About the arrow

**can**: Tensor product of series → Double series, its usage in C.S. and Combinatorics II.

(MO: 200442 & 201753), Schützenberger’s calculus (towards continuation of polylogarithms).


Collaboration at various stages of the work and in the framework of the Project

*Evolution Equations in Combinatorics and Physics*:

C. Bui, Q.H. Ngô, Vu Dinh, S. Goodenough.

CIP seminar, 09 February 2021 (rev. 09-02-2021 17:02)
The goal of this talk is threefold

A
Dualization of laws and co-laws of a bialgebra with the conc-bialgebras to begin with.

B
Pursue with the generality. How about a general bialgebra.

C
MRS factorisation(s): Local systems of coordinates for Hausdorff groups.
Transpose of a laws and dual laws

Original Problem

Let $\mathcal{B} = (\mathcal{B}, \mu, 1_\mathcal{B}, \Delta, \epsilon)$ be a bialgebra. We now will examine the dualization of it, i.e. ideally the existence of another bialgebra

$$\mathcal{B}_1 = (\mathcal{B}_1, \mu_1, 1_{\mathcal{B}_1}, \Delta_1, \epsilon_1)$$

and a pairing $\langle \cdot, \cdot \rangle : \mathcal{B}_1 \otimes \mathcal{B} \rightarrow k$ such that, identically

$$\langle x \mid \mu(y \otimes z) \rangle = \langle \Delta_1(x) \mid y \otimes z \rangle \otimes^2$$  \hspace{1cm} (1)

$$\langle \mu_1(x \otimes y) \mid z \rangle = \langle x \otimes y \mid \Delta(z) \rangle \otimes^2$$ \hspace{1cm} (2)

$$\epsilon(x) = \langle 1_{\mathcal{B}_1} \mid x \rangle ; \ \epsilon_1(x) = \langle x \mid 1_{\mathcal{B}} \rangle$$ \hspace{1cm} (3)

In addition, we require that (through $\langle \cdot, \cdot \rangle$) we get an embedding $(\mathcal{B}_1 \hookrightarrow \mathcal{B}^\vee)$ i.e. $\mathcal{B}_1^\perp = \{0\}$ and that there are sufficiently many elements in $\mathcal{B}_1$ to separate elements of $\mathcal{B}$ i.e. $\mathcal{B}_1^\perp = \{0\}$. We say that this pair is in separating duality (see discussion after MO question 179214).
Examples

The identities (1)-(3) mean that there is a correspondence between the elements of $\mathcal{B}$ and $\mathcal{B}_1$.

1. $\mathcal{B} = (k\langle X \rangle, \text{conc}, 1_{X^*}, \Delta_{\text{III}}, \epsilon)$; $\mathcal{B}_1 = (k\langle X \rangle, \text{III}, 1_{X^*}, \Delta_{\text{conc}}, \epsilon)$

   $\Delta_{\text{III}}(w) = \sum_{l+j=[1\ldots|w|]} w[l] \otimes w[j]$; $\Delta_{\text{conc}}(w) = \sum_{uv=w} u \otimes v$

2. **ϕ-shuffle.** $\varphi : k.X \otimes k.X \rightarrow k.X$ (associative without unit)

   
   For $a, b \in X$, $u, v \in X^*$

   $u \text{ III} 1_{X^*} = 1_{X^*} \text{ III} u = u$

   $a.u \text{ III} b.v = a.(u \text{ III} b.v) + b.(a.u \text{ III} v) + \varphi(a, b).(u \text{ III} v)$

   perturbation

With this law $\mathcal{B}_\varphi = (k\langle X \rangle, \text{III} \varphi, 1_{X^*}, \Delta_{\text{conc}}, \epsilon)$

is a Hopf algebra. The possibility of dualization of $\mathcal{B}_\varphi$ depends crucially on what $\varphi$ is.
<table>
<thead>
<tr>
<th>Name</th>
<th>Formula (recursion)</th>
<th>$\varphi$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shuffle</td>
<td>$au \triangleleft bv = a(u \triangleleft bv) + b(au \triangleleft v)$</td>
<td>$\varphi \equiv 0$</td>
<td>Ree</td>
</tr>
<tr>
<td>Stuffle</td>
<td>$x_i u \triangleleft x_j v = x_i(u \triangleleft x_j v) + x_j(x_i u \triangleleft v) + x_{i+j}(u \triangleleft v)$</td>
<td>$\varphi(x_i, x_j) = x_{i+j}$</td>
<td>Hoffman</td>
</tr>
<tr>
<td>Min-stuffle</td>
<td>$x_i u \uplus x_j v = x_i(u \uplus x_j v) + x_j(x_i u \uplus v) - x_{i+j}(u \uplus v)$</td>
<td>$\varphi(x_i, x_j) = -x_{i+j}$</td>
<td>Costermans</td>
</tr>
<tr>
<td>Muffle</td>
<td>$x_i u \bigtriangledown x_j v = x_i(u \bigtriangledown x_j v) + x_j(x_i u \bigtriangledown v) + x_{i\times j}(u \bigtriangledown v)$</td>
<td>$\varphi(x_i, x_j) = x_{i\times j}$</td>
<td>Enjalbert, HNM</td>
</tr>
<tr>
<td>q-shuffle</td>
<td>$x_i u \uplus q x_j v = x_i(u \uplus q x_j v) + x_j(x_i u \uplus q v) + q x_{i+j}(u \uplus q v)$</td>
<td>$\varphi(x_i, x_j) = q x_{i+j}$</td>
<td>Bui</td>
</tr>
<tr>
<td>q-shuffle2</td>
<td>$x_i u \uplus q x_j v = x_i(u \uplus q x_j v) + x_j(x_i u \uplus q v) + q^\bot x_{i+j}(u \uplus q v)$</td>
<td>$\varphi(x_i, x_j) = q^\bot x_{i+j}$</td>
<td>Bui</td>
</tr>
<tr>
<td>LDIAG(1, $q_s$)</td>
<td>$au \triangleleft bv = a(u \triangleleft bv) + b(au \triangleleft v) + q^</td>
<td>a</td>
<td></td>
</tr>
<tr>
<td>q-Infiltration</td>
<td>$au \uparrow bv = a(u \uparrow bv) + b(au \uparrow v) + q^\delta_a, b a(u \uparrow v)$</td>
<td>$\varphi(a, b) = q^\delta_a, b a$</td>
<td>Chen-Fox-Lyndon</td>
</tr>
<tr>
<td>AC-stuffle</td>
<td>$au \triangleleft \varphi bv = a(u \triangleleft \varphi bv) + b(au \triangleleft \varphi v) + \varphi(a, b)(u \triangleleft \varphi v)$</td>
<td>$\varphi(a, b) = \varphi(b, a)$</td>
<td>Enjalbert, HNM</td>
</tr>
<tr>
<td>Semigroup-stuffle</td>
<td>$x_t u \triangleleft_{\bot s} x_s v = x_t(u \triangleleft_{\bot s} x_s v) + x_s(x_t u \triangleleft_{\bot s} v) + x_{t\bot s}(u \triangleleft_{\bot s} v)$</td>
<td>$\varphi(x_t, x_s) = x_{t\bot s}$</td>
<td>Deneufchâtel</td>
</tr>
<tr>
<td>$\varphi$-shuffle</td>
<td>$au \triangleleft \varphi bv = a(u \triangleleft \varphi bv) + b(au \triangleleft \varphi v) + \varphi(a, b)(u \triangleleft \varphi v)$</td>
<td>$\varphi(a, b)$ law of AAU</td>
<td>Manchon, Paycha</td>
</tr>
</tbody>
</table>

**Common pattern**

\[ w \bigtriangleup \varphi 1 x^\ast = 1 x^\ast \bigtriangleup \varphi w = w \text{ and} \]

\[ au \bigtriangleup \varphi bv = a(u \bigtriangleup \varphi bv) + b(au \bigtriangleup \varphi v) + \varphi(a, b)(u \bigtriangleup \varphi v) \]
We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) $\mu : A\langle X \rangle \otimes_A A\langle X \rangle \to A\langle X \rangle$ can be described through its structure constants w.r.t. to the basis of words, i.e. for $u, v, w \in X^*$, $\Gamma^w_{u,v} := \langle \mu(u \otimes v) | w \rangle$ so that $\mu(u \otimes v) = \sum_{w \in X^*} \Gamma^w_{u,v} w$.

2. In the case when $\Gamma^w_{u,v}$ is locally finite in $w$, we say that the given law is dualizable, the arrow $t\mu$ restricts nicely to $A\langle X \rangle \hookrightarrow A\langle \langle X \rangle \rangle$ and one can define on the polynomials a comultiplication by the finite sum $\Delta_\mu(w) := \sum_{u,v \in X^*} \Gamma^w_{u,v} u \otimes v$.

3. When the law $\mu$ is dualizable, we have

$$\begin{array}{ccc}
A\langle \langle X \rangle \rangle & \xrightarrow{t\mu} & A\langle \langle X^* \otimes X^* \rangle \rangle \\
\uparrow \text{can} & & \uparrow \Phi|_{A\langle X \rangle \otimes_A A\langle X \rangle} \\
A\langle X \rangle & \xrightarrow{\Delta_\mu} & A\langle X \rangle \otimes_A A\langle X \rangle
\end{array}$$

(still when $\mu$ is dualizable), the arrow $\Delta_\mu$ is unique to be able to close the rectangle and $\Delta_\mu(P)$ is defined as above.
Now, let us give a family of (counter) examples. We start with $X = Y_S$ where $S \subset \mathbb{C}$ is an additive subsemigroup (instances of $S$ are $\mathbb{N}, \mathbb{N}_+, \mathbb{R}_+, [2, +\infty[, \mathbb{Z}$ and the upper-quater plane $\mathcal{P} = \mathbb{N} \oplus i\mathbb{N}$). Building, with $Y_S = \{y_s\}_{s \in S}$ and $\varphi(y_s, y_t) := y_{s+t}$, the $\varphi$-shuffle $\varpi_{\varphi}$ as above (slide 5).

With the above, we get the following table

<table>
<thead>
<tr>
<th>$S$</th>
<th>$\mathbb{N}$</th>
<th>$\mathbb{N}_+$</th>
<th>$\mathbb{R}_+$</th>
<th>$\mathbb{Z}$</th>
<th>$\mathcal{P}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dualizable ?</td>
<td>Y</td>
<td>Y</td>
<td>N</td>
<td>N</td>
<td>Y</td>
</tr>
</tbody>
</table>

the test is simple: $\varpi_{\varphi}$ is dualizable iff $S$ iff $M$ satisfies condition (D) in [Bourba89, Ch. III, §2.10]) which is

$$(\forall r \in S)(\{(s, t) \in S^2 | r = st\} \text{ is finite}) \quad (4)$$
Our last example will be the bialgebra of a monoid $M$. It is with $\Delta_{\circ}$, the Hadamard (pointwise) coproduct $B = (k[M], \mu_M, 1_M, \Delta_{\circ}, \varepsilon)$ where $\mu$ is the standard product in the algebra $k[M]$, $\Delta_{\circ}(m) = m \otimes m$ and $\varepsilon(f) = \sum_{m \in M} \langle f | m \rangle$.

Then, if $M$ satisfies condition (D) of Bourbaki, $B$ is dualizable with $\Delta_1(m) = \sum_{pq = m} p \otimes q$ and $f \circ g := \sum_{m \in M} \langle f | m \rangle \langle g | m \rangle m$ (pointwise product), and $\varepsilon_1(f) = \langle f | 1_M \rangle$, we have

$$B_1 = (k[M], \circ, \chi_M, \Delta_1, \varepsilon_1)$$

where $\chi_M$ is the characteristic function of $M$ ($m \mapsto 1_k$, $\forall m \in M$).

To end with, we remark that the dualization of a comultiplication is always possible, the difficulty being to dualize a product. We now consider separately this problem.
We start with a \( k \) a field and \( \mathcal{A} \) a \( k \) – AAU. Let \( \mu : \mathcal{A} \otimes_k \mathcal{A} \rightarrow \mathcal{A} \), we have

\[
\begin{array}{c}
\mathcal{A}^\vee & \xrightarrow{t_\mu} & (\mathcal{A} \otimes_k \mathcal{A})^\vee \\
\uparrow \text{can} & & \uparrow \Phi \\
? & \xrightarrow{\Delta_\mu} & \mathcal{A}^\vee \otimes_k \mathcal{A}^\vee
\end{array}
\]

where \( \Phi(S \otimes T) \) is the linear form such that

\[
\langle \Phi(S \otimes T) | u \otimes v \rangle := \langle S | u \rangle \langle T | v \rangle \quad \text{(5)}
\]

Due to the fact that \( k \) is a field, the arrow \( \Phi \) is into.

The set \( ? \) is the set of elements \( f \in \mathcal{A}^\vee \) such that \( t_\mu(f) \in \text{Im}(\Phi) \). One has a very simple criterium to characterize them.
We have the following (proof is left as an exercise).

**Theorem A (Sweedler, Abe)**

Let $k$ be a field and $A$ be a $k$-AAU $(A, \mu, 1_A)$, we use also infix notation $\mu(u \otimes v) = u \ast v$ and define the left-right-shifts by $\langle x \triangleright f \triangleleft y \mid z \rangle := \langle f \mid yzx \rangle$ (one-sided shifts are derived by $x, y = 1_A$). Then TFAE

1. $t_\mu(f) \in \text{Im}(\Phi)$

2. There exists a double (finite) sequence $(g_i, h_i)_{1 \leq i \leq n}$ such that for all $x, y \in A$,

   $$\langle f \mid x \ast y \rangle = \sum_{i=1}^{n} \langle g_i \mid x \rangle \langle h_i \mid y \rangle$$

3. The left-shifts $(x \triangleright f)_{x \in A}$ form a family of finite rank.

4. The right-shifts $(f \triangleleft x)_{x \in A}$ form a family of finite rank.

5. The bi-shifts $(x \triangleright f \triangleleft y)_{x, y \in A}$ form a family of finite rank.
End of the theorem

**Theorem A (Sweedler, Abe), cont’d**

It exists a matrix representation $\mu : \mathcal{A} \rightarrow \mathbb{k}^{n \times n}$ and vectors $\lambda \in \mathbb{k}^{1 \times n}$, $\tau \in \mathbb{k}^{n \times 1}$ such that, for all $a \in \mathcal{A}$

$$\langle f \mid a \rangle = \lambda \mu(a) \tau$$  \hspace{1cm} (6)

**Remarks**

i) Condition 2 in “Theorem A (Sweedler, Abe)” is exactly $^t\mu(f) \in \text{Im}(\Phi)$ so that, equivalence 1 $\iff$ 2 is just a reformulation.

ii) Property 6 allows to prove that, if $^t\mu(f) \in \text{Im}(\Phi)$, in fact $\Delta_\mu(f) \in \mathcal{A}^\circ \otimes \mathcal{A}^\circ$. So the commutative square in slide 9 give rise to a ladder which stops at the first step.
We start with a \( k - \mathbf{AAU} \) (\( k \) a field) \( A \), dualizing \( \mu : A \otimes_k A \to A \), we have

\[
\begin{align*}
A^\vee & \xrightarrow{t \mu} (A \otimes_k A)^\vee \\
\uparrow \text{can} & \quad \uparrow \Phi \\
A^\circ & \xrightarrow{\Delta \mu} A^\vee \otimes_k A^\vee \\
\uparrow \text{can} & \quad \uparrow j \otimes j \\
A^{\circ\circ} & \xrightarrow{\Delta \mu} A^\circ \otimes_k A^\circ
\end{align*}
\]

In fact, as said in the remarks above (slide 11), one sees that the “descent” stops at first step

and then \( A^{\circ\circ} = A^\circ \) this space will be defined as Sweedler’s dual of \( A \).
Case of the shuffle algebra/1

With the example of $\mathcal{A} = (\mathbb{C}\langle X \rangle, \varpi, 1_{X^*})$, the square

\[
\begin{array}{ccc}
\mathcal{A}^\vee & \xrightarrow{t_\mu} & (\mathcal{A} \otimes_k \mathcal{A})^\vee \\
\uparrow \text{can} & & \uparrow \Phi \\
\mathcal{A}^\circ & \xrightarrow{\Delta_\mu} & \mathcal{A}^\circ \otimes_k \mathcal{A}^\circ
\end{array}
\]

remarking that $\mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \simeq \mathbb{C}[X^* \otimes X^*]$ becomes

\[
\begin{array}{ccc}
\mathbb{C}\langle\langle X \rangle\rangle & \xrightarrow{t_\text{III}} & \mathbb{C}[\[X^* \otimes X^*\]] \\
\uparrow \text{can} & & \uparrow \Phi \\
\mathcal{A}^\circ & \xrightarrow{\Delta_{\text{III}}} & \mathcal{A}^\circ \otimes_\mathbb{C} \mathcal{A}^\circ
\end{array}
\]
So that, in this case [shuffle algebra], condition 2 in Theorem A reads for all $P, Q \in \mathbb{C}\langle X \rangle$ (equivalent, by bilinearity, to “for all $u, v \in X^*$”)

$$\langle R \mid u \shuffle v \rangle = \sum_{i=1}^{n} \langle S_i \mid u \rangle \langle T_i \mid v \rangle$$

Now considering the identity $(\alpha.x)^* \shuffle (\beta.x)^* = ((\alpha + \beta).x)^*$ with $c = \alpha + \beta \in \mathbb{N}_{\geq 2}$, we get $(c.x)^* = (a.x)^* \shuffle (b.x)^*$ for $a + b = c$ and then identities like

$$(c.x)^* = \frac{1}{c+1} \sum_{a+b=c, \ a,b \in \mathbb{N}} (a.x)^* \shuffle (b.x)^*$$

$$(c.x)^* = \frac{1}{c-1} \sum_{a+b=c, \ a,b \in \mathbb{N}_+} (a.x)^* \shuffle (b.x)^*$$

(7)
But, in spite of these identities, there is no formula of the type
\[ \Delta_{III}((c.x)^*) = \sum_{i=1}^{n} S_i \otimes T_i \]

Let us compute the shifts (defined as in slide 10) \( x^k \triangleright_{III} (c.x)^* \). We have

\[
x^k \triangleright_{III} (c.x)^* = \sum_{n \geq 0} \langle (c.x)^* | x^n III x^k \rangle x^n =
\]

\[
\sum_{n \geq 0} \langle (c.x)^* | \binom{n+k}{k} x^{n+k} \rangle x^n =
\]

\[
\frac{c^k}{k!} \sum_{n \geq 0} Q_k(n) c^n x^n
\]

with \( Q_k \in \mathbb{C}[x] \) is of degree \( k \) (exactly). This proves that the shifts \( x^k \triangleright_{III} (c.x)^* \) are all \( \mathbb{C} \)-linearly independent.

This shows that there is no hope that identities (7) could be dualized.
Computation of $\Delta_\|((c.x)^*)/1$

As said above $\Delta_\|((c.x)^*) \in \mathbb{C}\langle\langle X^* \otimes X^*\rangle\rangle$ but, as was proved $\Delta_\|((c.x)^*) \notin \mathbb{C}\langle\langle X^*\rangle\rangle \otimes \mathbb{C}\langle\langle X^*\rangle\rangle$, so

$$\Delta_\|((c.x)^*) = \sum_{u,v \in X^*} c(u,v) u \otimes v \quad (9)$$

Firstly, we remark that $\mathbb{C}\langle\langle X^* \otimes X^*\rangle\rangle$ (the algebra of functions on $X^* \otimes X^*$, the total algebra of the monoid $X^* \otimes X^*$, see “total algebra of a monoid” in [Bourba89, Ch. III, §2.10]) comes with a filtration due to the gradation of $X^* \otimes X^*$ as follows

1. If $M$ is a $\mathbb{N}$-graded monoid$^a$
2. So is the “double” of $M$ ($M \times M \simeq M \otimes_{\mathbb{k}} M \subset \mathbb{k}\langle M\rangle \otimes \mathbb{k}\langle M\rangle$) with

$$(M \otimes_{\mathbb{k}} M)_n := \bigsqcup_{p+q=n} M_p \otimes M_q \quad (10)$$

$^a$That is $M = \bigsqcup_{n \geq 0} M_n$ with $M_p. M_q \subset M_{p+q}$. 
Computation of $\Delta_{\text{III}}((c.x)^*)/2$

Computation continued

So comes the algebra

$$k[M] \otimes k[M] \simeq k[M \otimes_k M]$$

with

$$(k[M] \otimes k[M])_n = \bigoplus_{n=p+q} (k[M])_p \otimes (k[M])_q$$  \hspace{1cm} (11)$$

So it is a general fact that the dual of a graded algebra comes with a natural (decreasing) filtration\(^a\) given by $(k\langle\langle M \otimes M\rangle\rangle \geq n)$ to be the linear forms that have their support in $(M \otimes_k M)_{\geq n}$.

\(^a\)and even the dual of a increasingly filtered algebra, see MO question 310354.
Computation of $\Delta_{III}((c.x)^{*})/3$

Computation continued

In general, a family $(S_i)_{i \in I}$ in $k^M$ is said summable if, for each $m \in M$, the function $i \mapsto \langle S_i \mid m \rangle$ is finitely supported.

The end of the computation is left as an exercise using the following ingredients

- $(c.x)^* = \sum_{n \geq 0} (c.x)^n$
- The family $(\Delta_{III}((c.x)^n))_{n \geq 0}$ is summable
- If a family $(S_n)_{n \geq 0}$ is summable, so is $(\Delta_{III}(S_n))_{n \geq 0}$ and $\Delta_{III}$ commutes with the infinite sums.


THANK YOU FOR YOUR ATTENTION!