Lie Objects

Matthieu Deneufchâtel

Laboratoire d'Informatique de Paris Nord,
Université Paris 13

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Outline

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Lie Algebra

k a field of characteristic zero.

Definition

A Lie algebra $\mathscr{G}$ is a vector space endowed with a bilinear operation $[\cdot,\cdot] : \mathscr{G} \times \mathscr{G} \to \mathscr{G}$ satisfying the following relations, $\forall a, b, c \in \mathscr{G}$:

$$[a, a] = 0;$$

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0.$$

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Each associative algebra $\mathscr{A}$ has a natural Lie algebra structure $\mathscr{A}_L$ with the bracket defined by :

$$[a, b] = ab - ba.$$

In terms of categories :

$$f : k - \text{UAA} \longrightarrow k - \text{Lie algebra}.$$

Enveloping algebra

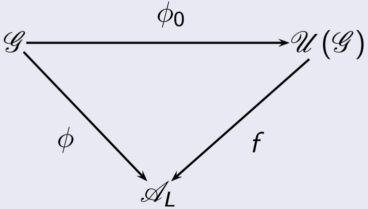
Let $\mathscr{G}$ be a Lie algebra. It is possible to associate to $\mathscr{G}$ an associative algebra called enveloping algebra of $\mathscr{G}$ denoted by $\mathscr{U}(\mathscr{G})$.

Universal problem.

There exists a unital associative algebra $\mathscr{U}(\mathscr{G})$ and a Lie algebra homomorphism $\phi_0 : \mathscr{G} \to \mathscr{U}(\mathscr{G})_L$ such that, for

- any associative algebra $\mathscr{A}$,
- any Lie algebra homomorphism $\phi : \mathscr{G} \to \mathscr{A}_L$,

there is a unique algebra homomorphism $f : \mathscr{U}(\mathscr{G}) \to \mathscr{A}$ making the following diagram commute:



Enveloping algebra

$g : k -$ Lie algebra $\longrightarrow k -$ UAA. g is the left adjoint of f.

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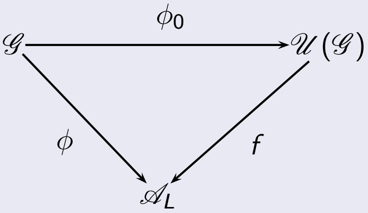

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Example : Free Lie Algebra

X an alphabet.

Theorem

There exists a Lie algebra $\mathrm{Lie}_k\langle X\rangle$ over k unique up to isomorphism and freely generated by X. It is called Free Lie Algebra.

Construction :

- Lie Monomials :
  - $\forall x \in X$, x is a Lie monomial ;
  - if u and v are Lie monomials, then so is $[u, v] = uv - vu$ (concatenation product).

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  - if u and v are Lie monomials, then so is $[u, v] = uv - vu$ (concatenation product).
- Lie Polynomials and Series : respectively finite and infinite k-linear combinations of Lie monomials.
- $\rightarrow \mathrm{Lie}_k\langle X\rangle$.

Link with the free associative algebra $k\langle X\rangle$

For $P, Q \in k\langle X\rangle$, their Lie bracket is $[P, Q] = PQ - QP$.
The smallest submodule of $k\langle X\rangle$ closed under this bracket and containing X is the free Lie algebra $\mathrm{Lie}_k\langle X\rangle$.
$k\langle X\rangle$ is the enveloping algebra of $\mathrm{Lie}_k\langle X\rangle$:

$$k\langle X\rangle = \mathscr{U}(\mathrm{Lie}_k\langle X\rangle).$$

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Coproduct Δ on $k\langle X\rangle$ (homomorphism of k-algebra defined on the letters):

$$\Delta(x) = x \otimes 1 + 1 \otimes x.$$

Lie polynomials (Friedrich)

The following conditions are equivalent :

- $P \in k\langle X\rangle$ is a Lie polynomial ;
- $\Delta(P) = P \otimes 1 + 1 \otimes P$ (P is primitive).

Outline

Example of $k\langle X\rangle$

$(k\langle X\rangle, \mathrm{conc}, 1_{X^*}, \Delta, \epsilon)$ is a cocommutative graded bialgebra :

$$k\langle X\rangle = \bigoplus_{n\geq 0} k_{=n}\langle X\rangle,$$

where $P \in k_{=n}\langle X\rangle$ means that $P = \sum_{|w|=n} \langle P|w\rangle w$.

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Lie polynomials are primitive elements :

$$\forall P \in \mathrm{Lie}_k(X),\ \Delta(P) = P\otimes 1 + 1\otimes P.$$

CQMM Theorem

Let $\mathscr{B}$ be a bialgebra. It is

- graded if :
  - $\mathscr{B} = \bigoplus_{n\geq 0} \mathscr{B}_n$;
  - $\mu\left(\mathscr{B}_p, \mathscr{B}_q\right) \subset \mathscr{B}_{p+q}, \forall\, p, q \in \mathbb{N}$;
  - $\Delta(\mathscr{B}_n) \subset \bigoplus\limits_{p+q=n} \mathscr{B}_p \otimes \mathscr{B}_q, \forall\, n \in \mathbb{N}$;
- connected if $\mathscr{B}_0 = k1_{\mathscr{B}}$.

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Let $\mathscr{B}$ be a cocommutative graded connected bialgebra.

Cartier-Quillen-Milnor-Moore Theorem

$\mathscr{B}$ is the enveloping algebra of its primitive elements.

Theorem

$(X, <)$, $\mathrm{Lyn}(X)$.
$\mathrm{Lyn}(X)$ is a (totally ordered) basis of $\mathrm{Lie}_k\langle X\rangle$.

Poincaré-Birkhoff-Witt

Let $(g_i)_{i\in\mathscr{I}}$ be a totally ordered basis of a Lie algebra $\mathscr{G}$.
Then the "decreasing" products

$$g^\alpha = g_{i_1}^{\alpha_1} \dots g_{i_p}^{\alpha_p},\ i_1 > \dots > i_p,\ \alpha_i \in \mathbb{N},$$

form a basis of $\mathscr{U}(\mathscr{G})$.

Thus, $\mathrm{Lyn}(X)$ induces a basis of $k\langle X\rangle = \mathscr{U}(\mathrm{Lie}_k(X))$ in the following way:

PBW Basis

For $l \in \mathrm{Lyn}(X)$, let us define $(P_l)_{l\in\mathrm{Lyn}(X)}$ by :

$$P_l = \begin{cases} l & \text{if } |l| = 1; \\ [l_1, l_2] & \text{otherwise, with } l = l_1 l_2 \text{ the standard factorization of } l. \end{cases}$$

If $w = l_{i_1}^{\alpha_1} \dots l_{i_k}^{\alpha_k}$ with $l_{i_1} > \dots > l_{i_k}$,

$$P_w = P_{l_{i_1}}^{\alpha_1} \dots P_{l_{i_k}}^{\alpha_k}.$$

- P_w is homogeneous for the multidegree (finely homogeneous).
- $P_w = w + \sum_{v > w \in X^*} *v$ where the star denotes coefficients in $\mathbb{Z}$.

$(P_l)_{l\in\mathrm{Lyn}(X)}$ is a basis of $\mathrm{Lie}_k(X)$ and $(P_w)_{w\in X^*}$ is a basis of $k\langle X\rangle$.

Outline

Dual basis

Duality bracket : $\langle u|v\rangle = \delta_{u,v} \Rightarrow k\langle\langle X\rangle\rangle \sim (k\langle X\rangle)^*$:

$$\langle S|P\rangle = \sum_{w\in X^*} \langle S|w\rangle\langle P|w\rangle.$$

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Duality bracket : $\langle u|v\rangle = \delta_{u,v}$

$$S_w = \begin{cases} w & \text{if } |w| = 1; \\ xS_u & \text{if } w = xu \text{ and } w \text{ is a Lyndon word;} \\ \dfrac{S_{l_{i_1}}^{ш\alpha_1} ш \ldots ш S_{l_{i_k}}^{ш\alpha_k}}{\alpha_1! \ldots \alpha_k!} & \text{otherwise, if } w = l_{i_1}^{\alpha_1} \ldots l_{i_k}^{\alpha_k} \end{cases}$$

with $S^{ш k} = S ш S^{k-1}$ for $k > 0$ and $S^{ш 0} = 1$.

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with $S^{\sqcup\!\sqcup k} = S \sqcup\!\sqcup S^{k-1}$ for $k > 0$ and $S^{\sqcup\!\sqcup 0} = 1$.

Theorem

$$\langle S_u | P_v \rangle = \delta_{u,v}.$$

Outline

Definition

$S \in k\langle\langle X\rangle\rangle$ is a Lie exponential iff there exists a Lie series $L \in \mathrm{Lie}_k\langle\langle X\rangle\rangle$ such that $S = \exp(L)$.

If $S \neq 0$, this is equivalent to the following properties :

- $\forall\, u, v \in X^*$, $\langle S | u ш v\rangle = \langle S|u\rangle\langle S|v\rangle$;
- $\Delta(S) = S \otimes S$.

Any Lie exponential S can be factored as an infinite product of “elementary“ Lie exponentials:

$$S = \prod_{l \in \mathrm{Lyn}(X)}^{\searrow} \exp\left(\langle S|S_l\rangle P_l\right).$$

Example : Polylogarithms

Definition

$$\mathrm{Li}_1(z) = 1,\ \mathrm{Li}_{x_1}(z) = \int_0^z \frac{dt}{1-t} = -\ln(1-z) \text{ and } \mathrm{Li}_{x_0}(z) = \ln(z).$$

Then

$$\mathrm{Li}_{x_0 w}(z) = \int_0^z \mathrm{Li}_w \frac{dt}{t};$$
$$\mathrm{Li}_{x_1 w}(z) = \int_0^z \mathrm{Li}_w \frac{dt}{1-t}.$$

Generating series of polylogarithms :

$$L(z) = \sum_{w \in X^*} L_w(z) w$$

is a Lie exponential, $\forall z \in \mathbb{C} \setminus (]-\infty, 0] \cup [1, +\infty[)$.

Outline

Characters

Definition

χ is a character of the k-algebra $\mathfrak{A}$ iff $\chi \in \mathrm{Hom}_{k-\mathrm{Alg}}(k\langle X\rangle, k)$:

- $\chi(a+b) = \chi(a) + \chi(b)$;
- $\chi(ab) = \chi(a)\chi(b)$;
- $\chi(1_{\mathfrak{A}}) = 1$;

Properties : Let $\mathscr{H}$ be a Hopf algebra and $\mathfrak{A}$ an AAU. Then

1. $\mathrm{Hom}_k(\mathscr{H}, \mathfrak{A})$ is an algebra for the convolution product ;
2. Moreover, if $\mathfrak{A}$ is commutative $\mathrm{Hom}_{k-\mathrm{Alg}}(\mathscr{H}, \mathfrak{A})$ is a group.

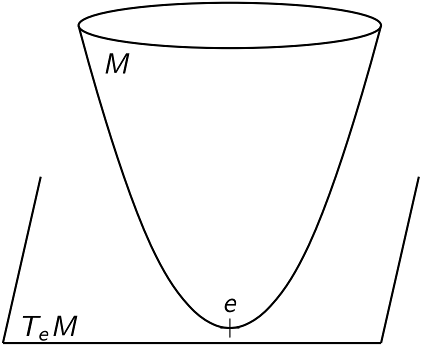
Characters of $k\langle X \rangle$

$(k\langle X\rangle, ш, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon, S)$ is a Hopf algebra.

Property

The set $\chi_k\,(k\langle X\rangle, ш, 1_{X^*})$ is a Lie group whose Lie algebra is obtained with infinitesimal characters.

- **Lie group :** A Lie group G is a differentiable manifold endowed with two operations that are smooth functions on G :
  - $G \times G \to G$ (product)
  - $G \to G$ (inversion)
- **Lie algebra associated to a Lie group :** its vector space is T_eG the tangent space of G at e (unit of the group).



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- **Infinitesimal characters :** $\delta \in \chi_k(k\langle X\rangle, ш, 1_{X^*})$ such that

$$\delta(xy) = \delta(x)\epsilon(y) + \epsilon(x)\delta(y).$$

Lemma and Proof

Lemma *(Minh)* **:**

$$\sum_{w \in X^*} \chi(w) w = \prod_{l \in \mathrm{Lyn}(X)}^{\searrow} \exp\left(\chi(l)\hat{l}\right) = \prod_{l \in \mathrm{Lyn}(X)}^{\searrow} \exp\left(\chi(l) P_l\right).$$

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- $$\langle \chi(S_l) P_l | u \sqcup\!\sqcup v \rangle = \chi(S_l)(\langle P_l | u \rangle \delta_{1,v} + \delta_{1,u} \langle P_l | v \rangle.$$

  $\Rightarrow \chi(S_l) P_l$ is an infinitesimal character.

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  $\Rightarrow \chi(S_l) P_l$ is an infinitesimal character.

- $$\sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w$$
  $$= \prod_{l \in \mathrm{Lyn}(X)} \exp\left(S_l \otimes P_l\right).$$

- Apply $\chi \otimes I$ to the previous equation.

Characters of $k\langle X\rangle$

$(k\langle X\rangle, ш, 1_{X^*}, \Delta_{\mathrm{conc}}, \epsilon, S)$ is a Hopf algebra.

Property

The set $\chi_k(k\langle X\rangle, ш, 1_{X^*})$ is a Lie group whose Lie algebra is obtained with infinitesimal characters.

Question : Does this property hold for larger classes of algebras ?
(Krajewski)

Thank you for your attention!