The Saddle point method in combinatorics asymptotic analysis: successes and failures (A personal view)

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The Saddle point method in combinatorics asymptotic analysis
The number of inversions in permutations

Let \( a_1 \ldots a_n \) be a permutation of the set \( \{1, \ldots, n\} \). If \( a_i > a_k \) and \( i < k \), the pair \( (a_i, a_k) \) is called an inversion; \( I_n(j) \) is the number of permutations of length \( n \) with \( j \) inversions.

Here, we show how to extend previous results using the saddle point method. This leads, e.g., to asymptotics for \( I_{\alpha n+\beta}(\gamma n + \delta) \), for integer constants \( \alpha, \beta, \gamma, \delta \) and more general ones as well.

With this technique, we will also show the known result that \( I_n(j) \) is asymptotically normal, with additional corrections.

The generating function for the numbers \( I_n(j) \) is given by

\[
\Phi_n(z) = \sum_{j \geq 0} I_n(j) z^j = (1 - z)^{-n} \prod_{i=1}^{n} (1 - z^i).
\]

By Cauchy’s theorem,

\[
I_n(j) = \frac{1}{2\pi i} \int_{\mathcal{C}} \Phi_n(z) \frac{dz}{z^{j+1}},
\]

where \( \mathcal{C} \) is, say, a circle around the origin.
The Gaussian limit, \( j = m + x\sigma, m = n(n - 1)/4 \)

Actually, we obtain here local limit theorems with some corrections (=lower order terms).

The Gaussian limit of \( I_n(j) \) is easily derived from the generating function \( \Phi_n(z) \) (using the Lindeberg-Lévy conditions. Indeed, this generating function corresponds to a sum for \( i = 1, \ldots, n \) of independent, uniform \([0..i−1]\) random variables. As an exercise, let us recover this result with the saddle point method, with an additional correction of order \( 1/n \). We have, for the random variable \( X_n \) characterized by

\[
\mathbb{P}(X_n = j) = J_n(j),
\]

with \( J_n := I_n/n! \),

\[
m := \mathbb{E}(X_n) = n(n - 1)/4,
\]

\[
\sigma^2 := \mathbb{V}(X_n) = n(2n + 5)(n − 1)/72.
\]
We know that
\[ l_n(j) = \frac{1}{2\pi i} \int_{\Omega} e^{S(z)} \, dz \]
where \( \Omega \) is inside the analyticity domain of the integrand, encircles the origin, passes through the saddle point \( \tilde{z} \) and
\[ S = \ln(\Phi_n(z)) - (j + 1) \ln z, \]
\( \tilde{z} \) is the solution of
\[ S^{(1)}(\tilde{z}) = 0. \] (1)
Figure 1 shows the real part of \( S(z) \) together with a path \( \Omega \) through the saddle point.
Figure 1: Real part of $S(z)$. Saddle-point and path, $n = 10$
We have

\[ J_n(j) = \frac{1}{n!2\pi i} \int_\Omega \exp \left[ S(\tilde{z}) + S^{(2)}(\tilde{z})(z - \tilde{z})^2/2! + \sum_{l=3}^\infty S^{(l)}(\tilde{z})(z - \tilde{z})^l/l! \right] dz \]

(note carefully that the linear term vanishes). Set \( z = \tilde{z} + i\tau \). This gives

\[ J_n(j) = \frac{1}{n!2\pi} \exp[S(\tilde{z})] \int_{-\infty}^{\infty} \exp \left[ S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^\infty S^{(l)}(\tilde{z})(i\tau)^l/l! \right] d\tau. \] (2)

We can now compute (2), for instance by using the classical trick of setting

\[ S^{(2)}(\tilde{z})(i\tau)^2/2! + \sum_{l=3}^\infty S^{(l)}(\tilde{z})(i\tau)^l/l! = -u^2/2. \]
To justify this procedure, we proceed in three steps (to simplify, we use \( \tilde{z} = 1 \)).

- setting \( z = e^{i\theta} \), we must show that the tail integral

\[
\int_{\theta_0}^{2\pi-\theta_0} e^{S(z)} d\theta
\]

is negligible for some \( \theta_0 \),

- we must insure that a central Gaussian approximation holds:

\[
S(z) \sim S(\tilde{z}) + S^{(2)}(\tilde{z})(z - \tilde{z})^2/2!
\]

in the integration domain \(|\theta| \leq \theta_0\), by choosing for instance \( S^{(2)}(\tilde{z})\theta_0^2 \to \infty, S^{(3)}(\tilde{z})\theta_0^3 \to 0, n \to \infty\),

- we must have a tail completion: the incomplete Gaussian integral must be asymptotic to a complete one.
We split the exponent of the integrand as

\[ S := S_1 + S_2, \]  

\[ S_1 := \sum_{i=1}^{n} \ln(1 - z^i), \]  

\[ S_2 := -n \ln(1 - z) - (j + 1) \ln z. \]

Set

\[ S^{(i)} := \frac{d^i S}{dz^i}. \]

Set \( \tilde{z} := z^* - \varepsilon \), where \( z^* = \lim_{n \to \infty} \tilde{z} \). Here, \( z^* = 1 \). (This notation always means that \( z^* \) is the approximate saddle point and \( \tilde{z} \) is the exact saddle point; they differ by a quantity that has to be computed to some degree of accuracy.) This leads, to first order, to

\[ \frac{(n + 1)^2}{4} - \frac{3n}{4} - \frac{5}{4} - j \]  

\[ + \left[ -\frac{(n + 1)^3}{36} + 7\frac{(n + 1)^2}{24} - \frac{49n}{72} - \frac{91}{72} - j \right] \varepsilon = 0. \]
Set \( j = m + x\sigma \) in (4). This shows that, asymptotically, \( \varepsilon \) is given by a Puiseux series of powers of \( n^{-1/2} \), starting with \( -6x/n^{3/2} \).

To obtain the next terms, we compute the next terms in the expansion of (1), i.e., we first obtain

\[
\begin{align*}
\left[ \frac{(n + 1)^2}{4} - 3n/4 - 5/4 - j \right] & + \left[ -(n + 1)^3/36 + 7(n + 1)^2/24 - 49n/72 - 91/72 - j \right] \varepsilon \\
& + \left[ -j - 61/48 - (n + 1)^3/24 + 5(n + 1)^2/16 - 31n/48 \right] \varepsilon^2 = 0.
\end{align*}
\]

(5)
More generally, even powers $\varepsilon^{2k}$ lead to a $O(n^{2k+1}) \cdot \varepsilon^{2k}$ term and odd powers $\varepsilon^{2k+1}$ lead to a $O(n^{2k+3}) \cdot \varepsilon^{2k+1}$ term. Now we set $j = m + x\sigma$, expand into powers of $n^{-1/2}$ and equate each coefficient with 0. This leads successively to a full expansion of $\varepsilon$. Note that to obtain a given precision of $\varepsilon$, it is enough to compute a given finite number of terms in the generalization of (5). We obtain

$$
\varepsilon = -6x/n^{3/2} + (9x/2 - 54/25x^3)/n^{5/2} - (18x^2 + 36)/n^3
+ x[-30942/30625x^4 + 27/10x^2 - 201/16]/n^{7/2} + O(1/n^4).
$$

(6)
Let us first analyze $S(\tilde{z})$. We obtain

\begin{align*}
S_1(\tilde{z}) &= \sum_{i=1}^{n} \ln(i) + \left[-\frac{3}{2} \ln(n) + \ln(6) + \ln(-x)\right]n \\
&\quad + \frac{3}{2}x\sqrt{n} + \frac{43}{50}x^2 - \frac{3}{4} \\
&\quad + \left[\frac{3x}{8} + \frac{6}{x} + \frac{27}{50}x^3\right]/\sqrt{n} \\
&\quad + \left[\frac{5679}{12250}x^4 - \frac{9}{50}x^2 + \frac{173}{16}\right]/n + \mathcal{O}(n^{-3/2}), \\
S_2(\tilde{z}) &= \left[\frac{3}{2} \ln(n) - \ln(6) - \ln(-x)\right]n - \frac{3}{2}x\sqrt{n} - \frac{34}{25}x^2 + \frac{3}{4} \\
&\quad - \left[\frac{3x}{8} + \frac{6}{x} + \frac{27}{50}x^3\right]/\sqrt{n} \\
&\quad - \left[\frac{5679}{12250}x^4 - \frac{9}{50}x^2 + \frac{173}{16}\right]/n + \mathcal{O}(n^{-3/2}),
\end{align*}

and so

\[ S(\tilde{z}) = -x^2/2 + \ln(n!) + \mathcal{O}(n^{-3/2}). \]
Also,

\[ S^{(2)}(\tilde{z}) = \frac{n^3}{36} + (1/24 - 3/100x^2)n^2 + O(n^{3/2}), \]
\[ S^{(3)}(\tilde{z}) = O(n^{7/2}), \]
\[ S^{(4)}(\tilde{z}) = -\frac{n^5}{600} + O(n^4), \]
\[ S^{(l)}(\tilde{z}) = O(n^{l+1}), \quad l \geq 5. \]

We compute \( \tau \) as a \text{truncated series} in \( u \), setting \( d\tau = \frac{d\tau}{du} du \),
expanding w.r.t. \( n \) and integrating on \( [u = -\infty..\infty] \). This amounts to the \text{reversion} of a series. Finally (2) leads to

\[ J_n \sim e^{-x^2/2} \cdot \exp \left[ \left( -\frac{51}{50} + \frac{27}{50x^2} \right) / n + O(n^{-3/2}) \right] / (2\pi n^3/36)^{1/2}. \]

(7)

Note that \( S^{(3)}(\tilde{z}) \) does not contribute to the \( 1/n \) correction.
To check the effect of the correction, we first give in Figure 2, for $n = 60$, the comparison between $J_n(j)$ and the asymptotics (7), without the $1/n$ term. Figure 3 gives the same comparison, with the constant term $-51/(50n)$ in the correction. Figure 4 shows the quotient of $J_n(j)$ and the asymptotics (7), with the constant term $-51/(50n)$. The “hat” behaviour, already noticed by Margolius, is apparent. Finally, Figure 5 shows the quotient of $J_n(j)$ and the asymptotics (7), with the full correction.
The number of inversions in permutations Median versus $A$

A (large) for a Luria-Delbruck-like distribution, with parameter $A$

Sum of positions of records in random permutations

Merten's theorem for toral automorphisms

Representations of numbers as $\sum_{n \leq x} \varepsilon_k k$

The q-Catalan numbers

A simple case of the Mahonian statistic

Asymptotics of the Stirling numbers of the first kind revisited

0.001
0.002
0.003
0.004
0.005

0.001
0.002
0.003
0.004
0.005

700 800 900 1000 1100

\[ J_n(j) \] (circle) and the asymptotics (7) (line), without the $1/n$

term, $n = 60$

Figure 2: $J_n(j)$ (circle) and the asymptotics (7) (line), without the $1/n$

term, $n = 60$
The number of inversions in permutations

Median versus $A(A)$ for a Luria-Delbruck-like distribution, with parameter $A$

Sum of positions of records in random permutations

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Figure 3: $J_n(j)$ (circle) and the asymptotics (7) (line), with the constant in the $1/n$ term, $n = 60$
Figure 4: Quotient of $J_n(j)$ and the asymptotics (7), with the constant in the $1/n$ term, $n = 60$
The number of inversions in permutations Median versus $A$

A large

Sum of positions of records in random permutations Merten's theorem for toral automorphisms Representations of numbers as $\sum n^k = -n^\varepsilon k$
The q-Catalan numbers A simple case of the Mahonian statistic Asymptotics of the Stirling numbers of the first kind revisited

Figure 5: Quotient of $J_n(j)$ and the asymptotics (7), with the full $1/n$ term, $n = 60$
Case $j = n - k$

It is easy to see that here, we have $z^* = 1/2$. We obtain, to first order,

$$[C_{1,n} - 2j - 2 + 2n] + [C_{2,n} - 4j - 4 - 4n] \epsilon = 0$$

with

$$C_{1,n} = C_1 + O(2^{-n}),$$

$$C_1 = \sum_{i=1}^{\infty} \frac{-2i}{2^i - 1} = -5.48806777751 \ldots,$$

$$C_{2,n} = C_2 + O(2^{-n}),$$

$$C_2 = \sum_{i=1}^{\infty} 4 \frac{i(i2^i - 2^i + 1)}{(2^i - 1)^2} = 24.3761367267 \ldots.$$
Set \( j = n - k \). This shows that, asymptotically, \( \varepsilon \) is given by a Laurent series of powers of \( n^{-1} \), starting with \( (k - 1 + C_1/2)/(4n) \). We next obtain

\[
[C_1 - 2j - 2 + 2n] + [C_2 - 4j - 4 - 4n]\varepsilon + [C_3 + 8n - 8j - 8]\varepsilon^2 = 0
\]

for some constant \( C_3 \). More generally, powers \( \varepsilon^{2k} \) lead to a \( \mathcal{O}(1) \cdot \varepsilon^{2k} \) term, powers \( \varepsilon^{2k+1} \) lead to a \( \mathcal{O}(n) \cdot \varepsilon^{2k+1} \) term. This gives

\[
\varepsilon = (k - 1 + C_1/2)/(4n) + (2k - 2 + C_1)(4k - 4 + C_2)/(64n^2) + \mathcal{O}(1/n^3).
\]

Now we derive

\[
S_1(\tilde{z}) = \ln(Q) - C_1(k - 1 + C_1/2)/(4n) + \mathcal{O}(1/n^2)
\]

with \( Q := \prod_{i=1}^{\infty}(1 - 1/2^i) = .288788095086 \ldots \). Similarly

\[
S_2(\tilde{z}) = 2 \ln(2) n + (1 - k) \ln(2) + (-k^2/2 + k - 1/2 + C_1^2/8)/(2n) + \mathcal{O}(1/n^2)
\]
and so

\[ S(\tilde{z}) = \ln(Q) + 2 \ln(2)n + (1 - k) \ln(2) + (A_0 + A_1 k - k^2 / 4) / n + O(1/n^2) \]

with

\[ A_0 := -(C_1 - 2)^2 / 16, \]
\[ A_1 := (-C_1 / 2 + 1) / 2. \]

Finally, we obtain

\[ I_n(n - k) \sim e^{2 \ln(2)n + (1 - k) \ln(2)} \frac{Q}{(2\pi S(2,1))^{1/2}} \times \]
\[ \times \exp \left\{ \left[ (A_0 + 1/8 + C_2/16) + (A_1 + 1/4) k - k^2 / 4 \right] / n + O(1/n^2) \right\} . \]

Figure 6 shows, for \( n = 300 \), the quotient of \( I_n(n - k) \) and the asymptotics (8).
Figure 6: Quotient of $I_n(n-k)$ and the asymptotics (8), $n = 300$
Case \( j = \alpha n - x, \alpha > 0 \)

Of course, we must have that \( \alpha n - x \) is an integer. For instance, we can choose \( \alpha, x \) integers. But this also covers more general cases, for instance \( I_{\alpha n + \beta}(\gamma n + \delta) \), with \( \alpha, \beta, \gamma, \delta \) integers. We have here \( z^* = \alpha/(1 + \alpha) \). We derive, to first order,

\[
C_{1,n}(\alpha) - (j + 1)(1 + \alpha)/\alpha + (1 + \alpha)n \nonumber \\
+ [C_{2,n}(\alpha) - (j + 1)(1 + \alpha)^2/\alpha^2 - (1 + \alpha)^2n] \varepsilon = 0
\]

with, setting \( \varphi(i, \alpha) := [\alpha/(1 + \alpha)]^i \),

\[
C_{1,n}(\alpha) = C_1(\alpha) + \mathcal{O}([\alpha/(1 + \alpha)]^{-n}), \\
C_1(\alpha) = \sum_{i=1}^{\infty} \frac{i(1 + \alpha)\varphi(i, \alpha)}{\alpha[\varphi(i, \alpha) - 1]}, \\
C_{2,n}(\alpha) = C_2(\alpha) + \mathcal{O}([\alpha/(1 + \alpha)]^{-n}), \\
C_2(\alpha) = \sum_{i=1}^{\infty} \varphi(i, \alpha)i(1 + \alpha)^2(i - 1 + \varphi(i, \alpha))/[(\varphi(i, \alpha) - 1)^2\alpha^2].
\]
\[ \hat{Q}(\alpha) := \prod_{i=1}^{\infty} (1 - \varphi(i, \alpha)) = \prod_{i=1}^{\infty} \left(1 - \left(\frac{\alpha}{1 + \alpha}\right)^i\right) = Q\left(\frac{\alpha}{1 + \alpha}\right). \]

We finally derive
\[
I_n(\alpha n - x) \sim e^{-\ln(1/(1+\alpha)) - \alpha \ln(\alpha/(1+\alpha))} n + (x - 1) \ln(\alpha/(1+\alpha)) \times \\
\frac{\hat{Q}(\alpha)}{(2\pi S(2,1))^{1/2}} \exp \left[ \left\{ -(1 + 3\alpha + 4\alpha^2 - 12\alpha^2 C_1 + 6 C_1^2 \alpha^2 \right. \right. \\
+ \alpha^4 + 3\alpha^3 - 6 C_2 \alpha^2 - 12 C_1^3 \alpha)/[12\alpha(1 + \alpha)^3] \right. \\
+ x(2\alpha^2 - 2 C_1 \alpha + 3\alpha + 1)/[2\alpha(1 + \alpha)^2] \right. \\
- x^2/[2\alpha(1 + \alpha)] \left. \right\} / n + \mathcal{O}(1/n^2) \].
\]

(9)

Figure 7 shows, for \( \alpha = 1/2, n = 300 \), the quotient of \( I_n(\alpha n - x) \) and the asymptotics (9).
Figure 7: Quotient of $\ln(\alpha n - x)$ and the asymptotics (9), $\alpha = 1/2$, $n = 300$
The moderate Large deviation, $j = m + x n^{7/4}$

Now we consider the case $j = m + x n^{7/4}$. We have here $z^* = 1$. We observe the same behaviour as in the Gaussian limit for the coefficients of $\varepsilon$ in the generalization of (5). Finally we obtain

$$J_n \sim e^{-18x^2\sqrt{n}-2916/25x^4} \times$$

$$\times \exp \left[ x^2(-189568/625x^4+1161/25)/\sqrt{n} \right.$$

$$+ (-51/50 - 183660096/15625x^8 + 17637426/30625x^4)/n$$

$$+ O(n^{-5/4}) \right] / (2\pi n^3/36)^{1/2}. \quad (10)$$

Note that $S^{(3)}(\tilde{z})$ does not contribute to the correction and that this correction is equivalent to the Gaussian case when $x = 0$. Of course, the dominant term is null for $x = 0$. The exponent $7/4$ that we have chosen is of course not sacred; any fixed number below 2 could also have been considered.
Figure 8: Quotient of $J_n(j)$ and the asymptotics (10), with the $1/\sqrt{n}$ and $1/n$ term, $n = 60$
Large deviations, \( j = \alpha n(n - 1), 0 < \alpha < 1/2 \)

Here, again, \( z^* = 1 \). Asymptotically, \( \varepsilon \) is given by a Laurent series of powers of \( n^{-1} \), but here the behaviour is quite different: all terms of the series generalizing (5) contribute to the computation of the coefficients. It is convenient to analyze separately \( S_1^{(1)} \) and \( S_2^{(1)} \). This gives, by substituting
\[
\tilde{z} := 1 - \varepsilon, \quad j = \alpha n(n - 1), \quad \varepsilon = a_1/n + a_2/n^2 + a_3/n^3 + \mathcal{O}(1/n^4),
\]
and expanding w.r.t. \( n \),
\[
S_2^{(1)}(\tilde{z}) \sim (1/a_1 - \alpha)n^2 + (\alpha - \alpha a_1 - a_2/a_1^2)n + \mathcal{O}(1),
\]
\[
S_1^{(1)}(\tilde{z}) \sim \sum_{k=0}^{n-1} f(k),
\]
\[
f(k) := -(k + 1)(1 - \varepsilon)^k/[1 - (1 - \varepsilon)^{k+1}]
= -(k + 1)(1 - [a_1/n + a_2/n^2 + a_3/n^3 + \mathcal{O}(1/n^4)])^k
\]
\[
/\{1 - (1 - [a_1/n + a_2/n^2 + a_3/n^3 + \mathcal{O}(1/n^4)])^{k+1}\}.\]
This immediately suggests to apply the Euler-Mac Laurin summation formula, which gives, to first order,

$$S_1^{(1)}(\tilde{z}) \sim \int_0^n f(k)dk - \frac{1}{2}(f(n) - f(0)),$$

so we set $k = -un/a_1$ and expand $-f(k)n/a_1$. This leads to

$$\int_0^n f(k)dk$$

$$\sim \int_0^{-a_1} \left[ -\frac{ue^u}{a_1^2(1-e^u)}n^2 \right. + \frac{e^u[2a_1^2 - 2e^u a_1^2 - 2u^2 a_2 - u^2 a_1^2 + 2e^u u a_1^2]}{2a_1^3(1-e^u)^2}n \left. \right] du$$

$$+ O(1) - \frac{1}{2}(f(n) - f(0)).$$
This readily gives
\[
\int_0^n f(k)\,dk \sim -\text{dilog}(e^{-a_1})/a_1^2 n^2
\]
\[+ \left[2a_1^3 e^{-a_1} + a_1^4 e^{-a_1} - 4a_2 \text{dilog}(e^{-a_1}) + 4a_2 \text{dilog}(e^{-a_1})e^{-a_1} + 2a_1^2 e^{-a_1} - 2a_1^2 + 2a_1^2 e^{-a_1}\right]/[2a_1^3(e^{-a_1} - 1)]n + O(1).
\]
Combining \(S_1^{(1)}(\tilde{z}) + S_2^{(1)}(\tilde{z}) = 0\), we see that \(a_1 = a_1(\alpha)\) is the solution of
\[-\text{dilog}(e^{-a_1})/a_1^2 + 1/a_1 - \alpha = 0.
\]
We check that \(\lim_{\alpha\to0} a_1(\alpha) = \infty\), \(\lim_{\alpha\to1/2} a_1(\alpha) = -\infty\).
Similarly, \(a_2(\alpha)\) is the solution of the linear equation
\[
\alpha - \alpha a_1 - a_2/a_1^2 + e^{-a_1}/[2(1 - e^{-a_1})] - 1/(2a_1)
\]
\[+ \left[2a_1^3 e^{-a_1} + a_1^4 e^{-a_1} + 4a_2 \text{dilog}(e^{-a_1})(e^{-a_1} - 1) + 2a_1^2 e^{-a_1} - 2a_1^2 + 2a_1^2 e^{-a_1}\right]/[2a_1^3(e^{-a_1} - 1)]
\]
\[
= 0
\]
and \(\lim_{\alpha\to0} a_2(\alpha) = -\infty\), \(\lim_{\alpha\to1/2} a_2(\alpha) = \infty\).
We could proceed in the same manner to derive $a_3(\alpha)$ but the computation becomes quite heavy. So we have computed an approximate solution $\tilde{a}_3(\alpha)$.

Finally,

$$J_n(\alpha n(n - 1))$$

$$\sim e^{[1/72a_1(a_1-18+72\alpha)]n+C_0} \frac{1}{(2\pi n^3/36)^{1/2}} \times$$

$$\times \exp \left[(1/72a_2^2 + 1/36a_1a_3 - 1/4a_3 + 1/4a_2 - 1/2a_1 + a_3\alpha + a_1a_2\alpha + 1/3a_1^3\alpha - a_2\alpha - 1/2a_1^2\alpha - 5/24a_1a_2 + 1/24a_1^2a_2 + 1139/18000a_1^2 - 1/16a_1^3 + 87/25 - 18\alpha)/n + O(1/n^2)\right]. \quad (11)$$

$C_0 := 1/72a_1^3 - 1/4a_2 + 1/4a_1 - a_1\alpha - 5/48a_1^2 + 1/36a_1a_2 + a_2\alpha + 1/2a_1^2\alpha$

Note that, for $\alpha = 1/4$, the $1/n$ term gives $-51/50$, again as expected.

Figure 9 shows, for $n = 80$ and $\alpha \in [0.15..0.35]$, the quotient of $J_n(\alpha n(n - 1))$ and the asymptotics (11).
Figure 9: Quotient of $J_n(\alpha n(n - 1))$ and the asymptotics (11), $n = 80$
Median versus $A$ ($A$ large) for a Luria-Delbruck-like distribution, with parameter $A$

The distribution studied by Zheng, $p(n)$, depending on two parameters, $A, k$, is defined as follows. Set $\theta := 1/A$. We have

$$
\xi(n) = \frac{1}{1 + \theta} \frac{1}{n} \left[ -\frac{1}{n+1} + \sum_{j=1}^{n-1} j \xi(j) \frac{1}{(n-j)(n-j+1)} \right], \quad n \geq 1,
$$

$$
\xi(0) = \ln(1 + 1/\theta).
$$

Note that

$$
\frac{1}{n} \sum_{j=1}^{n-1} j \frac{1}{(n-j)(n-j+1)} \sim 1 - \frac{\ln n}{n}.
$$

Also, with some extra integer parameter $k$, we set

$$
p(n) = -\frac{k}{n} \sum_{j=1}^{n} j \xi(j) p(n-j), \quad n \geq 1, \quad p(0) = \frac{1}{(1 + 1/\theta)^k}.
$$

A picture of $p(n)$ ($A = 20, k = 2$) is given in Figure 10.

Guy Louchard

The Saddle point method in combinatorics asymptotic analysis
The number of inversions in permutations

Median versus $A$ (a large)

for a Luria-Delbruck-like distribution, with parameter $A$

Sum of positions of records in random permutations

Merten's theorem for toral automorphisms

Representations of numbers as $\sum_{n,k=-\infty}^{\infty}$

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Asymptotics of the Stirling numbers of the first kind revisited

Figure 10: $p(n)$
We know that the GF of $\xi(n)$ is given by

$$G(z) = \sum_{0}^{\infty} \xi(n)z^n = \ln \left[ 1 - \frac{1}{\theta} (-1 + \varphi(z)) \right], \quad G(1) = 0, \quad (13)$$

with

$$\varphi(z) = \sum_{1}^{\infty} \frac{z^n}{n(n+1)} = 1 + \left( \frac{1}{z} - 1 \right) \ln(1 - z).$$

Set

$$F(z) := \sum_{1}^{\infty} z^n p(n).$$
Now

\[ zF'z = \sum_{n=1}^{\infty} z^n np(n) = -k \sum_{n=1}^{\infty} \sum_{j=1}^{n} z^j j \xi(j) z^{n-j} p(n-j) \]

\[ = -k \sum_{n=1}^{\infty} z^n n \xi(n) p(0) - k \sum_{j=1}^{\infty} z^j j \xi(j) \sum_{n=j+1}^{\infty} z^{n-j} p(n-j) \]

\[ = -kzG'(z)p(0) - kzG'(z)F(z). \]

Solving, this gives

\[ F(z) = \left[ \int_0^{\theta} z \left( \frac{\theta + 1}{\theta} \right)^{-k} d\theta + C \right] \frac{z^k}{(\theta z - \ln(1-z) + z \ln(1-z))^k}. \]
For instance, for $k = 2$, we have

$$F(z) = - \left[ 2z^2 \ln(1 - z)\theta - 2\theta z^2 - 2z \ln(1 - z)\theta + \ln(1 - z)^2 - z^2 - 2z \ln(1 - z) + z^2 \ln(1 - z)^2 \right] \theta^2 / [(1 + \theta)^2[\theta z - \ln(1 - z) + z \ln(1 - z)]^2].$$

Of course

$$p(0) + F(1) = 1.$$ 

The GF of $\sum_n p(j)$ is given by

$$FD(z) = \sum_1^\infty z^n \sum_n^\infty p(j) = \frac{z}{1 - z} [1 - p(0) - F(z)].$$
How to find $n^*$ such that

$$[z^{n^*}]FD(z) = 1/2.$$ 

Of course, we can’t use the singularity of $FD(z)$ at $z = 1$, as it gives

$$[z^n]FD(z)$$

for large $n$, but FIXED $\theta$. 

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The Saddle point method in combinatorics asymptotic analysis
The Saddle point

We have

\[ [z^j]FD(z) = \frac{1}{2\pi i} \int_{\Omega} \frac{FD(z)}{z^{j+1}} \, dz = \frac{1}{2\pi i} \int_{\Omega} \frac{e^{\ln(FD(z))}}{z^{j+1}} \, dz. \]  

(14)

It appears, after some experiments, that the values of \( j \) such that \([z^j]FD(z) = 1/2\) are such that

\[ j \sim \alpha^* A \ln(A) = -\frac{\alpha^* \ln(\theta)}{\theta}, \]

for some constant \( \alpha^* \). Note that this does not confirm Zheng’s conjecture. For instance, \( k = 1 \) leads to \( \alpha^* = 0.92 \ldots \), \( k = 2 \) to \( \alpha^* = 0.252 \ldots \), \( k = 3 \) to \( \alpha^* = 4.25 \ldots \). For now on, we set for instance \( k = 2 \).
Set

\[ G(z) := \ln(FD(z)) - \left(-\frac{\alpha^* \ln(\theta)}{\theta} + 1\right) \ln(z). \]

Set \( z \) to \( C \) and expand \( G'(z) \) into \( \theta \). This gives, to first order,

\[ \frac{\alpha \ln(\theta)}{C \theta} + \frac{1}{1 - C} = 0, \]

or

\[ C = 1 + \frac{\theta}{\alpha \ln(\theta)} + O(\theta^2). \]

So we try

\[ \tilde{z} = 1 + \frac{\beta \theta}{\ln(\theta)}. \quad (15) \]
This gives, after some complicated manipulations (done by Maple)

\[ G'(\tilde{z}) \sim \frac{[-\beta + \alpha \beta^2 + 2\alpha + 3\beta \alpha - 3] \ln(\theta)}{(1 + \beta)(2 + \beta)\theta}. \]

Solving \( G'(\tilde{z}) = 0 \) gives

\[ \beta = \frac{1 - 3\alpha + \sqrt{1 + 6\alpha + \alpha^2}}{2\alpha}. \]

But this is impossible: \( \beta \) is decreasing as expected, but

**PROBLEM 1:** \( \beta(3/2) = 0! \). The same problem appears for all \( k \).

To be convinced that (15) is the correct asymptotics, we have computed numerically the solution \( z_n \) of \( G'(z_n) = 0 \) for different values of \( \theta \) and \( \alpha \) and we have extracted the corresponding values of \( \beta \) from (15). For instance, for \( \theta = 1/1000 \), we obtain in Figure 11 the graph of \( \beta \) vs \( \alpha \). And the values of \( \theta \): 1/100, 1/500, 1/1500 give the same function, with some remarkable fit.
The number of inversions in permutations

Median versus $A$ (large) for a Luria-Delbruck-like distribution, with parameter $A$

Sum of positions of records in random permutations

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Representations of numbers as $\sum n^k = -n^\varepsilon k$

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Asymptotics of the Stirling numbers of the first kind revisited

Figure 11: $\beta$ vs $\alpha$, $\theta = 1/1000, k = 2$
Independently of PROBLEM 1, it appears, after some experiments, that the next term would be

\[ \tilde{z} = 1 + \frac{\beta \theta}{\ln(\theta)} + \frac{\gamma \theta \ln(-\ln(\theta))}{\ln(\theta)^2}. \]

This leads to some equation

\[ \varphi(\gamma, \alpha) = 0. \]

Also, we obtain, to first order,

\[ G''(\tilde{z}) \sim f(\beta) \frac{\ln(\theta)^2}{\theta^2}, \quad f(\beta) = \frac{7 + 6\beta + \beta^2}{(2 + \beta)^2(1 + \beta)^2}, \]

and

\[ G(\tilde{z}) \sim -\ln(\theta) + g(\beta) + \ln(-\ln(\theta)) + \alpha \beta, \quad g(\beta) = \ln \left( \frac{2 + \beta}{(1 + \beta)^2} \right). \]
Hence, by standard manipulations,

\[ [z^j]FD(z) \sim \frac{e^{\alpha\beta + g(\beta)}}{\sqrt{2\pi} \sqrt{f(\beta)}} = \phi(\alpha) \text{ say}, \]

independent of \( \theta \).

If we knew \( \beta(\alpha) \), this would give \( \alpha^* \) from \( \phi(\alpha^*) = 1/2 \).
The statistic $s_{rec}$ is defined as the sum of positions of records in random permutations. The generating function (GF) of $s_{rec}$ is given by

$$G(z) = \prod_{i=1}^{n} (z^i + i - 1)$$

(16)

and the probability generating function (PGF) is given by

$$Z(z) = \frac{\prod_{i=1}^{n} (z^i + i - 1)}{n!}.$$  

(17)

This statistic has been the object of recent interest in the literature. Kortchemski obtains the GF (16) and also proves that

$$J_n(j) := z^j [G(z)] \sim e^{n \ln(n) y}, \text{ where } j = \frac{n(n + 1)}{2} x, \quad x = 1 - y^2,$$

(18)

with an error $O(1/\ln(n))$. 
The large deviation $j = \frac{n(n+1)}{2}(1 - y^2)$

By Cauchy’s theorem,

$$J_n(j) = \frac{1}{2\pi i} \int_{\Omega} \frac{G(z)}{z^{j+1}} \, dz$$

$$= \frac{1}{2\pi i} \int_{\Omega} e^{S(z)} \, dz,$$

where $\Omega$ is inside the analyticity domain of the integrand and encircles the origin and

$$S(z) = \sum_{i=1}^{n} \ln(z^i + i - 1) - \left( \frac{n(n+1)}{2}(1 - y^2) + 1 \right) \ln(z).$$

Set

$$S^{(i)} := \frac{d^i S}{dz^i}.$$
First we must find the solution of

\[ S^{(1)}(\tilde{z}) = 0 \]

with smallest module. This leads to

\[
\sum_{i=1}^{n} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1} - \left( \frac{n(n+1)}{2} (1 - y^2) + 1 \right) = 0. \tag{20}
\]

In some previous sections, we simply tried \( \tilde{z} = z^* + \varepsilon \), plug into (20), and expanded into \( \varepsilon \). Here it appears that we cannot get this expansion. So we expand first (20) itself. But we must be careful. There exists some \( \tilde{i} \) such that \( \tilde{z}^{\tilde{i}} = \tilde{i} \). Some numerical experiments suggest that \( \tilde{i} = \mathcal{O}(n) \). So we set \( \tilde{i} = \alpha n \), \( 0 < \alpha < 1 \) and we must now compute \( \alpha \).
We obtain $\tilde{z} = e^\xi$, with

$$\xi = \frac{L + \ln(\alpha)}{\alpha n},$$

where here and in the sequel, $L := \ln(n)$. Note that this leads to

$$\tilde{z}^n = \exp \left( \frac{L + \ln(\alpha)}{\alpha} \right) = n^{1/\alpha} \alpha^{1/\alpha}.$$

Using the classical splitting of the sum technique (20) leads to (we provide in the sequel only a few terms in the expansions, but Maple knows and uses more)

$$\Sigma_1 := \sum_{i=1}^{\tilde{i}} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1},$$

$$\Sigma_2 := \sum_{i=\tilde{i}+1}^{n} \frac{i\tilde{z}^i}{\tilde{z}^i + i - 1} - \left( \frac{n(n+1)}{2} (1 - y^2) + 1 \right).$$
As
\[ \frac{\tilde{z}^i - 1}{i} < \frac{\tilde{z}^i - 1}{\tilde{i}} < \frac{\tilde{z}^i}{\tilde{i}} = 1, \ i < \tilde{i}, \]

we have

\[ \Sigma_1 = \sum_{i=1}^{\tilde{i}} \frac{\tilde{z}^i}{1 + \frac{\tilde{z}^i - 1}{i}} = \sum_{i=1}^{\tilde{i}} \tilde{z}^i \left[ 1 - \frac{\tilde{z}^i - 1}{i} + \left( \frac{\tilde{z}^i - 1}{i} \right)^2 + \ldots \right] \]

(21)

The first summation is immediate

\[ \sum_{i=1}^{\tilde{i}} \tilde{z}^i = \frac{\tilde{z}^{\tilde{i}+1} - 1}{\tilde{z} - 1} - \frac{\tilde{z}}{\tilde{z} - 1}. \]
For the next summations, we again use Euler-Mac Laurin summation formula. First of all, the correction (to first order) is given by

\[
\frac{1}{2} + \frac{1}{2} \frac{\tilde{i}}{1 + \frac{\tilde{i} - 1}{\tilde{i}}} = \frac{1}{2} + \frac{1}{2} \frac{\alpha n}{2 - 1/(\alpha n)} \sim \frac{1}{4} \alpha n. \tag{22}
\]

Next, we must compute integrals such as

\[
\int_{1}^{\tilde{i}} \tilde{z}^i \left( \frac{\tilde{z}^i - 1}{i} \right)^k \, di. \tag{23}
\]

But we know that

\[
\int_{1}^{\tilde{i}} \tilde{z}^i \left( \frac{\tilde{z}^j - 1}{i} \right) \, di = \int_{1}^{\tilde{i}} e^{\xi i} \left( \frac{e^{\xi i} - 1}{i} \right) \, di = Ei(1, -2\xi) - Ei(1, -\xi) + Ei(1, -i\xi) - Ei(1, -2i\xi)
\]

\[
= Ei(1, -2\xi) - Ei(1, -\xi) + Ei(1, -(L + \ln(\alpha))) - Ei(1, -2(L + \ln(\alpha)))
\]

where \(Ei(x)\) is the exponential integral.
Setting $L_1 := L + \ln(\alpha)$, we have

\[ \Re(Ei(1, -\xi)) = -\gamma - \ln(\xi) - \xi - \frac{\xi^2}{4} + \ldots, \]

\[ \Re(Ei(1, -L_1)) = e^{L_1} \left[ -\frac{1}{L_1} - \frac{1}{L_1^2} + \ldots \right] = \alpha n \left[ -\frac{1}{L_1} - \frac{1}{L_1^2} + \ldots \right]. \]

We use similar expansions for terms like (23). This finally leads, with (22), to

\[ \Sigma_1 = n^2 \left[ \frac{7\alpha^2}{12L} + \frac{\alpha^2(-84 \ln(\alpha) - 31)}{144L^2} + \ldots \right] + O(n). \]
Now we turn to $\Sigma_2$. As

$$\frac{i - 1}{\tilde{z}^i} < \frac{i + 1 - 1}{\tilde{z}^{i+1}} < \frac{i}{\tilde{z}^i} = 1, i > \tilde{i},$$

we have

$$\Sigma_2 = \sum_{i=\tilde{i}+1}^{n} \frac{i}{1 + \frac{i-1}{\tilde{z}^i}} - \left( \frac{n(n+1)}{2} (1 - y^2) + 1 \right)$$

$$= \sum_{i=\tilde{i}+1}^{n} i \left[ 1 - \frac{i-1}{\tilde{z}^i} + \left( \frac{i-1}{\tilde{z}^i} \right)^2 + \ldots \right] - \left( \frac{n(n+1)}{2} (1 - y^2) + 1 \right)$$

$$= n^2 \left[ \frac{1}{2} - \frac{\alpha^2}{2} - \frac{47\alpha^2}{60L} + \ldots \right] + O(n)$$

$$+ \frac{n^3}{n^{1/\alpha}} \left[ \frac{\alpha}{\alpha^{1/\alpha}L} + \ldots \right] - \left( \frac{n(n+1)}{2} (1 - y^2) + 1 \right) + \ldots$$
So

\[ S'(\tilde{z}) = \sum_{1} + \sum_{2} = n^2 \left[ \frac{1}{2} - \frac{\alpha^2}{2} - \frac{\alpha^2}{5L} + \ldots - \frac{(1 - y^2)}{2} \right] + O(n) \]

\[ + \frac{n^3}{n^{1/\alpha}} \left[ \frac{\alpha}{\alpha^{1/\alpha} L} + \ldots \right] + \ldots = 0 \]  \hspace{1cm} (24)

Putting the coefficient of \( n^2 \) to 0, and solving wrt \( \alpha \) gives

\[ \alpha^* = y - \frac{y}{5L} + \frac{-3199y/1800 + \ln(y)y/5}{L^2} + \ldots \]  \hspace{1cm} (25)

Now we must consider the other terms of (24). First we must compare \( n \) with \( \frac{n^3}{n^{1/\alpha}} \).

If \( \alpha > \frac{1}{2} \), \( n^{3-1/\alpha} > n \) and vice-versa. The most interesting case is the case \( \alpha > \frac{1}{2} \) (the other one can be treated by similar method). Note hat there are also other terms in (24) of order \( n^{k-(k-2)/\alpha} \), \( k \geq 4 \). This is greater than \( n \) if \( \alpha > (k - 2)/(k - 1) \).
Returning to (24), we first compute \( n^{1/\alpha} = n^{1/y} \varphi(y, L) \), with

\[
\varphi(y, L) = e^{L(1/\alpha - 1/y)} = e^{1/(5y)} - \frac{e^{1/(5y)}(-3271 + 360 \ln(y))}{1800yL} + \ldots
\]

So we obtain from (24) the term

\[
\frac{n^3}{n^{1/y} \varphi(y, L)} \left[ \frac{\alpha}{\alpha^{1/\alpha} L} + \ldots \right],
\]

and with (25),

\[
\frac{n^3}{n^{1/y}} \left[ \frac{y}{y^{1/y} e^{1/(5y)} L} + \ldots \right].
\]
Now we set $\alpha = \alpha^* + \frac{Cn}{n^{1/y}}$, plug into (24) (ignoring the $O(n)$ term), and expand. The $n^2$ term must theoretically be 0. Actually, it is given by a series of large powers of $1/L$ as we only use a finite number of terms in (25). Solving the coefficient of $\frac{n^3}{n^{1/y}}$ wrt $C$, we obtain

$$C = \frac{e^{-1/(5y)}y^{(3y-1)/y}}{Ly^3} + \ldots$$

and

$$\alpha = \alpha^* + \frac{Cn}{n^{1/y}} + \ldots \quad (27)$$

$$\tilde{J}_n(j) = \frac{e^{S(\tilde{z})}}{\sqrt{2\pi S''(\tilde{z})}} \quad (28)$$

In Figure 12, we give, for $n = 150$, a comparison between $\ln(J_n(j))$ (circle) and $\ln(\tilde{J}_n(j))$ (line). The fit is quite good, but when $y$ is near 1. But $j$ is then small and our asymptotics are no more very efficient. We also show the first approximation (18): $nLy$ (line blue) which is only efficient for very large $n$. 

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The Saddle point method in combinatorics asymptotic analysis
**Figure 12:** Comparison between $\ln(J_n(j))$ (circle) and $\ln(\tilde{J}_n(j))$ (line), $n = 150$. Also it shows the first approximation (1): $nLy$ (line blue)
The central region \( j = yn \)

From (17), we see that, as expected, \( Z(1) = 1 \). Moreover

\[
Z'(z) = \sum_{i=1}^{n} \frac{iz^{i-1}}{z^i + i - 1} Z(z),
\]

and

\[
Z''(z) = \sum_{i=1}^{n} \frac{i[z^{i-2}i^2 - 2z^{i-2} - z^{2i-2} + z^{i-2}]}{(z^i + i - 1)^2} Z(z) \\
+ \left( \sum_{i=1}^{n} \frac{iz^{i-1}}{z^i + i - 1} \right)^2 Z(z).
\]
So

\[ \mathbb{E}(s_{rec}) = Z'(1) = n, \]

and the variance is given by

\[ \mathbb{V}(s_{rec}) = Z''(1) + n - n^2 = \sum_{i=1}^{n} (i - 2) + n^2 + n - n^2 = \frac{n(n-1)}{2}. \]

Of course \( Z(z) \) corresponds to a sum of independent non

identically distributed random variables, but it is clear that the

Lindeberg-Lévy conditions are not satisfied here. The distribution is

not asymptotically Gaussian. As will be clear later on, it is

convenient to separate the cases \( j \geq 1 \) and \( j < 1 \).
The case $y \geq 1$

Again, by Cauchy’s theorem,

$$
\mathbb{P}(\text{srec} = j) = \frac{1}{2\pi \text{i}} \int_{\Omega} \frac{Z(z)}{z^{j+1}} \, dz
$$

$$
= \frac{1}{2\pi \text{i}} \int_{\Omega} e^{S(z)} \, dz,
$$

where $\Omega$ is inside the analyticity domain of the integrand and encircles the origin and

$$
S(z) = \sum_{i=1}^{n} \ln(z^i + i - 1) - (yn + 1) \ln(z) - \ln(n!).
$$

First we must find the solution of

$$
S^{(1)}(\tilde{z}) = 0
$$

with smallest module.
This leads to
\[
\sum_{i=1}^{n} \frac{i \tilde{z}^i}{\tilde{z}^i + i - 1} - (yn + 1) = 0. \tag{30}
\]

Set \( \tilde{z} := z^* + \varepsilon \), where, here, it is easy to check that \( z^* = 1 \). Set \( j = yn \).
This leads, to first order (keeping only the \( \varepsilon \) term in (30)), to
\[
\varepsilon := \frac{2(y - 1)}{n} + \frac{-4 + 10y - 4y^2}{n^2} + O(1/n^3).
\]

This shows that, asymptotically, \( \varepsilon \) is given by a **Laurent series** of powers of \( n^{-1} \). To obtain more precision, we set again \( j = yn \), expand (30) into powers of \( \varepsilon \) (we use 9 terms), set
\[
\varepsilon = \frac{a_1}{n} + \sum_{i=2}^{4} \frac{a_i}{n^i}
\]
expand in powers of \( n^{-1} \), and equate each coefficient to 0.
This gives, for the coefficient of $n$

$$1 - y + 1/120a_1^4 + 1/2a_1 + 1/720a_1^5 + 1/362880a_1^8 + 1/6a_1^2$$
$$+ 1/5040a_1^6 + 1/3628800a_1^9 + 1/24a_1^3 + 1/40320a_1^7 = 0.$$

We observe that all terms of the expansion of (30) into $\varepsilon$ contribute to the computation of the coefficients. We have already encountered this situation in analyzing the number in inversions in permutations. So we must turn to another approach. Setting $i = k + 1$, (30) becomes

$$\sum_{k=0}^{n-1} f(k) - (yn + 1) = 0,$$

(31)

where

$$f(k) = \frac{(k + 1)\tilde{z}^{k+1}}{\tilde{z}^{k+1} + k}.$$
This gives, always by Euler-Mac Laurin summation formula, to first order,

$$\int_0^n f(k) dk - \frac{1}{2} (f(n) - f(0)) - (yn + 1) = 0,$$

so we set $k = un/a_1, \tilde{z} = 1 + a_1/n$ and expand $f(k)n/a_1$. This leads to

$$\int_0^{a_1} e^u \frac{ndu}{a_1} - yn + \mathcal{O}(1) = 0,$$

or

$$e^{a_1} = ya_1 + 1. \quad (32)$$
Note that

\[ a_1(y) \sim \ln(y), \quad y \to \infty, \]
\[ a_1(1) = 0. \]

The explicit solution of (5) is easily found, we have

\[ e^{a_1} = ya_1 + 1 = y[a_1 - 1/y] = e^{a_1+1/y} e^{-1/y}, \]
\[ -e^{-[a_1+1/y]}[a_1 + 1/y] = -e^{-1/y}/y, \]
\[ -[a_1 + 1/y] = W(-1, -e^{-1/y}/y), \]
\[ a_1 = -1/y - W(-1, -e^{-1/y}/y), \]

where \( W(-1, x) \) is the suitable branch of the Lambert equation:
\[ W(x)e^{W(x)} = x. \]

**PROBLEM 2: THE THIRD DERIVATIVE IS OF ORDER \( n^3 \)**
**ALTHOUGH THE SECOND DERIVATIVE IS OF ORDER \( n^2 \).**
Finally, by standard technique, (29) should lead to

\[ \mathbb{P}(srec = j) \sim \frac{e^{F(y)}}{\sqrt{2\pi n^2 F_2(y)}}, \quad (33) \]

and

\[ \mathbb{P}(srec = n) \sim \frac{1}{\sqrt{2\pi n^2 / 2}}. \quad (34) \]

Also

\[ \mathbb{P}(srec = j) \sim \frac{e^{-y \ln(y)}}{\sqrt{2\pi n^2 y}}, \quad y \to \infty. \]
The case $y < 1$

We have

$$\mathbb{P}(srec = j) = \frac{1}{n!} [z^j] \prod_{i=1}^{j} [z^i + i - 1] \prod_{u=j}^{n-1} u,$$

and, if $j$ is large, by (34),

$$[z^j] \prod_{i=1}^{j} [z^i + i - 1] \sim j! \frac{1}{\sqrt{\pi j}}.$$

So

$$\mathbb{P}(srec = j) \sim \frac{1}{n!} \frac{(j - 1)!}{\sqrt{\pi}} \prod_{u=y}^{n-1} u = \frac{1}{n \sqrt{\pi}}.$$

For $j$ large enough, $\mathbb{P}(srec = j)$ is constant and given by (34).

We have made a numerical comparison of

$\mathbb{P}(srec = j), n = 200, j = 1..3n$ with (34) and (33). This is given in Figure 13. This is quite satisfactory.
Figure 13: Comparison between $\mathbb{P}(srec = j), n = 200, j = 1..3n$ (circle) and the asymptotics (34) and (33) (line)
Merten’s theorem for toral automorphisms

Let

\[ \varphi_n(z) := \prod_{1}^{n} (1 - z^k)(1 - z^{-k}) \]

What is the asymptotic behaviour of

\[ [z^j] \varphi_n(z), \]

in particular the asymptotic value of

\[ [z^0] \varphi_n(z). \]

A plot of \([z^j] \varphi_n(z), n = 15\) is given in Figure 14. This seems to have a **Gaussian envelope**.
Figure 14: $[z^j] \varphi_n(z)$, $n = 15$
But

\[ S(z) := \ln(\varphi_n(z)) - \ln(z) = \sum_{1}^{n} (\ln(1 - z^k) + \ln(1 - z^{-k})) - \ln(z) \]

doesn’t appear to posses zeroes, two plots of

\[ |S(z)|^2 \]

given in Figures 15, 16, reveal a quite \textit{irregular behaviour}. 
The number of inversions in permutations Median versus $A$

A large

$\sum n_k = -n \epsilon_k$

The q-Catalan numbers A simple case of the Mahonian statistic Asymptotics of the Stirling numbers of the first kind revisited

Figure 15: $|S(z)|^2$, $n = 15$
The number of inversions in permutations

Median versus $A$ (\(A\) large) for a Luria-Delbruck-like distribution, with parameter $A$.

Sum of positions of records in random permutations

Merten's theorem for toral automorphisms

Representations of numbers as $\sum n^k = -n \epsilon_k k$

The q-Catalan numbers

A simple case of the Mahonian statistic

Asymptotics of the Stirling numbers of the first kind revisited

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure16.png}
\caption{$|S(z)|^2, \; n = 15$}
\end{figure}
Representations of numbers as $\sum_{k=-n}^{n} \varepsilon_k k$

We consider the number of representations of $m$ as $\sum_{k=-n}^{n} \varepsilon_k k$, where $\varepsilon_k \in \{0, 1\}$. For $m = 0$, this is sequence A000980 in Sloane’s encyclopedia. This problem has a long history. Here, we extend previous ranges a bit, to $O(n^{3/2})$. But we improve at the same time the quality of the approximation.

The generating function of the number of representations for fixed $n$ is given by

$$C_n(z) = 2 \prod_{k=1}^{n} (1 + z^k)(1 + z^{-k}), \quad C_n(1) = 2 \cdot 4^n.$$ 

By normalisation, we get the probability generating function of a random variable $X_n$:

$$F_n(z) = 4^{-n} \prod_{k=1}^{n} (1 + z^k)(1 + z^{-k}),$$
The Gaussian limit

We obtain mean $\mathbb{M}$ and variance $\mathbb{V}$:

$$\mathbb{M}(n) = 0, \quad \sigma^2 := \mathbb{V}(n) = \frac{n(n + 1)(2n + 1)}{12}.$$  

We consider values $j = x\sigma$, for $x = \mathcal{O}(1)$ in a neighbourhood of the mean 0.

The Gaussian limit of $X_n$ can be obtained by using the Lindeberg-Lévy conditions, but we want more precision.

We know that

$$P_n(j) = \frac{1}{2\pi i} \int_{\Omega} e^{S(z)} \, dz$$

where

$$S(z) := \ln(F_n(z)) - (j + 1) \ln z$$

with

$$\ln(F_n(z)) = \sum_{i=1}^{n} [\ln(1 + z^i) + \ln(1 + z^{-i}) - \ln 4].$$

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The Saddle point method in combinatorics asymptotic analysis
Set $\tilde{z} := z^* - \varepsilon$, where, here, $z^* = 1$.
Finally this leads to

$$P_n(j) \sim e^{-x^2/2} \cdot \exp \left( \frac{-39/40 + 9/20x^2 - 3x^4/40}{n + \mathcal{O}(n^{-3/2})} \right) / (2\pi n^3/6)$$

Note again that $S^{(3)}(\tilde{z})$ does not contribute to the $1/n$ correction.
Note also that, unlike in the instance of the number of inversions in permutations, we have an $x^4$ term in the first order correction.
Figure 17 shows, for $n = 60$, $Q_3$: the quotient of $P_n(j)$ and the asymptotics (35), with the constant term $-39/(40n)$ and the $x^2$ term $9x^2/(20n)$, and $Q_4$: the quotient of $P_n(j)$ and the full asymptotics (35). $Q_4$ gives indeed a good precision on a larger range.
Figure 17: $Q_3$ (blue) and $Q_4$ (red)
The case $j = n^{7/4} - x$

Now we show that our methods are strong enough to deal with probabilities that are far away from the average; viz. $n^{7/4} - x$, for fixed $x$. Of course, they are very small, but nevertheless we find asymptotic formulæ for them. Later on, $n^{7/4} - x$ will be used as an integer.

This finally gives

$$P_n(j) \sim e^{-3n^{1/2} - 27/10} \left[ 1 + \frac{369}{175} n^{1/2} + \frac{931359}{245000} n + \frac{1256042967}{471625000} n^{3/2} + \frac{4104x}{175} n^{7/4} - \frac{9521495156603}{2145893750000} n^2 + \frac{7561917x}{122500} n^{9/4} + \left( -\frac{235974412468129059}{34139218750000} + \frac{18x^2}{n^{5/2}} \right) n^{3/2} + \cdots \right]$$

$$\bigg/ (2\pi n^{3/6})^{1/2}. \quad (36)$$
Figure 18: $P_n(j)$ (circle) and the full asymptotics (36) up to the $n^{-5/2}$ term, (line), $n = 150$, scaling $= 10^{21}$
The q-Catalan numbers

The q-Catalan numbers \( C_n(q) \) are defined as

\[
C_n(q) = \frac{1 - q}{1 - q^{n+1}} \frac{(q; q)_{2n}}{(q; q)_n(q; q)_{2n}},
\]

with \( (q; q)_n = (1 - q)(1 - q^2) \ldots (1 - q^n) \). Note that \( C_n = C_n(1) \), a Catalan number, and the polynomial \( F_n(q) = C_n(q)/C_n \) is the probability generating function of a distribution \( X_n \) that we call the **Catalan distribution**. It has been shown recently that this distribution is asymptotically normal.

Since we attack \( F_n(q) \) and \( C_n(q) \) mostly with analytic methods, we find it more appropriate to replace the letter \( q \) by \( z \); note that

\[
C_n(z) = \prod_{i=1}^{n-1} \frac{1 - z^{n+i+1}}{1 - z^{i+1}}.
\]
It is easy to get the mean $m$ and variance $\sigma^2$ of the random variable $X_n$ characterized by

$$P_n(i) := \mathbb{P}(X_n = i) := \frac{[z^i]C_n(z)}{C_n},$$

namely

$$m = \frac{n(n - 1)}{2},$$

$$\sigma^2 = \frac{n(n^2 - 1)}{6}.$$

(In the full paper we sketch how these moments and higher ones can be computed quite easily.)
The Gaussian limit

Set $j = m + x \sigma$, with $m = n(n - 1)/2$ and $\sigma = \sqrt{n(n^2 - 1)/6}$. Then, for fixed $x$, the following approximation holds:

$$P_n(j) \sim e^{-x^2/2} \cdot \exp\left((-9/40 + 9/20x^2 - 3x^4/40)/n\right)/\left(2\pi n^3/6\right)^{1/2}.$$  

(37)

Note that, like in the Representations of numbers as $\sum_{k=-n}^{n} \epsilon_k k$, we have a $x^4$ term in the first order correction. Figure 19 shows, for $n = 60$, $Q_3$: the quotient of $P_n(j)$ and the asymptotics (37), with the constant term $-9/(40n)$ and the $x^2$ term $9x^2/(20n)$ and $Q_4$: the quotient of $P_n(j)$ and the full asymptotics (37). $Q_4$ gives indeed a good precision on a larger range.
The number of inversions in permutations

Median versus $A$ (large)

A large

Sum of positions of records in random permutations

Merten's theorem for toral automorphisms

Representations of numbers as

$\sum_{n} k = -n \epsilon k$

The q-Catalan numbers

A simple case of the Mahonian statistic

Asymptotics of the Stirling numbers of the first kind revisited

Figure 19: $Q_3$ (blue) and $Q_4$ (red)
The case \( j = n - k \)

We deal with probabilities that are far away from the average; viz. \( j = n - k \), for fixed \( k \). Again, they are very small, but nevertheless we find asymptotic formulæ for them.

We have \( z^* = 1 \) and, as we will see, \( \varepsilon \) is again given by a Puiseux series of powers of \( n^{-1/2} \). But the approach of previous Sections is doomed to failure: all terms of the generalization of (5) contribute to the computation of the coefficients. So we have to turn to another technique. We have

\[
P_n(n-k) = [z^{n-k}] \prod_{i=1}^{n-1} \frac{1 - z^{n+i+1}}{1 - z^{i+1}} \bigg/ C_n.
\]

So we set

\[
S := S_1 + S_2 + S_3 + S_4,
\]
with

\[ S_1 := \sum_{i=1}^{n-1} \ln(1 - z^{n+i+1}), \]

\[ S_2 := - \sum_{i=1}^{n-1} \ln(1 - z^{i+1}) = - \sum_{i=2}^{n} \ln(1 - z^{i}), \]

\[ S_3 := -(n + 1 - k) \ln z, \quad S_4 := - \ln(C_n). \]

Set \( \tilde{z} = z^* - \varepsilon = 1 - \varepsilon \). We must have \( S'(\tilde{z}) = 0 \), with

\[ S'(z) = S'_1(z) + S'_2(z) + S'_3(z), \]

\[ S'_1(z) = \sum_{i=1}^{n-1} \frac{-(n + i + 1)z^{n+i}}{1 - z^{n+i+1}}, \]

\[ S'_2(z) = \sum_{i=1}^{n-1} \frac{(i + 1)z^{i}}{1 - z^{i+1}}, \]

\[ S'_3(z) = - \frac{n + 1 - k}{z}. \]
It is clear that we must have either $S'_1(\tilde{z}) = \mathcal{O}(n)$ or $S'_2(\tilde{z}) = \mathcal{O}(n)$. Set

$$\varepsilon = \frac{f(n)}{n},$$

for a function still to be determined. Let us first (roughly) solve

$$S'_1(\tilde{z}) = n.$$ 

We have

$$S'_1(\tilde{z}) \sim n \frac{\tilde{z}^n}{1 - \tilde{z}} \sim \frac{n^2 e^{-f(n)}}{f(n)}.$$ 

This leads to the equation

$$\frac{n^2 e^{-f(n)}}{f(n)} = n, \text{ i.e. } e^{f(n)} f(n) = n,$$
which is solved by

\[ f(n) = W(n), \]

where \( W \) is the \textbf{Lambert function}. But we know that

\[ W(n) \sim \ln(n) - \ln \ln(n) + \mathcal{O}\left(\frac{\ln \ln(n)}{\ln(n)}\right). \]

So

\[ \varepsilon \sim \frac{\ln(n)}{n}. \]

But then

\[ S'_2(\tilde{z}) \sim \frac{1}{(1 - \tilde{z})^2} \sim \frac{n^2}{\ln(n)^2} = \Omega(n). \]
So we must turn to the other choice, i.e.

\[ S'_2(\tilde{z}) \sim O \left( \frac{1}{(1 - \tilde{z})^2} \right) \sim O \left( \frac{n^2}{f(n)^2} \right). \]

The equation

\[ \frac{n^2}{f(n)^2} = n \]

leads now to

\[ f(n) = \sqrt{n}, \]

and

\[ S'_1(\tilde{z}) \sim \frac{n^2 e^{-\sqrt{n}}}{\sqrt{n}} = n^{3/2} e^{-\sqrt{n}}, \]

which is exponentially negligible.
This rough analysis leads to the more precise asymptotics

$$
\varepsilon = \frac{a_1}{n^{1/2}} + \frac{a_2}{n} + \frac{a_3}{n^{3/2}} + \cdots
$$

and we must solve

$$
S'_2(\tilde{z}) + S'_3(\tilde{z}) = 0,
$$

i.e.

$$
\sum_{j=1}^{n-1} \frac{(j + 1)\tilde{z}^j}{1 - \tilde{z}^{j+1}} - \frac{n + 1 - k}{\tilde{z}} = 0.
$$

We rewrite this equation as

$$
\sum_{j=2}^{n} \frac{j\tilde{z}^j}{1 - \tilde{z}^j} = M,
$$

and will later replace $M$ by $n + 1 - k$. It is not hard to see that we can solve

$$
\sum_{j=2}^{\infty} \frac{j\tilde{z}^j}{1 - \tilde{z}^j} = M,
$$

since the extra terms introduce an exponentially small error.
Now we replace $\tilde z$ by $e^{-t}$:

$$g(t) = \sum_{j=2}^{\infty} \frac{je^{-jt}}{1 - e^{-jt}},$$

and compute its Mellin transform:

$$g^*(s) = (\zeta(s - 1) - 1)\zeta(s)\Gamma(s). \quad (39)$$

The original function can be recovered by a contour integral:

$$g(t) = \frac{1}{2\pi i} \int_{3-i\infty}^{3+i\infty} (\zeta(s - 1) - 1)\zeta(s)\Gamma(s)t^{-s} ds.$$
This integral can be approximately (in a neighbourhood of $t = 0$) solved by taking residues into account:

$$g(t) = \frac{\pi^2}{6t^2} - \frac{3}{2t} + \frac{13}{24} - \frac{t}{12} + \frac{t^3}{720} - \frac{t^5}{30240} + \mathcal{O}(t^7).$$

Now we solve

$$\frac{\pi^2}{6t^2} - \frac{3}{2t} + \frac{13}{24} - \frac{t}{12} + \frac{t^3}{720} - \frac{t^5}{30240} \sim M,$$ 

and find

$$t(M) \sim \frac{\pi}{\sqrt{6}} M^{-1/2} - \frac{3}{4} M^{-1} + \frac{(81 + 13\pi^2) \sqrt{6}}{288\pi} M^{-3/2} - \left(\frac{13}{32} + \frac{\pi^2}{144}\right) M^{-2}.$$

To simplify the next expressions, we will set $n = w^2$. 
Now, since $1 - \varepsilon = e^{-t}$, we set $M = n + 1 - k$, expand and reorganize:

\[
\varepsilon \sim \frac{\pi}{\sqrt{6}} w^{-1} - \left(\frac{\pi^2}{12} + \frac{3}{4}\right) w^{-2} + \frac{\sqrt{6}(4\pi^4 + 3\pi^2 + 72\pi^2 k + 243)}{864\pi} w^{-3}.
\]

(40)

Note that $k$ appears (linearly) only in the coefficient of $w^{-3}$. The quadratic term in $k$ appears in the coefficient of $w^{-5}$. 
This finally leads to

\[
P_n(n - k) \sim e^{T_4} \frac{2^{-1/2} \pi^{3/2}}{12} \left( 1 - 2^{1/2} 3^{1/2} \frac{216 + 13\pi^2 + 24\pi^2 k}{144\pi w} \right) \\
+ \frac{19008\pi^2 + 20736\pi^2 k + 31104 + 217\pi^4 + 624\pi^4 k + 576\pi^4 k^2}{6912\pi^2 w^2} \\
- 3^{1/2} 2^{1/2} \left[ -229635 + 11104128\pi^2 + 1907712\pi^4 k + 11197440\pi^2 k \right. \\
\left. + 4069\pi^6 + 771984\pi^4 + 1244160\pi^4 k^2 + 15624\pi^6 k \right. \\
\left. + 13824\pi^6 k^3 + 22464\pi^6 k^2 \right] / \left[ 2985984\pi^3 w^3 \right] \)
\]

\[
T_4 := -2 \ln(2) w^2 + \frac{\sqrt{6\pi} w}{3}.
\]

The quality of the approximation is given in Figure 20, with the \( w^{-2} \) term (and \( k^2 \) contribution) and in Figure 21, with the \( w^{-2} \) and \( w^{-3} \) terms (and \( k^3 \) contribution). The fit is rather good: the curves cover the exact graph above and below.
Figure 20: $P_n(n - k), n = 60$ exact (circle), asymptotics (line), with $w^{-2}$ term
Figure 21: $P_n(n - k), n = 60$ exact (circle), asymptotics (line), with $w^{-2}$ and $w^{-3}$ terms
A simple case of the Mahonian statistic

Canfield, Janson and Zeilberger have analyzed the Mahonian distribution on multiset permutations: classic permutations on \( m \) objects can be viewed as words in the alphabet \( \{1, \ldots, m\} \). If we allow repetitions, we can consider all words with \( a_1 \) occurences of 1, \( a_2 \) occurences of 2, \ldots, \( a_m \) occurences of \( m \). Let \( J_m \) denote the number of inversions. Assuming that all words are equally likely, the probability generating function of \( J_m \) is given, setting \( N = a_1 + \cdots + a_m \), by

\[
\phi_{a_1, \ldots, a_m}(z) = \frac{\prod_{i=1}^{m} a_i! \prod_{i=1}^{N} (1 - z^i)}{N! \prod_{j=1}^{m} \prod_{i=1}^{a_j} (1 - z^i)}.
\]

The mean \( \mu \) and variance \( \sigma^2 \) are given by

\[
\mu(J_m) = e_2(a_1, \ldots, a_m)/2, \quad \sigma^2(J_m) = \frac{(e_1 + 1)e_2 - e_3}{12},
\]

where \( e_k(a_1, \ldots, a_m) \) is the degree \( k \) elementary symmetric function.
Let $a^* = \max_j a_j$ and $N^* = N - a^*$. Recently it was proved that, if $N^* \to \infty$ then the sequence of normalized random variables

$$\frac{J_m - \mu(J_m)}{\sigma(J_m)}$$

tends to the standard normal distribution. The authors also conjecture a local limit theorem and prove it under additional hypotheses.

In this talk, we analyze simple examples of the Mahonian statistic, for instance, we consider the case

$$m = 2, a_1 = an, a_2 = bn, n \to \infty.$$
We analyze the central region \( j = \mu + x\sigma \) and one large deviation \( j = \mu + x n^{7/4} \). The exponent 7/4 that we have chosen is of course again not sacred, any fixed number below 2 and above 3/2 could also have been considered. We have here

\[
\phi(z) = \frac{(an)!(bn)! \prod_{i=1}^{(a+b)n}(1 - z^i)}{((a + b)n)! \prod_{i=1}^{an}(1 - z^i) \prod_{i=1}^{bn}(1 - z^i)},
\]

\[
\mu = \frac{abn^2}{2},
\]

\[
\sigma^2 = \frac{ab(a + b + 1/n)n^3}{12}.
\]

Guy Louchard

The Saddle point method in combinatorics asymptotic analysis
The Gaussian limit

To compute $S_1(\tilde{z})$, we first compute the asymptotics of the $i$ term, this leads to a $\ln(i)$ contribution, which will be cancelled by the factorials. We obtain

$$Z_2(j) \sim e^{-x^2/2}.$$ 

$$\exp \left[ \left[ -\frac{3a^2 + 13ab + 3b^2}{20ab(a + b)} + \frac{3(a^2 + b^2 + ab)x^2}{10ab(a + b)} \right] / n + \mathcal{O}(n^{-3/2}) \right]$$

$$/ (\pi ab(a + b)n^3/6)^{1/2}.$$ 

(41)
The Large deviation, $j = \mu + xn^{7/4}$

We derive

$$Z_2(j) \sim \exp \left[ -\frac{6x^2}{ab(a+b)} n^{1/2} - \frac{36(a^2 + ab + b^2)x^4}{5[ab(a+b)]^3} ight.$$ 

$$+ C_9(x, a, b)/n^{1/2} + C_{10}(x, a, b)/n + \ldots \left. \right]$$

$$\sqrt{(\pi ab(a+b)n^3/6)^{1/2}}. \quad (42)$$

To check the effect of the correction, we first give in Figure 22, for $n = 50$, $a = b = 1/2$ and $x \in [0..0.2]$, the comparison between $Z_2(j)$ and the asymptotics (42). Figure 23 shows the quotient of $Z_2(j)$ and the asymptotics (42)
Figure 22: The comparison between $Z_2(j)$ and the asymptotics (42).
Figure 23: The quotient of $Z_2(j)$ and the asymptotics (42)
Asymptotics of the Stirling numbers of the first kind revisited

Let \( \begin{bmatrix} n \\ j \end{bmatrix} \) be the Stirling number of the first kind (unsigned version). Their generating function is given by

\[
\phi_n(z) = \prod_{i=0}^{n-1} (z + i) = \frac{\Gamma(z + n)}{\Gamma(z)}, \quad \phi_n(1) = n!.
\]

Consider the random variable \( J_n \), with probability distribution

\[
\mathbb{P}[J_n = j] = Z_n(j),
\]

\[
Z_n(j) := \frac{\begin{bmatrix} n \\ j \end{bmatrix}}{n!}.
\]
\[ M := \mathbb{E}(J_n) = \sum_{0}^{n-1} \frac{1}{1+i} = H_n = \psi(n + 1) + \gamma, \]

\[ \sigma^2 := \mathbb{V}(J_n) = \sum_{0}^{n-1} \frac{i}{(1+i)^2} = \psi(1, n+1) + \psi(n+1) - \frac{\pi^2}{6} + \gamma, \]

where \( \psi(k, x) \) is the \( k \)th polygamma function, and

\[ M \sim \ln(n) + \gamma + \frac{1}{2n} + \mathcal{O} \left( \frac{1}{n^2} \right), \]

\[ \sigma^2 \sim \ln(n) - \frac{\pi^2}{6} + \gamma + \frac{3}{2n} + \mathcal{O} \left( \frac{1}{n^2} \right). \]

It is convenient to set

\[ A_n := \ln(n) - \frac{\pi^2}{6} + \gamma = \ln \left( ne^{\gamma - \pi^2/6} \right), \]

and to consider all our next asymptotics \((n \to \infty)\) as functions of \(A_n\). All asymptotics can be reformulated in terms of \(\ln(n)\).
We have

\[ M \sim A_n + \frac{\pi^2}{6} + O\left(\frac{1}{n}\right), \]

\[ \sigma^2 \sim A_n + O\left(\frac{1}{n}\right). \]

A celebrated central limit theorem of Goncharov says that

\[ J_n \sim \mathcal{N}(M, \sigma), \]

where \( \mathcal{N} \) is the Gaussian distribution, with a rate of convergence \( O(1/\sqrt{\ln(n)}) \).
In this Section, we want to obtain a more precise local limit theorem for \( J_n \) in terms of \( x := \frac{J_n - M}{\sigma} \) and \( A_n \). Actually, we obtain the following result, where we use \( B_n := \sqrt{A_n} \) to simplify the expressions.

\[
Z_n(j) \sim R_1,
\]

\[
R_1 := \frac{1}{\sqrt{2\pi B_n}} e^{-x^2/2}.
\]

\[
\frac{1}{B_n} \left[ 1 + \frac{x^3}{6} - \frac{x}{2} \right] + \frac{3x^2}{8} - \frac{x^4}{6} - \frac{1}{12} + \frac{x^6}{72} \cdot B_n^2
\]

\[
+ \frac{1}{B_n^3} \left[ -\pi^2 x^3/18 + 37x^5/240 - 355x^3/144 + x/8 - x^7/48 
\right.
\]

\[
+ x^9/1296 + \pi^2 x/6 - \zeta(3)x + \zeta(3)x^3/3 \right] + \ldots \right] .
\]

A comparison of \( Z_n(j) \) with \( \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ - \left( \frac{j-M}{\sigma} \right)^2 / 2 \right] \) with \( Z_n(j)/R_1 \), with 2 terms in \( R_1 \), is given in Figure 24.
The number of inversions in permutations

Median versus $A$

A large for a Luria-Delbruck-like distribution, with parameter

Sum of positions of records in random permutations

Merten's theorem for toral automorphisms

Representations of numbers as $\sum n_k = -n \varepsilon_k$

The q-Catalan numbers

A simple case of the Mahonian statistic

Asymptotics of the Stirling numbers of the first kind revisited

Figure 24: $Z_n(j) \left/ \left[ \frac{1}{\sqrt{2\pi \sigma}} \exp \left[ - \left( \frac{j-M}{\sigma} \right)^2 / 2 \right] \right] \right.,$ color=red, $Z_n(j)/R_1,$ color=blue, $n = 3000$
The precision of $R_1$ is of order $10^{-2}$. Using 3 terms in $R_1$ leads to a less good result: $A_n$ is not large enough to take advantage of the $A_n^{-3/2}$ term: $A_n = 6.94$ here, we deal with asymptotic series, not necessarily convergent ones. More terms can be computed in $R_1$ (which is almost automatic with Maple).
The justification of the integration procedure goes as follows. We proceed as in Flajolet and Sedgewick (Analytic combinatorics, FS), ch. VIII. We can choose here \( \tilde{z} = 1 \). This leads, with \( z = e^{i\theta} \), to

\[
S(z) \sim S_0(z) + O\left(\sqrt{\ln(n)\theta}\right) + \text{constant term},
\]

with

\[
S_0(z) = \sum_{k=0}^{n-1} \ln[e^{i\theta} + k] - H_n i\theta
\]

\[
\sim \sum_{k=0}^{n-1} \frac{1}{1 + k} [e^{i\theta} - 1] - \frac{1}{2} \sum_{k=0}^{n-1} \left[ \frac{1}{1 + k} [e^{i\theta} - 1] \right]^2 - H_n i\theta + O(\theta^3)
\]

\[
\sim H_n [e^{i\theta} - 1 - i\theta] + O(\theta^2).
\]
Set

\[ h(\theta) := e^{i\theta} - 1 - i\theta. \]

We have

\[ h(\theta) \sim -\frac{\theta^2}{2}, \]

The function \( h(\theta) \) is the same as in FS, Ex. VIII.3, which proves the validity of our integration procedure: we use here \( H_n \sim \ln(n) \) instead of \( n \). The complete asymptotic expansion is justified as in FS, Ex. VIII.4.
Large deviation, $j = n - n^\alpha$, $1 > \alpha > 1/2$

We have

$$G_n(z) := \frac{\Gamma(z + n)}{\Gamma(z)z^{j+1}} = \exp[S(z)],$$

with

$$S(z) = S_1(z) + S_2(z), \quad S_1(z) = \sum_{0}^{n-1} \ln(z+i), \quad S_2(z) = -(j+1)\ln(z).$$

Some experiments with some values for $\alpha$ ($\alpha = 5/8$ is a good choice) show that $\tilde{z}$ must be a combination of $x = n^\alpha$ and $y = n^{1-\alpha}$ and $x \gg y \gg 1$. Note that both $x$ and $y$ are large. We will derive series of powers of $x^{-1}$, where each coefficient is a series of powers of $y^{-1}$. 
First, by bootstrapping, we obtain (we give the first terms)

\[
\tilde{z} = \frac{ny}{2} \left[ 1 - \frac{4}{3y} + \frac{2}{9y^2} + \frac{8}{135y^3} + \frac{8}{405y^4} \\
+ \frac{16}{1701y^5} + \frac{232}{45525y^6} + \frac{64}{18225y^7} + \ldots \right]
\]

\[
+ \frac{1}{x} \left[ 1 - \frac{1}{y} + \frac{4}{9y^2} - \frac{16}{135y^3} + \ldots \right]
\]

\[
+ \frac{1}{x^2} \left[ 1 - \frac{1}{y} + \frac{0}{y^2} + \ldots \right]
\]

\[
+ \frac{1}{x^3} [1 + \ldots] + O \left( \frac{1}{x^4} \right). \tag{43}
\]

Note that the choice of dominant terms in the bracket of (43) depends on \( \alpha \). For instance, for \( \alpha = 3/4 \), the dominant terms (in decreasing order) are

\[
1, \frac{1}{y}, \frac{1}{y^2}, \left\{ \frac{1}{x}, \frac{1}{y^3} \right\}, \left\{ \frac{1}{xy}, \frac{1}{y^4} \right\}, \left\{ \frac{1}{xy^2}, \frac{1}{y^5} \right\}, \left\{ \frac{1}{x^2}, \frac{1}{xy^3}, \frac{1}{y^6} \right\}, \ldots
\]
We obtain the following result

\[ [z^j] \phi_n(z) \sim \frac{1}{\sqrt{2\pi}} \frac{y^2 \sqrt{x}}{2} e^{S(\bar{z})} T_4 \]

\[ T_4 = 1 - \frac{2}{3y} - \frac{2}{9y^2} + \ldots + \frac{1}{x} \left( \frac{5}{12} - \frac{11}{18y} + \ldots \right) \]

\[ + \frac{1}{x^2} \left( \frac{73}{288} - \frac{133}{432y} + \ldots \right) + \frac{1}{x^3} \left( \frac{721}{576} + \ldots \right) + \mathcal{O}\left( \frac{1}{x^4} \right) \]
We have made several experiments with (44), with \( n \) up to 500. The result is unsatisfactory, only values of \( x \) of order \( \sqrt{n} \) give reasonable results. Actually, only very large values of \( n \) lead to good precision. So we turn to another formulation: instead of using an expansion for \( e^{S(\tilde{z})} \), we plug directly \( \tilde{z} \) into \( G_n(z) \), i.e. we set

\[
T_7 = G_n(\tilde{z}),
\]

leading to

\[
[z^j] \phi_n(z) \sim \frac{1}{\sqrt{2\pi}} \frac{y^2 \sqrt{x}}{2} T_7 T_4 =: T_8 \text{ say}.
\]

For \( n = 500 \), using two and three terms in \( T_4 \), we give in Figures 25 and 26, the quotient \([z^j] \phi_n(z) / T_8\). The precision is now of order \( 10^{-5} \).
Figure 25: The quotient \( [z^j] \phi_n(z)/T_8 \), two terms in \( T_4 \), as function of \( j \), \( n = 500 \)
Figure 26: The quotient $[z^j]\phi_n(z)/T_8$, three terms in $T_4$, as function of $j$, $n = 500$