Evolution, Localisation and Wronskians.
Lie theoretic aspects of NCDE.

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Collaboration at various stages of the work
and in the framework of the Project

*Evolution Equations in Combinatorics and Physics*:
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CIP seminar,
Plan

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Review of categories useful in combinatorics.

Below a quick list of the categories of use in combinatorics ($k$ is a given field), morphisms are standard.

1. **St**, the category of sets
2. **Mon**, the category of monoids
3. **CMon**, the category of commutative monoids
4. **Gp**, the category of groups
5. **Ring**, the category of rings
6. **CRing**, the category of commutative rings
7. **Vect$_k$**, the category of $k$-vector spaces
8. **Lie$_k$**, the category of $k$-Lie algebras
9. **AAU$_k$**, the category of $k$-Associative Algebras with Unit
10. **CAAU$_k$**, the category of $k$-Associative and Commutative Algebras with Unit
Useful categories/2

11. $\text{Mg}$, the category of Magmas i.e. sets with only a binary law (without conditions)

12. $\text{Alg}_k$, the category of $k$-Algebras (without conditions)

13. $\text{DiffAlg}_k$, the category of $k$-Associative Differential Algebras with Unit.

14. $\text{CDiffAlg}_k$, the category of $k$-Associative Commutative Differential Algebras with Unit.

15. $\text{DiffRing}$, the category of Differential rings.

16. $\text{CDiffRing}$, the category of Commutative Differential Rings.

Remarks. –

i) All of these have a standard forgetful functor to $\text{St}$. They usually compose and factor nicely. See also [19].

ii) For $k = \mathbb{Z}$, one has

$$\text{DiffAlg}_\mathbb{Z} = \text{DiffRing} \text{ and } \text{CDiffAlg}_\mathbb{Z} = \text{CDiffRing}.$$
The Magnus group is the set of series with constant term $1_{X^*}$, the Hausdorff (sub)-group, is the (conc-)group of $\mathbb{III}$-characters. When $k$ is a field, these are the group-like series for $\Delta_{\mathbb{III}}$. When $k \supset \mathbb{Q}$ these are also Lie exponentials (here $A, B$ are Lie series and $\exp(A)\exp(B) = \exp(H(A, B)))$. 
Possibly subgroups:

Lie Group $G$

$\mathbf{L}(G)$ (Lie algebra)

Magnus, Hausdorff
Continuous, Algebraic
Closed subgroups (Cartan theorem)

<table>
<thead>
<tr>
<th>of $Gl(n, k)$</th>
<th>Zariski</th>
<th>U($n, k$), D($n, k$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>of $Gl(n, k)$</td>
<td>Topology ($k = \mathbb{R}, \mathbb{C}$)</td>
<td>$Gl_+(n, \mathbb{R})$</td>
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<tr>
<td>of $Mag(n, \mathbb{C})$</td>
<td>Topology (Formal)</td>
<td>($\text{Haus}_\varphi(n, \mathbb{C})$)</td>
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<tr>
<td>$U(n, k)$</td>
<td>Equations</td>
<td>$k = \mathbb{C}, X^*X = I$</td>
</tr>
<tr>
<td>$O(n, k)$</td>
<td>Equations</td>
<td>$k = \mathbb{R}, (tX)X = I$</td>
</tr>
<tr>
<td>$D(n, k)$</td>
<td>Equations</td>
<td>$a_{i,j} = 0, i \neq j$</td>
</tr>
<tr>
<td>$N(n, k)$</td>
<td>Equations</td>
<td>$a_{i,j} = 0, i &gt; j$ $a_{ii} = 1$</td>
</tr>
</tbody>
</table>

Remarks

i) Here, the formal topology is defined by summability conditions.
ii) For formal aspects of DE, have a look at the chapter “Combinatorial differential equations” in [1].
Local coordinates on (infinite dimensional) Lie groups, factorization of Riemann zeta functions

Given a (finite dimensional) Lie group $G$ (real $k = \mathbb{R}$ or complex $k = \mathbb{C}$) and its Lie algebra $\mathfrak{g}$, one can prove (a basis $B = (b_i)_{1 \leq i \leq n}$ of $\mathfrak{g}$ being given) that there exists a neighbourhood $W$ of $1_G$ (in $G$) and $n$ local coordinate analytic functions

$$W \to k, \ (t_i)_{1 \leq i \leq n}$$

such that, for all $g \in W$

$$g = \prod_{1 \leq i \leq n} e^{t_i(g)b_i} = e^{t_1(g)b_1}e^{t_2(g)b_2}\cdots e^{t_n(g)b_n}$$

(1)

to see this, just remark that

$$(t_1, t_2, \cdots t_n) \to \exp(t_1 b_1) \exp(t_2 b_2) \cdots \exp(t_n b_n)$$

is a local diffeomorphism from $k^n$ to $G$ in a neighbourhood of 0 and take the inverse.

This is the local Wei-Norman's theorem.

My questions are the following

Let us loosely call infinite dimensional a Lie group whose Lie algebra is not finite dimensional (this includes the example below and infinite dimensional Banach-Lie groups for instance).

Q1) Can you provide examples of infinite dimensional Lie groups where the exponential map
Theorem (Wei-Norman theorem)

Let $G$ be a $k$-Lie group (of finite dimension) ($k = \mathbb{R}$ or $k = \mathbb{C}$) and let $\mathfrak{g}$ be its $k$-Lie algebra. Let $B = \{b_i\}_{1 \leq i \leq n}$ be a (linear) basis of it. Then, there is a neighbourhood $W$ of $1_G$ (within $G$) and $n$ analytic functions (local coordinates)

$$W \to k, \ (t_i)_{1 \leq i \leq n}$$

such that, for all $g \in W$

$$g = \prod_{1 \leq i \leq n} e^{t_i(g)b_i} = e^{t_1(g)b_1}e^{t_2(g)b_2}\ldots e^{t_n(g)b_n}.$$
We take $G = G\text{l}_+(2, \mathbb{R})$ ($G\text{l}_+(2, \mathbb{R})$, connected component of 1 within $G\text{l}(2, \mathbb{R})$),

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad (1)$$

We will practically compute the Wei-Norman coefficients through an Iwasawa decomposition

$$M = \text{unitary} \times \text{diagonal} \times \text{unitriangular}$$

and compute $MTDU = I_2$ through the following elementary operations

1. (Orthogonalisation)
2. (Normalisation)
3. (Unitarisation)
\[ M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (C_1, C_2) = (C_1^{(1)}, C_2^{(1)}) e^{\frac{\langle C_1 | C_2 \rangle}{||C_1||^2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

\[ = e^{\arctan\left(\frac{a_{21}}{a_{11}}\right)} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} e^{\log(||C_1^{(1)}||)} \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} e^{\log(||C_2^{(1)}||)} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} e^{\frac{\langle C_1 | C_2 \rangle}{||C_1||^2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

We then get a Wei-Norman decomposition w.r.t. the following basis of \( \mathfrak{gl}(2, \mathbb{R}) \):

\[ \mathfrak{gl}(2, \mathbb{R}) : \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

**Remark.** – Iwasawa decomposition is general for \( \text{Gl}(n, k) \), \( k \) being one of the fields \( \mathbb{R}, \mathbb{C}, \mathbb{H} \), see [4]. For \( \mathbb{R} \) and non-Archimedean fields in the same book, see [2].
Bits and pieces for the BTT

Theorem (DDMS [1])

Let \((A, d)\) be a \(k\)-commutative associative differential algebra with unit and \(C\) be a differential subfield of \(A\) (i.e. \(d(C) \subseteq C\)). We suppose that \(S \in A\langle\langle X\rangle\rangle\) is a solution of the differential equation

\[
\text{d}(S) = MS ; \langle S|1_{X^*}\rangle = 1_A \tag{2}
\]

where the multiplier \(M\) is a homogeneous series (a polynomial in the case of finite \(X\)) of degree 1, i.e.

\[
M = \sum_{x \in X} u_x x \in C\langle\langle X\rangle\rangle \tag{3}
\]

The following conditions are equivalent:

(i) The family \((\langle S|w \rangle)_{w \in X^*}\) of coefficients of \(S\) is free over \(\mathcal{C}\).

(ii) The family of coefficients \((\langle S|y \rangle)_{y \in X \cup \{1_{X^*}\}}\) is free over \(\mathcal{C}\).

(iii) The family \((u_x)_{x \in X}\) is such that, for \(f \in \mathcal{C}\) and \(\alpha_x \in k\)

\[
d(f) = \sum_{x \in X} \alpha_x u_x \implies (\forall x \in X)(\alpha_x = 0) .
\]  

(4)

(iv) The family \((u_x)_{x \in X}\) is free over \(k\) and

\[
d(\mathcal{C}) \cap \text{span}_k ((u_x)_{x \in X}) = \{0\} .
\]  

(5)
Possibly subgroups:

- Lie Group $G$

$L(G)$ (Lie algebra)

Magnus, Hausdorff
Continuous, Algebraic
Why BTT and NCDE? : Review of the facts

- \( \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \) \((\Re(s) > 1)\)
- when one multiplies two of these, one gets quantities like
  \[
  \zeta(s_1) \zeta(s_2) = \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2}} = \zeta(s_1, s_2) + \zeta(s_1 + s_2) + \zeta(s_2, s_1)
  \]
- and, with several of them, we are led to the following definition of MultiZeta Values (MZV), converging in
  \[
  \mathcal{H}_r = \{(s_1, \ldots, s_r) \in \mathbb{C}^r \mid \forall m = 1, \ldots, r, \Re(s_1) + \ldots + \Re(s_m) > m\}.
  \]
  \[
  \zeta(s_1, \ldots, s_k) := \sum_{n_1 > \ldots > n_k \geq 1} \frac{1}{n_1^{s_1} \ldots n_k^{s_k}} \] \( (6) \)
- On the other hand, one has the classical polylogarithms defined, for \( k \geq 1, |z| < 1 \), by
  \[
  - \log(1 - z) = \text{Li}_1 = \sum_{n \geq 1} \frac{z^n}{n^1}; \quad \text{Li}_2 = \sum_{n \geq 1} \frac{z^n}{n^2}; \ldots; \quad \text{Li}_k(z) := \sum_{n \geq 1} \frac{z^n}{n^k}
  \]
The analogue of the classical polylogarithms for MZV reads

\[ \text{Li}_{y_1\ldots y_k}(z) := \sum_{n_1 > \ldots > n_k \geq 1} \frac{z^{n_1}}{n_1^{s_1} \ldots n_k^{s_k}} ; \ |z| < 1 \]

They satisfy the recursion (ladder stepdown)

\[ z \frac{d}{dz} \text{Li}_{y_1\ldots y_k} = \text{Li}_{y_1\ldots y_{k-1}} \text{ if } s_1 > 1 \]

\[ (1 - z) \frac{d}{dz} \text{Li}_{y_1y_2\ldots y_k} = \text{Li}_{y_2\ldots y_k} \text{ if } k > 1 \]

which, with \( s_i \in \mathbb{N}_{\geq 1}, \ k \geq 1 \), ends at the “seed”

\[ \text{Li}_{y_1}(z) = \text{Li}_1(z) = \log \left( \frac{1}{1 - z} \right) \]

For the next step, we code the moves \( z \frac{d}{dz} \) (resp. \( (1 - z) \frac{d}{dz} \)) - or more precisely sections \( \int_0^z \frac{f(s)}{s} ds \) (resp. \( \int_0^z \frac{f(s)}{1-s} ds \)) - with \( x_0 \) (resp. \( x_1 \)).
Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $*_0 = 0$

\[
\langle S|x_1^n \rangle = \frac{(-\log(1-z))^n}{n!} ; \quad \langle S|x_0 x_1 \rangle = \text{Li}_2(z) = \text{Li}_{x_0 x_1}(z) = \sum_{n \geq 1} \frac{z^n}{n^2}
\]

\[
\langle S|x_0^2 x_1 \rangle = \text{Li}_3(z) = \text{Li}_{x_0^2 x_1}(z) = \sum_{n \geq 1} \frac{z^n}{n^3} ; \quad \langle S|x_1 x_0 x_1 \rangle = \text{Li}_{x_1 x_0 x_1}(z) = \text{Li}[1,2](z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}}{n_1 n_2^2}
\]

\[
\langle S|x_0 x_1^2 \rangle = \text{Li}_{x_0 x_1^2}(z) = \text{Li}[2,1](z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}}{n_1^2 n_2^2} ; \quad \text{above "cl. not." stands for "classical notation"}
\]
Why BTT and NCDE? Review of the facts

- Calling $S$ the prospective generating series:

$$S = \sum_{w \in X^*} \langle S \mid w \rangle w; \ X = \{x_0, x_1\} \quad (9)$$

V. Drinfel’d [1] indirectly proposed a way to complete the tree:

$$\begin{cases} 
\mathbf{d}(S) = (\frac{x_0}{z} + \frac{x_1}{1-z})S \\
\lim_{z \to 0} S(z)e^{-x_0 \log(z)} = 1_{\mathcal{H}(\Omega)}\langle X \rangle 
\end{cases} \quad (NCDE) \quad (Asympt. \ Init. \ Cond.) \quad (10)$$

From the general theory, this system has a unique solution which is precisely $\text{Li}$ (called $G_0$ in [1]) ; $S \mapsto \mathbf{d}(S)$ being the term by term derivation of the coefficients.

- Minh [2] indicated a way to effectively compute this solution through (improper) iterated integrals.

Explicit construction of Drinfeld’s $G_0$

Given a word $w$, we note $|w|_{x_1}$ the number of occurrences of $x_1$ within $w$

\[
\alpha_0^z(w) = \begin{cases} 
1_{\Omega} & \text{if } w = 1x_* \\
\int_0^z \alpha_0^s(u) \frac{ds}{1-s} & \text{if } w = x_1u \\
\int_1^z \alpha_0^s(u) \frac{ds}{s} & \text{if } w = x_0u \text{ and } |u|_{x_1} = 0 (w \in x_0^*) \\
\int_0^z \alpha_0^s(u) \frac{ds}{s} & \text{if } w = x_0u \text{ and } |u|_{x_1} > 0 (w \in x_0X^*x_1x_0^*)
\end{cases}
\]

The third line of this recursion implies

\[
\alpha_0^z(x_0^n) = \frac{\log(z)^n}{n!}
\]

one can check that (a) all the integrals (although improper for the fourth line) are well defined (b) the series $S = \sum_{w \in X^*} \alpha_0^z(w)w$ is Li ($G_0$ in [1]).
Some coefficients with $X = \{x_0, x_1\}$; $u_0(z) = \frac{1}{z}$; $u_1(z) = \frac{1}{1-z}$, $t_0 = 0$

\[
\langle S|x_1^n \rangle = \frac{(-\log(1-z))^n}{n!}; \quad \langle S|x_0x_1 \rangle = \text{Li}_2(z) = \text{Li}_{x_0x_1}(z) = \sum_{n \geq 1} \frac{z^n}{n^2} \\
\langle S|x_0^2x_1 \rangle = \text{Li}_3(z) = \text{Li}_{x_0^2x_1}(z) = \sum_{n \geq 1} \frac{z^n}{n^3} ; \quad \langle S|x_1x_0x_1 \rangle = \text{Li}_{x_1x_0x_1}(z) = \text{Li}_{[1,2]}(z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}}{n_1 n_2^2} \\
\langle S|x_0^2 \rangle = \text{Li}_{x_0^2}(z) = \text{Li}_{[2,1]}(z) = \sum_{n_1 > n_2 \geq 1} \frac{z^{n_1}}{n_1^2 n_2} ; \quad \langle S|x_0^n \rangle = \frac{\log^n(z)}{n!}
\]
The series $S_{Pic}^{z_0}$ ($z_0 \in \Omega$) can be computed by Picard’s process:

$$S_0 = 1_{\chi^*} ; \quad S_{n+1} = 1_{\chi^*} + \int_{z_0}^{z} M.S_n$$

and its limit is $S_{Pic}^{z_0} := \lim_{n \to \infty} S_n \left( = \sum_{w \in X^*} \alpha_{z_0}^z(w) w \text{ this afternoon} \right)$. One has,

**Proposition**

i) Series $S_{Pic}^{z_0}$ is the unique solution of

$$\left\{ \begin{array}{ll}
\mathbf{d}(S) &= M.S \text{ with } M = \sum_{i=1}^{n} \frac{x_i}{z-a_i} \\
S(z_0) &= 1_{\mathcal{H}(\Omega)\langle\langle\chi\rangle\rangle}
\end{array} \right. \quad (11)$$

ii) The complete set of solutions of $\mathbf{d}(S) = M.S$ is $S_{Pic}^{z_0} \cdot \mathbb{C}\langle\langle\chi\rangle\rangle$. 
About solutions of NCDE

1. The set $S$ of series satisfying (NCDE) has a lot of nice combinatorial properties.
   - Right $\mathbb{C}\langle\langle X \rangle\rangle$ module of rank one ($S = S_0.\mathbb{C}\langle\langle X \rangle\rangle$, where $S_0$ is any solution with non-zero constant term, such a solution can be constructed by Picard process).
   - Linear independence of the coefficients (when non-zero).

2. The ones like $L_i$ or constructed through Picard’s process (Chen series, i.e. limit of $S_0 = 1X^*; S_{n+1} = 1X^* + \int_{z_0}^{z} M.S_n$) have moreover
   - Shuffle property
   - Factorisation
   - Extension to rational functions (some of them for $L_i$, all for $S_{Pic}^{z_0}$).

Now, as the lists are coded by words, it is possible to use the rich allowance of notations invented by algebraists, computer scientists, combinatorialists and physicists about NonCommutative Formal Power Series (NCFPS\(^1\)).

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\(^1\)This was the initial intent of the series of conferences FPSAC.
What is so special with solutions like $Li$ and $S_{Pic}^{Z_0}$.

The following general theorem explains (a) why $Li$ and $S_{Pic}^{Z_0}$ have the shuffle property and (b) why $Li$ is unique.

**Theorem (Analyse et Géometrie, Cargèse, IESC, 21-24 Nov. 2017)**

Let

$$(TSM) \quad dS = M_1S + SM_2. \quad (12)$$

with $S \in \mathcal{H}(\Omega)\langle\langle X\rangle\rangle$, $M_i \in \mathcal{H}(\Omega)_+\langle\langle X\rangle\rangle$

(i) **Solutions of** $(TSM)$ **form a $\mathbb{C}$-vector space.**

(ii) **Solutions of** $(TSM)$ **have their constant term (as coefficient of** $1_X^*$ **) which are constant functions (on** $\Omega$ **); there exists solutions with constant coefficient $1_\Omega$ **(hence invertible).**

(iii) **If two solutions coincide at one point** $z_0 \in \Omega$ **(or asymptotically), they coincide everywhere.**
What is so special with solutions like $\text{Li}$ and $S_{Pic}^{\zeta_0}/2$

**Theorem (cont’d)**

(iv) Let be the following one-sided equations

\[
(LM_1) \quad dS = M_1 S \quad (RM_2) \quad dS = SM_2. \tag{13}
\]

and let $S_1$ (resp. $S_2$) be a solution of $(LM_1)$ (resp. $(LM_2)$), then $S_1 S_2$ is a solution of $(TSM)$. Conversely, every solution of $(TSM)$ can be constructed so.

(v) Let $S_{Pic,LM_1}^{\zeta_0}$ (resp. $S_{Pic,RM_2}^{\zeta_0}$) the unique solution of $(LM_1)$ (resp. $(RM_2)$) s.t. $S(z_0) = 1_{\mathcal{H}(\Omega) + \langle X \rangle}$ then, the space of all solutions of $(TSM)$ is

\[
S_{Pic,LM_1}^{\zeta_0} \cdot \mathbb{C} \langle X \rangle \cdot S_{Pic,RM_2}^{\zeta_0}
\]

(vi) If $M_i$, $i = 1, 2$ are primitive for $\Delta_\Pi{}^a$ and if $S$, a solution of $(TSM)$, is group-like at one point (or asymptotically), it is group-like everywhere (over $\Omega$).

---

\[{}^a\Delta_\Pi \] is the canonical comultiplication of $\mathbb{C} \langle X \rangle$ viewed as an enveloping algebra.
The categories \textbf{DiffRing, CDiffRing, DiffAlg}_k, \textbf{CDiffAlg}_k

1. We begin with \textbf{DiffAlg}_k
Let \( k \) be a ring \textbf{DiffAlg}_k is the category of pairs \((A, \partial)\) where \( A \in \text{AAU}_k \) and \( \partial \in \text{Der}(A) \). An arrow \( f : (A, \partial_A) \to (B, \partial_B) \) is an arrow \( f \in \text{Hom}_k(A, B) \) such that \( f \partial_A = \partial_B f \).

2. For \((A, \partial_A) \in \textbf{DiffAlg}_k\), \( \ker(\partial_A) \) is a \( k \)-subalgebra of \( A \) called that of constants of \( A \).

We now describe the free objects

\[
\text{St} \xleftarrow{F} \text{DiffAlg}_k
\]

\[
X \xrightarrow{f} A
\]

\[
j_X \xrightarrow{\hat{f}} k\langle\{X\}\rangle
\]

Figure: A solution of the universal problem w.r.t. the natural forgetful functor from \textbf{DiffAlg}_k to \textbf{St}.
Construction of $\mathbf{k}\langle\{X\}\rangle$ and $\mathbf{k}\{X\}$

1. We describe the structure. Let $X$ be an alphabet. The free object $\mathbf{k}\langle\{X\}\rangle$ is:
   - a free algebra $\mathbf{k}\langle X \times \mathbb{N}\rangle$ where, for all $x \in X$, is noted $(x, n) = x^{[n]}$ and, for convenience, $x^{[0]} = x$. This algebra is equipped with the derivation $\partial$ such that $\partial(x^{[k]}) = x^{[k+1]}$.

2. Existence of $\partial$ as a derivation is standard (see e.g. [5], Ch I, §2.8 Extension of derivations).

3. The construction is similar to what is to be found in [20], but in the noncommutative realm.

We now say a word of the construction in [20]

\[
\begin{array}{ccc}
\text{St} & \xleftarrow{F} & \text{CDiffAlg}_k \\
\downarrow{f} & & \downarrow{A} \\
X & \xrightarrow{jx} & \mathbf{k}\{X\}
\end{array}
\]
Construction of $k\{X\}$

1. Construction of $k\{X\}$ is very similar to that of $k\langle\{X\}\rangle$ but
2. It is devoted to the category $\text{CDiffAlg}_k$ (commutative differential $k$-algebras)
3. It uses commutative polynomials i.e. the basic algebra is $k[X \times \mathbb{N}]$ (and not $k\langle X \times \mathbb{N}\rangle$) with the same notations $((x, n) = x^{[n]}$ and $x^{[0]} = x$).
4. It is the one used for Proposition 2 in Vu’s talk (and, in fact, the construction can be done using $k\{X\}$ with $Y_i^{[j]} = Y_{ij}$ and a suitable ideal).
5. We recall Proposition 2.

**Proposition 2**

Let $F$ be a differential field with algebraically closed field of constants $C_F$ and $\mathcal{L}(Y) = Y^{(n)} + a_{n-1} Y^{(n-1)} + ... + a_1 Y' + a_0 Y = 0$ be defined over $F$. Then there exists a Picard-Vessiot extension $L$ of $F$ for $\mathcal{L}$, that is unique up to differential $F$-isomorphism.
Application: Cartan theorem in Banach algebras (without transversality nor Lipschitz condition)

See https://mathoverflow.net/questions/356531 for motivation.

**Theorem** Let $B$ be a Banach algebra (with unit $e$) and $G$ be a closed subgroup of $B^{-1}$ (the group of multiplicative inverses). Let $L(G)$ be the tangent space of $G$ and $m : I \to L(G)$ be a continuous function ($I \subset \mathbb{R}$ is an open interval containing $0_{\mathbb{R}}$), then

i) The following system

\[ y'(t) = m(t)y(t) ; \quad y(0) = e \]

admits a unique solution, say $s(t)$.

ii) The trajectory of $s$ is entirely in $G$ (in other words $t \mapsto s(t)$ is a path drawn on $G$). My questions are the following:

Q1) Is it known? (I expect so, at least of the specialists)

Q2) If yes, is there a sound reference? (not general, but about this very simple and precise property).
The Magnus group is the set of series with constant term $1_{X^*}$, the Hausdorff (sub)-group, is the group of group-like series for $\Delta_{\text{III}}$. These are also Lie exponentials (here $A, B$ are Lie series and $\exp(A)\exp(B) = \exp(H(A, B))$).
Let \((\mathbb{C}\langle\{X\}\rangle, \partial)\) be the differential algebra freely generated by \(X\) (a single formal variable). We define a comultiplication \(\Delta\) by asking that all \(X^k\) be primitive note that \(\Delta\) commutes with the derivation. Setting, in \(\mathbb{C}\langle\{X\}\rangle\), \(D = \partial(e^X)e^{-X}\), direct computation shows that \(D\) is primitive and hence a Lie series\(^2\), which can therefore be written as a sum of (evaluations of) Dynkin trees.

On the other hand, the formula

\[
D = \sum_{k \geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} X^l(\partial X)X^{k-1-l} \cdot \sum_{n \geq 0} \frac{(-X)^n}{n!} \tag{14}
\]

suggests that all bidegrees, in \((X, \partial X)\), are of the form \([n, 1]\) and thus, there exists an univariate series \(\Phi(Y) = \sum_{n \geq 0} a_n Y^n\) such that \(D = \Phi(adX)[\partial X]\).

\(^2\)Which would be trivial, if we were in \(\mathbb{C}\{X\}\) (i.e. \(X\) commutes with \(\partial X\), as there \(D = \partial(X)\), but this is not the case within \(\mathbb{C}\langle\{X\}\rangle\) as shows the computation (14).
About Magnus expansion and Poincaré-Hausdorff formula/2

Using left and right multiplications by $X$ (resp. noted $g$, $d$), we can rewrite (14) as

$$D = \left( \sum_{k \geq 1} \frac{1}{k!} \sum_{l=0}^{k-1} g^l d^{k-1-l} [\partial X] \right) e^{-X}$$

(15)

but, from the fact that $g$, $d$ commute, the inner sum $\sum_{l=0}^{k-1} g^l d^{k-1-l}$ is ruled out by the following identity (in $\mathbb{C}[Y, Z]$, but computed within $\mathbb{C}(Y, Z)$) and

$$\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{((Y - Z) + Z)^k - Z^k}{Y - Z} = \sum_{j=1}^{k} \binom{k}{j} (Y - Z)^j Z^{k-j}$$
\[
\sum_{l=0}^{k-1} Y^l Z^{k-1-l} = \frac{Y^k - Z^k}{Y - Z} = \frac{((Y - Z) + Z)^k - Z^k}{Y - Z} = \sum_{j=1}^{k} \binom{k}{j} (Y - Z)^j Z^{k-j}
\]

Taking notice that \((g - d) = ad_X\) and plugging (16) into (14), one gets

\[
D = \left( \sum_{k \geq 1} \frac{1}{k!} \sum_{j=1}^{k} \binom{k}{j} (ad_X)^{j-1} d^{k-j} [\partial X] \right) e^{-X} =
\]

\[
\frac{1}{ad_X} \left( \sum_{k \geq 1} \sum_{j=1}^{k} \frac{1}{j!(r-j)!} (ad_X)^j d^{k-j} [\partial X] \right) e^{-X} = \frac{e^{ad_X} - 1}{ad_X} [X']
\]

which is Poincaré-Hausdorff formula (of course \(\frac{e^{ad_X} - 1}{ad_X}\) stands for the substitution of \(ad_X\) in the formal series corresponding to the entire function \(e^z - 1\)).
Abstract BTT theorem towards localisation

Theorem (DDMS.\(^1\) “Linz”)

Let \((A, d)\) be a \(k\)-commutative associative differential algebra with unit \((\ker(d) = k \text{ is a field})\) and \(C\) be a differential subfield of \(A\) (i.e. \(d(C) \subset C\)). We suppose that \(S \in A\langle\langle X\rangle\rangle\) is a solution of the differential equation

\[
d(S) = MS; \langle S|1_\ast \rangle = 1_A
\]

(18)

where the multiplier \(M\) is a homogeneous series (a polynomial in the case of finite \(X\)) of degree 1, i.e.

\[
M = \sum_{x \in X} u_x x \in C\langle\langle X\rangle\rangle.
\]

(19)

The following conditions are equivalent:

Theorem (cont’d)

i) The family \((\langle S|w\rangle)_{w\in X^*}\) of coefficients of \(S\) is free over \(\mathcal{C}\).

ii) The family of coefficients \((\langle S|y\rangle)_{y\in X\cup\{1_{X^*}\}}\) is free over \(\mathcal{C}\).

iii) The family \((u_x)_{x\in X}\) is such that, for \(f \in \mathcal{C}\) and \(\alpha_x \in k\)

\[
d(f) = \sum_{x\in X} \alpha_x u_x \implies (\forall x \in X)(\alpha_x = 0). \quad (20)
\]

iv) The family \((u_x)_{x\in X}\) is free over \(k\) and

\[
d(\mathcal{C}) \cap \text{span}_k((u_x)_{x\in X}) = \{0\}. \quad (21)
\]
Need for localization

In practical cases, we only have a differential subalgebra of $\mathcal{C}_0 \subset \mathcal{H}(\Omega)$ (as image, through $\text{Li}$, of a shuffle subalgebra of $\text{Dom}(\text{Li})$).

- $\mathbb{C}[z]$
- $\mathbb{C}[z, z^{-1}, (1 - z)^{-1}]$
- $\mathbb{C}[z^\alpha(1 - z)^{-\beta}]_{\alpha, \beta \in \mathbb{C}} = \mathbb{C}_c$

Realizing the fraction field $\text{Fr}(\mathcal{C}_0)$ as (differential) field of germs makes the computation difficult to handle. It is easier to check the freeness of the “basic triangle” directly with the algebra. For instance, for the polylogarithms, we just have to show that, given $P_i \in \mathbb{C}_c$,

$$P_1(z) + P_2(z) \log(z) + P_3(z)(\log\left(\frac{1}{1 - z}\right)) = 0_{\Omega} \implies P_i \equiv 0 \quad (22)$$

which can be done using deck transformations (see below).
Localization

**Theorem (Thm1 in "Linz", Localized form)**

Let \((\mathcal{A}, d)\) be a commutative associative differential ring \((\ker(d) = k\text{ being a field})\) and \(\mathcal{C}\) be a differential subring (i.e. \(d(\mathcal{C}) \subset \mathcal{C}\)) of \(\mathcal{A}\) which is an integral domain containing the field of constants.

We suppose that, for all \(x \in X\), \(u_x \in \mathcal{C}\) and that \(S \in \mathcal{A}\langle\langle X\rangle\rangle\) is a solution of the differential equation (18) and that \((u_x)_{x \in X} \in \mathcal{C}^X\).

The following conditions are equivalent:

1. The family \((\langle S|w\rangle)_{w \in X^*}\) of coefficients of \(S\) is free over \(\mathcal{C}\).
2. The family of coefficients \((\langle S|y\rangle)_{y \in X \cup \{1_{X^*}\}}\) is free over \(\mathcal{C}\).
3. For all \(f_1, f_2 \in \mathcal{C}, f_2 \neq 0\) and \(\alpha \in k(X)\), we have the property
   \[W(f_1, f_2) = f_2^2 \left( \sum_{x \in X} \alpha_x u_x \right) \iff (\forall x \in X)(\alpha_x = 0).\] (23)

where \(W(f_1, f_2)\), the wronskian, stands for \(d(f_1)f_2 - f_1 d(f_2)\).
In fact, in the localized form and with $C$ not a differential field, ($iii$) is strictly weaker than ($iii'$), as shows the following family of counterexamples

1. $\Omega = \mathbb{C} \setminus (]-\infty, 0])$
2. $X = \{x_0\}$, $u_0 = z^\beta$, $\beta \notin \mathbb{Q}$
3. $C_0 = \mathbb{C}\{z^\beta\} = \mathbb{C}.1_\Omega \oplus \text{span}_\mathbb{C}\{z^{(k+1)\beta-l}\}_{k,l \geq 0}$
4. $S = 1_\Omega + \left(\sum_{n \geq 1} \frac{z^{n(\beta+1)}}{(\beta+1)n!}\right)$

Let us show that, for these data ($iii$) holds but not ($i$).
Firstly, we show that $C_0 = \mathbb{C}\{z^\beta\}$ corresponds to the given direct sum. We remark that the family $(z^\alpha)_{\alpha \in \mathbb{C}}$ is $\mathbb{C}$-linearly free (within $\mathcal{H}(\Omega)$), which is a consequence of the fact that they are eigenfunctions, for different eigenvalues, of the Euler operator $z \frac{d}{dz}$. 


Then
\[ \mathbb{C}\{\{z^\beta\}\} = \mathbb{C}1_\Omega \oplus \text{span}_\mathbb{C}\{z^{(k+1)\beta-l}\}_{k,l \geq 0} = \text{span}_\mathbb{C}\{z^{(k')\beta-l}\}_{k',l \geq 0} \]

comes from the fact that the RHS is a subset of the LHS as, for all, \( k, l \geq 0, z^{(k+1)\beta-l} \in \mathbb{C}\{\{z^\beta\}\} \). Finally \( 1_\Omega \in \mathbb{C}\{\{z^\beta\}\} \) by definition (\( \mathbb{C}\{\{X\}\} \) is a \( \mathbb{C}\)-AAU).

(iii) is fulfilled. Here
\[ u_0(z) = z^{\beta} \]
is such that, for any \( f \in C_0 \) and \( c_0 \) in \( \mathbb{C} \), we have
\[ c_0 u_0 = \partial_z(f) \implies (c_0 = 0) \quad (24) \]

But (i) is not. Because we have the following relation
\[ (\beta + 1)z^{\beta-1}\langle S|x_0\rangle - z^{2\beta}1_\Omega = 0 \]
Sketch of the proof

After some technicalities, we show that (18) can be transported in $\mathcal{A}[(C^\times)^{-1}]$ by means of the following commutative diagram and back.

\[
\begin{array}{ccc}
C & \xrightarrow{\phi_C} & Fr(C) \\
\downarrow d & & \downarrow \phi_A \\
\mathcal{A} & \xrightarrow{\phi_A} & \mathcal{A}[(C^\times)^{-1}] \\
\downarrow j_d & & \downarrow d_{frac} \\
C & \xrightarrow{\phi_C} & Fr(C) \\
\downarrow & & \downarrow \\
\mathcal{A} & \xrightarrow{j_{frac}} & \mathcal{A}[(C^\times)^{-1}]
\end{array}
\]

(25)
Proof that \([1_Ω, \log(z), \log(\frac{1}{1-z})]\) is \(C_\mathbb{C}\)-free.

Let us suppose \(P_i, i = 1 \ldots 3\) such that

\[
P_1(z) + P_2(z) \log(z) + P_3(z)(\log(\frac{1}{1-z})) = 0_Ω
\]

We first prove that \(P_2 = \sum_{i \in F} c_i z^{\alpha_i}(1 - z)^{\beta_i}\) is zero using the deck transformation \(D_0\) of index one around zero. One has \(D_0^n(\sum_{i \in F} c_i z^{\alpha_i}(1 - z)^{\beta_i}) = \sum_{i \in F} c_i z^{\alpha_i}(1 - z)^{\beta_i} e^{2i\pi n \alpha_i}\), the same calculation holds for all \(P_i\) which proves that all \(D_0^n(P_i)\) are bounded. But one has \(D_0^n(\log(z)) = \log(z) + 2i\pi n\) and then

\[
D_0^n(P_1(z) + P_2(z) \log(z) + P_3(z)(\log(\frac{1}{1-z}))) =
\]

\[
D_0^n(P_1(z)) + D_0^n(P_2(z))(\log(z) + 2i\pi n) + D_0^n(P_3(z)) \log(\frac{1}{1-z}) = 0
\]

It suffices to build a sequence of integers \(n_j \rightarrow +\infty\) such that \(\lim_{j \rightarrow \infty} D_0^{n_j}(P_2(z)) = P_2(z)\) which is a consequence of the following lemma.
**Lemma**

*Let us consider a homomorphism* $\varphi : \mathbb{N} \to G$ *where* $G$ *is a compact (Hausdorff) group, then it exists* $u_j \to +\infty$ *such that*

$$\lim_{j \to \infty} \varphi(u_j) = e$$

**Proof.**

*First of all, due to the compactness of* $G$, *the sequence* $\varphi(n)$ *admits a subsequence* $\varphi(n_k)$ *convergent to some* $\ell \in G$. *Now one can refine the sequence as* $n_{k_j}$ *such that*

$$0 < n_{k_1} - n_{k_0} < \ldots < n_{k_{j+1}} - n_{k_j} < n_{k_{j+2}} - n_{k_{j+1}} < \ldots$$

*With* $u_j = n_{k_{j+1}} - n_{k_j}$ *one has* $\lim_{j \to \infty} \varphi(u_j) = e$. 

**End of the proof** One applies the lemma to the morphism

$$n \mapsto (e^{2i \pi n \alpha_i})_{i \in F} \in \mathbb{U}^F$$
Conclusion

- For Series with variable coefficients, we have a theory of Noncommutative Evolution Equation sufficiently powerful to cover iterated integrals and multiplicative renormalisation.
- Use of combinatorics on words gives a necessary and sufficient condition on the “inputs” to have linear independance of the solutions over higher function fields.
- Picard (Chen) solutions admit enlarged indexing w.r.t. compact convergence on Ω (polylogarithmic case) but Drinfeld’s $G_0$ has a domain which includes only some rational series.
- Localization is possible (under certain conditions).
- Local BTT theorem allows to explore linear and algebraic independences w.r.t. subalgebras of $Dom(Li)$. 
Thank you for your attention.
Links

1. Categorical framework(s)
   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems
   https://ncatlab.org/nlab/show/universal+construction

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]
   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphisms
   - https://ncatlab.org/nlab/show/heteromorphism
   - D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside.
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   | https://hal.archives-ouvertes.fr/hal-01015295/document |
| 8 | State on a star-algebra  
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| 9 | Hilbert module  
   | https://ncatlab.org/nlab/show/Hilbert+module |


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