An algebraic Birkhoff decomposition for the continuous renormalization group

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Séminaire CALIN, LIPN Paris 13, 8th February 2011
What is the algebraic (geometric) structure underlying renormalization?

- Perturbative renormalization in QFT is a Birkhoff decomposition → Hopf algebra of Feynman diagrams. (*Connes-Kreimer* 2000)

- Exact renormalization is an *algebraic* Birkhoff decomposition → Hopf algebra of decorated rooted trees.
Program

- Birkhoff decomposition
- Exact Renormalization Group equations as fixed point equation
- Power series of trees
- Algebraic Birkhoff decomposition for the ERG


Birkhoff decomposition

Complex plane $\gamma(C)$

Lie group $G$

$\gamma(z) = \gamma_{-1}(z)\gamma_{+}(z), \quad z \in C$

where $\gamma_{\pm} : C_{\pm} \rightarrow G$ are holomorphic.

$\rightarrow G$ nice enough: exists for any loop $\gamma$, unique assuming $\gamma_{-}(\infty) = 1$.

$\rightarrow \gamma$ defined on $C_{+}$ with pole at $D$:

$\gamma \rightarrow \gamma_{+}(D)$

is a natural principle to extract finite value from singular expression $\gamma(D)$.

$\rightarrow$ dimensional regularization in QFT: $D$ is the dimension of space time, $G$ is the group of characters of the Hopf algebra of Feynman diagrams.
Birkhoff decomposition: Hopf algebra of Feynman diagrams

Coalgebra $C_o$: reverse the arrow!

Coproduct $\Delta : C_0 \mapsto C_0 \otimes C_0$, counity $\eta : C_0 \mapsto \mathbb{C}$,

\[ C_o \otimes C_o \otimes C_o \xrightarrow{\Delta \otimes \text{id}_C} C_o \otimes C_o \]

\[ C_o \otimes C_o \xleftarrow{\Delta} C_o \]

\[ \mathbb{C} \otimes C_o \xleftarrow{\eta \otimes \text{id}_C} C_o \otimes C_o \]

\[ C_o \otimes C \xrightarrow{\text{id}_C \otimes \eta} C_o \otimes C_o \]

\[ C_o \xrightarrow{\text{id}_C} C_o \]

\[ \mathbb{C} \xrightarrow{\Delta} C_o \otimes C_o \]

\[ C_o \xleftarrow{\text{id}_C} C_o \]
Birkhoff decomposition: Hopf algebra of Feynman diagrams

Bialgebra $\mathcal{B}$: algebra + coalgebra.

Antipode $S : \mathcal{B} \leftrightarrow \mathcal{B}$,

\[ \text{id}_\mathcal{B} \ast S = m(\text{id}_\mathcal{B} \otimes S)\Delta = \eta 1, \quad S \ast \text{id}_\mathcal{B} = m(S \ast \text{id}_\mathcal{B})\Delta = \eta 1. \]

Bialgebra with antipode = Hopf algebra $\mathcal{H}$.

→ 1PI-Feynman diagrams form an Hopf algebra,

→ Combinatorics of perturbative renormalization is encoded within the coproduct $\Delta$. 
Birkhoff decomposition: Hopf algebra of Feynman diagrams

The Hopf algebra $H_F$ of Feynman diagrams:

Algebra structure:
- product: disjoint union of graphs,
- unity: the empty set.

Hopf algebra structure:
- counity: $\eta(\emptyset) = 1$, $\eta(\Gamma) = 0$ otherwise,
- coproduct:
  $$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma, \gamma \neq \Gamma} \gamma \otimes \Gamma / \gamma$$

$$\Delta(\text{graph}) = \text{graph} \otimes 1 + 1 \otimes \text{graph} + \sum_{\gamma \subset \Gamma, \gamma \neq \Gamma} \gamma \otimes \Gamma / \gamma$$

- antipode: built by induction.
Birkhoff decomposition: perturbative renormalization

\( \mathcal{A} : \) complex functions in \( \mathbb{C} \), pole in \( D (= 4) \).

\( \mathcal{A}_+ : \) holomorphic functions in \( \mathbb{C} \).

\( \mathcal{A}_- : \) polynomial in \( \frac{1}{z - D} \) without constant term.

\[
\begin{align*}
\text{Feynman rules: } & H_F \xrightarrow{U} \mathcal{A} \\
\text{Counterterms: } & H_F \xrightarrow{C} \mathcal{A}_- \\
\text{Renormalized theory: } & H_F \xrightarrow{R} \mathcal{A}_+
\end{align*}
\]

\[ C \ast U = R \]

Compose with character \( \chi_z \) of \( \mathcal{A} \),

\[
\gamma(z) \doteq \chi_z \circ U, \quad \gamma_-(z) \doteq \chi_z \circ C, \quad \gamma_+(z) \doteq \chi_z \circ R,
\]

\( \gamma(z), z \in \mathbb{C} \) is a loop within the group \( G \) of characters of \( H_F \),

\[
\gamma(z) = \gamma_-^{-1}(z) \gamma_+(z).
\]

The renormalized theory is the evaluation at \( D \) of the positive part of the Birkhoff decomposition of the bare theory.
The Exact Renormalization Group equations govern the evolution of the parameters of the theory with respect to the scale of observation (e.g. energie $\Lambda$),

$$\Lambda \frac{\partial}{\partial \Lambda} S = \beta(\Lambda, S)$$

where $S(\Lambda) \in \mathcal{E}$, vector space of "actions".

- no analogous to the dimension $D$ where to localize the pole
- analogous to $C \ast U = R$.

**Definition (Connes, Kreimer, Kastler):** $H$ commutative Hopf algebra, $\mathcal{A}$ commutative algebra. $p_-$ projection onto a subalgebra $\mathcal{A}_-$. An algebra morphism $\gamma : H \rightarrow \mathcal{A}$ has a unique algebraic Birkhoff decomposition if there exist two algebra morphisms $\gamma_+, \gamma_-$ from $H$ to $\mathcal{A}$ such that

$$\gamma_+ = \gamma_- * \gamma$$

$$p_+ \gamma_+ = \gamma_+, \quad p_- \gamma_- = \gamma_-$$

with $p_+$ the projection on

$$\mathcal{A}_+ = \text{Ker } p_-.$$
ERG as fixed point equation

Dimensional analysis: \( \Lambda \rightarrow t, \ S \rightarrow x, \ \beta \mapsto X, \)

\[
\frac{\partial x}{\partial t} = Dx + X(x)
\]

\( x(t) \in \mathcal{E}, \ D \) diagonal matrix of dimensions, \( X \) smooth operator \( \mathcal{E} \rightarrow \mathcal{E}, \)

\[
X(x + y) = X(x) + X_x'(y) + X_x''(y, y) + \ldots + \frac{1}{n!}X_x^{[n]}(y, \ldots, y) + \mathcal{O}(\|y\|^{n+1})
\]

where \( X_x^{[n]} \) is a linear symmetric application from \( \mathcal{E}^{[n]} \) to \( \mathcal{E}. \)

\[
x(t) = e^{(t-t_0)D}x_0 + \int_{t_0}^{t} e^{(t-u)D}X(x(u))du.
\]

\( x \) belongs to the space \( \tilde{\mathcal{E}} \) of smooth maps from \( \mathbb{R}^{*+} \) to \( \mathcal{E}, \) as well as \n
\[
\tilde{x}_0: \ t \mapsto e^{(t-t_0)D}x_0.
\]

Define \( \chi_0, \) smooth map from \( \tilde{\mathcal{E}} \) to \( \tilde{\mathcal{E}}, \)

\[
\chi_0(x): t \mapsto \int_{t_0}^{t} e^{(t-u)D}X(x(u))du.
\]
Fixed point equation

\[ x = \tilde{x}_0 + \chi_0(x) \]

- \( x(t) \) represents the parameters at a scale \( t \).
- \( \tilde{x}_0 \) encodes the initial conditions at a fixed scale \( t_0 \).

Wilson’s ERG context: \( t_0 \) is an UV cutoff. One interested in \( t_0 \to +\infty \).
ERG as fixed point equation: mixed initial conditions

\[ \tilde{x}_0(t) = e^{(t-t_0)D}x_0 \]
\[
\begin{cases} 
\text{converges on } E^+ \\
\text{is constantly zero on } E^0 \quad \text{as } t_0 \to +\infty \\
\text{diverges on } E^- 
\end{cases}
\]

where \( E^+, E^0, E^- \) are proper subspaces of \( D \) corresponding to positive, zero and negative eigenvalues (irrelevant, marginal, relevant).

- Finiteness of \( x(t) \) at high scale by imposing initial conditions for relevant sector at scale \( t_1 \neq t_0 \).
- \( P \) orthogonal projection \( E \mapsto E^- \) allows mixed initial conditions

\[
x_R = P\tilde{x}_1 + (I - P)\tilde{x}_0 : 
\]

\[
\chi_R = P\chi_1 + (I - P)\chi_0 \text{ with } \chi_i(x) : t \mapsto \int_{t_i}^t e^{(t-u)D}X(x(u))du 
\]

\[
x(t) = x_R + \chi_R(x)
\]

Renormalization deals with change of initial condition in fixed point equation.
\[ \chi \text{ is a smooth operator from } \tilde{\mathcal{E}} \text{ to } \tilde{\mathcal{E}}: \]

\[ \chi(x + y) = \chi(x) + \chi_x'(y) + \chi_x''(y, y) + \ldots + \frac{1}{n!} \chi_x^{[n]}(y, \ldots, y) + O(\|y\|^{n+1}) \]

where \( \chi_x^{[n]} \) is a linear symmetric application from \( \tilde{\mathcal{E}}^{[n]} \) to \( \tilde{\mathcal{E}} \).

▶ **Physicists’ notations:** \( x = \{x^\mu\} \), \( \chi(x) = \{\chi^\mu(x)\} \),

\[ \chi_x'(y) = \partial_\nu \chi^\mu_{/x} \ y^\nu, \quad \chi_x''(y_1, y_2) = \partial_\nu \partial_\rho \chi^\mu_{/x} \ y_1^\nu \ y_2^\rho. \]

▶ **Coordinate free notations:** \( \chi'(\chi) \) is the map \( \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}} \)

\[ y \mapsto \chi'_y(\chi(y)). \]
Power series of trees: smooth non linear operators

\[ \chi^\emptyset \doteq \mathbb{I}, \quad \chi^\bullet \doteq \chi, \quad \chi \doteq \chi'(\chi), \quad \chi^\prime \doteq \frac{1}{2} \chi''(\chi, \chi) \ldots \]

Taylor expansion:

\[ \chi(\mathbb{I} + \chi) = \chi^\bullet + \chi + \chi^\prime + \ldots = \sum_T \phi(T) \chi^T = f_\phi[\chi] \]

where \( \phi(T) = 1 \) for any rooted tree \( T \), except \( \phi(\emptyset) = 0. \)
$H_T$ is a Hopf algebra with counit $\epsilon = 0$ except $\epsilon(1) = 1$, the antipode

$$S : \bullet \mapsto -\bullet$$

$$T \mapsto -T - \sum_{c \in C(T)} S(P_c(T))R_c(T)$$

and the coproduct

$$\Delta(T) = T \otimes 1 + 1 \otimes T + \sum_{c \in C(T)} P_c(T) \otimes R_c(T), \quad \Delta(1) = 1 \otimes 1.$$
Proposition: Butcher group, B-series; T.K, P.M.:

The group of formal power series starting with $I$ (i.e. $\phi(\emptyset) = 1$) is isomorphic to the opposite group of characters of $H_T$.

$$f_\phi[\chi] \circ f_\psi[\chi] = \sum_T \phi(T) \chi^T \left( \sum_{T'} \psi(T') \chi^{T'} h^{|T'|} \right) h^{|T|}$$

$$= \sum_T \psi * \phi(T) \chi^T h^{|T|}$$

$$= f_{\psi * \phi}[\chi].$$
Power series of trees: solution of fixed point equation

\[ x = x_0 + \chi_0(x) \iff x_0 = (I - \chi_0)(x). \]

\[ x = (I - \chi_0)^{-1}(x_0) = f_\psi[\chi_0]^{-1}(x_0) = f_{\phi_1}[\chi_0](x_0) \]

where \( \psi = 0 \) except \( \psi(\emptyset) = 1, \psi(\bullet) = -1 \) and \( \phi_1 = \psi^{-1} = 1 \).

\[ x = x_R + \chi_R(x) \implies x = f_{\phi_1}[\chi_R](x_R) \]

\[ \xi = I - (I - \chi_R) \circ (I - \chi_0)^{-1} \implies (I - \chi_R)^{-1} = (I - \chi_0)^{-1} \circ (I - \xi)^{-1} \]
Power series of trees: rooted trees with two decorations

\[
f_{\phi_1}[\chi_R] = f_{\phi_1}[\chi_0] \circ f_{\phi_1}[\xi]
\]

1 character, 2 operators \(\iff\) 1 operator, 2 characters :

\[
f_{\phi_+}[Y] = f_{\phi}[Y] \circ f_{\phi_-}[Y]
\]

\[
Y_\bullet = \chi_R, \quad Y^\bullet = \xi, \quad Y^{\circ\circ} = \chi_R''(\xi, \xi),
\]

\[
\phi = \phi_1^{-1} \ast \phi_+
\]

\[
\phi_-(T) = \begin{cases} 
\phi_1(T) & \text{if } T \in H_{\bullet} \\
0 & \text{if } T \notin H_{\bullet},
\end{cases}
\]

\[
\phi_+(T) = \begin{cases} 
\phi_1(T) & \text{if } T \in H_{\bullet} \\
0 & \text{if } T \notin H_{\bullet},
\end{cases}
\]

where \(H_{\bullet}, H_{\bullet}\) are the set of trees decorated with one decoration only so that

\[
f_{\phi_1}[\chi_R] = f_{\phi_+}[Y], \quad f_{\phi_1}[\xi] = f_{\phi_-}[Y]
\]

**Proposition:** \(\lim_{t_0 \to +\infty} f_{\phi_1}[\chi_R](\chi_R)\) is finite order by order and does not depend on \(x_0\).
Algebraic Birkhoff decomposition for the ERG

Perturbative renormalization: $H_F \xrightarrow{\text{Feyman rules}} A \xrightarrow{\text{evaluation at } z} G$.

Exact renormalization: $H_T \xrightarrow{\text{evaluation on decorations}} G$.

→ No Birkhoff decomposition since no loop in $G$.
→ Algebraic Birkhoff decomposition on which algebra?

As $U, C, R$ map a Feynman diagram to a meromorphic function, characters map a decorated rooted tree to a monomial in $Y^T$,

$$\gamma(T) = \phi(T) Y^T, \quad \gamma_{\pm}(T) = \phi_{\pm}(T) Y^T.$$

Unfortunately $\gamma, \gamma_{\pm}$ do not define an algebraic Birkhoff decomposition.

$$\gamma_+(\begin{array}{c}
\end{array}) = 0$$

$$\langle \gamma_-( \otimes \gamma, 1 \otimes \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \otimes 1 + \cdots \rangle$$

$$\text{essaibirk} + Y^\circ Y^\bullet.$$
Algebraic Birkhoff decomposition for the ERG

→ Algebraic Birkhoff decomposition with

- target
  \[ A = \{1, •, ■\}, \quad A_- = \{1, •\}. \]

- projection \( p_- : A \to A_- \)
  \[ p_-(1) = 1, \quad p_-(•) = •, \quad p_-(■) = 0. \]

- Algebra homorphism \( H_T \to A \)
  \[ \gamma(T) = \phi(T)\Gamma(T), \quad \gamma_{\pm}(T) = \phi_{\pm}(T)\Gamma(T). \]

where \( \phi = \phi_-^{-1} \ast \phi_+ \) and \( \Gamma \) counts the decoration

\[ \Gamma(\begin{array}{c} \star \hline \end{array} \begin{array}{c} \star \hline \end{array}) = •^3\square \]

[\[\gamma_+ = \gamma_- \ast \gamma\]
Perturbative renormalization with dimensional regularization has a nice description in terms of Birkhoff decomposition of a loop around the dimension $D$ of space time

- geometrical interpretation (bundles on the Riemann sphere),
- Galois theory for the renormalization group (Connes, Marcolli).

Analogous formulation for ERG, only at the algebraic level

- Is the algebra of decorations an artificial tool?
- Deeper structure (Rota-Baxter operator, cf Ebrahimi-Fard)?
- Signification of the characters?