A}}$ are of finite rank.

Proof 2/4 : $(v) \Rightarrow (iv)$

If the g_i's are in $f_{\mathfrak{A}}$, their shifts are also in this orbit (which is finite dimensional). Thus, they are of finite rank and $g_i(yz) = \sum_{j=1}^{m} g_{ij}^1(y) g_{ij}^2(z)$.

$$f(yzx) = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i(y) g_{ij}^1(y) g_{ij}^2(z),\ \forall\, z \in \mathfrak{A}.$$

This is equivalent to the following equation:

$$_x f_y = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i(y) g_{ij}^1(y) g_{ij}^2 \qquad (5)$$

Do we have $(g_i)_i \subset f_{\mathfrak{A}}$?

Proof 3/4 : $(v) \Rightarrow (iv)$

We assume now that n is minimal.
Since ${}^t\mu$ is bilinear, the first lemma applied to eq. (2) implies that the f_i's and g_i's form free families.

Proof 3/4 : $(v) \Rightarrow (iv)$

We assume now that n is minimal.
Since ${}^t\mu$ is bilinear, the first lemma applied to eq. (2) implies that the f_i's and g_i's form free families.

$(g_i)_i$ is a free family of minimal rank. Hence, $(g_i)_i$ is a basis of $\mathrm{span}_{x\in\mathfrak{A}}(f_x)$ and the g_i's are in $f_{\mathfrak{A}}$.

Eq. (5) implies that

$$({}_x f_y)_{x,y\in\mathfrak{A}} \text{ is of finite rank.}$$

Proof 3/4 : $(iii) \Rightarrow (vi)$

Let ${}_{x_1}f, {}_{x_2}f, \ldots, {}_{x_n}f$ be a basis of the orbit ${}_{\mathfrak{A}}f$ of f under the action of $\mathfrak{A}$. ${}_1f = f \in {}_{\mathfrak{A}}f$. Thus,

$$ {}_1f = (\lambda_1 \ldots \lambda_n) \begin{pmatrix} {}_{x_1}f \\ \vdots \\ {}_{x_n}f \end{pmatrix} $$

and

$$ {}_1f(y) = (\lambda_1 \ldots \lambda_n) \begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix}. $$

Proof 3/4 : $(iii) \Rightarrow (vi)$

Now, if we do another shift on the left, we stay in the same space generated by ${}_{x_1}f, {}_{x_2}f, \ldots, {}_{x_n}f$ and we have another decomposition which involves a (uniquely defined) matrix $\alpha(y)$:

$$
{}_{y}\begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix} = \begin{pmatrix} {}_{yx_1}f(y) \\ \vdots \\ {}_{yx_n}f(y) \end{pmatrix}
$$

with ${}_{yx_i}f(y) = \sum_{j=1}^{n} \alpha_{ij}(y)\, {}_{x_j}f$. Therefore,

$$
{}_{y}\begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix} = \begin{pmatrix} \alpha_{11}(y) & \ldots & \alpha_{1n}(y) \\ \vdots & & \vdots \\ \alpha_{n1}(y) & \ldots & \alpha_{nn}(y) \end{pmatrix} \begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix}
$$

Proof 3/4 : (*iii*) ⇒ (*vi*)

$f(z) = {}_z f(1)$ thus

$$
{}_z\left[(\lambda_1 \ldots \lambda_n) \begin{pmatrix} {}_{x_1}f \\ \vdots \\ {}_{x_n}f \end{pmatrix} \right]\Bigg|_{y=1} = (\lambda_1 \ldots \lambda_n) \; {}_z\begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix}\Bigg|_{y=1}
$$

$$
= \left[(\lambda_1 \ldots \lambda_n)\alpha(z) \begin{pmatrix} {}_{x_1}f(y) \\ \vdots \\ {}_{x_n}f(y) \end{pmatrix} \right]\Bigg|_{y=1}
$$

Finally,

$$
f(z) = (\lambda_1 \ldots \lambda_n)\alpha(z) \begin{pmatrix} {}_{x_1}f(1) \\ \vdots \\ {}_{x_n}f(1) \end{pmatrix}.
$$

Proof 4/4 : $(iii) \Rightarrow (i)$

If $({}_x f)_{x\in\mathfrak{A}}$ is of finite rank, there exists a linear representation $(\lambda, \alpha, \gamma)$ of f :

$$f(z) = (\lambda_1 \dots \lambda_n)\alpha(z) \begin{pmatrix} {}_{x_1}f(1) \\ \vdots \\ {}_{x_n}f(1) \end{pmatrix}$$

with :

$$\alpha(x+y) = \alpha(x) + \alpha(y),$$

$$\alpha(\delta x) = \delta\alpha(x),$$

$$\alpha(xy) = \alpha(x)\alpha(y).$$

Hence $f(z) = \sum_{k,\ell=1}^{n} \lambda_k \alpha_{k\ell}(z)\gamma_\ell$.

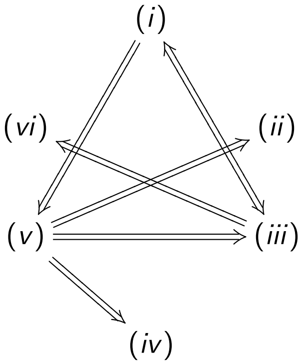
Proof 4/4 : (*iii*) ⇒ (*i*)

Therefore,

$$f(xy) = \sum_{k,\ell=1}^{n} \lambda_k \alpha_{k\ell}(xy)\gamma_\ell$$
$$= \sum_{k,\ell=1}^{n} \sum_{j=1}^{m} \lambda_k \alpha_{kj}(x)\alpha_{j\ell}(y)\gamma_\ell$$
$$= \sum_{j=1}^{m} \left(\sum_{k=1}^{n} \lambda_k \alpha_{kj}(x) \right) \left(\sum_{\ell=1}^{n} \alpha_{j\ell}(y)\gamma_\ell \right)$$

This equation tells us that $f(xy)$ and therefore ${}^t\mu(f)$ belongs to $\mathfrak{A}^* \otimes \mathfrak{A}^*$.

What was proved so far ?



The proofs of the following implications are straightforward now :
$(ii) \Rightarrow (i)$, $(iv) \Rightarrow (i)$ and $(vi) \Rightarrow (i)$.