Toward Uniform Random Generation in 1-safe Petri Nets

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Motivation

- Complexity and scale in software systems are increasing.
- The crucial factor is related to concurrency.
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  → probabilistic framework in a trace monoid
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• Difficulty: "Combinatorial explosion problems"
• Approach: Statistical model checking → probabilistic framework in a trace monoid
• Goal: Random generation for concurrent systems → 1-safe Petri nets
Concurrent models - 1-safe Petri nets

$M_0$

Reachability graph
Concurrent models - 1-safe Petri nets

![Diagram of M1 and Reachability graph]

**M1**

Reachability graph
Concurrent models - 1-safe Petri nets

- Concurrency :
- Casuality :
- Conflit :
Concurrent models - 1-safe Petri nets

- Concurrency: a, c
- Casuality:
- Conflict:
Concurrent models - 1-safe Petri nets

- **Concurrency**: a, c
- **Casuality**: a, b
- **Conflict**:

Reachability graph
Concurrent models - 1-safe Petri nets

- **Concurrency**: a, c
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- **Conflit**: b, c

Reachability graph
Concurrent models - 1-safe Petri nets

From $M_0$, $abacb$ is a valid firing sequence.
Concurrent models - 1-safe Petri nets

- From $M_0$, $abacb$ is a valid firing sequence.
- We lost the feature of concurrency by viewing the firing sequences as the sequential executions.
  ex: $abacb = abcab$
Concurrent models - trace monoids

**Trace monoid** $\mathcal{M}$
- **Alphabet**: $\Sigma = \{a, b, c\}$
- **Independent relation**: $\mathcal{I} = \{(a, c)\}$

**Heap of pieces**
- **Pieces**:
  - $a$
  - $b$
  - $c$
Concurrent models - trace monoids

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Heap of pieces
- Pieces:
  - $a$
  - $b$
  - $c$
- Example of heap:

\[
\begin{array}{c}
  \text{abacb} = \text{abcab}
\end{array}
\]
Concurrent models - trace monoids

**Trace monoid \( \mathcal{M} \)**
- **Alphabet**: \( \Sigma = \{ a, b, c \} \)
- **Independent relation**: \( \mathcal{I} = \{(a, c)\} \)
- **Canonical normal form**: \( abacb = a \cdot b \cdot ac \cdot b \)

**Heap of pieces**
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Heap of pieces
- Pieces:
  \[
  \begin{array}{ccc}
    a & b & c \\
  \end{array}
  \]
- Example of heap:
  \[
  \begin{array}{ccc}
    b \\
    a & c \\
    b \\
    a \\
  \end{array}
  \]

$abacb = abcab$
Framework - random sampling from a Markov chain

- Take account of "concurrency" and "states"

traces in a trace monoid  
words in an automaton
Framework - random sampling from a Markov chain

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Framework- random sampling from a Markov chain

- Take account of "concurrency" and "states"
- The executions of 1-safe Petri nets are understood up to traces.
Uniform measure on trace monoids

Trace monoid: \( \mathcal{M} = \langle a, b, c \mid a \cdot c = c \cdot a \rangle \)

- set of cliques \( C : \varepsilon, a, b, c, ac \)
- Möbius polynomial \( \mu(x) = \sum_{c \in C} (-1)^{|c|} x^{|c|} = 1 - 3x + x^2 \)
- Möbius inversion formula: \( G(x) = \sum_{u \in \mathcal{M}} x^{|u|} = \frac{1}{\mu(x)} \)
Uniform measure on trace monoids

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Theorem (Abbes, Mairesse 2015)

There exists a unique uniform measure \( \nu \) on \( \partial\mathcal{M} \), satisfying:

\[ \forall u \in \mathcal{M}, \quad \nu(\uparrow u) = p_0^{|u|} \]

\( p_0 \): the root of smallest modulus of \( \mu(x) \).
Uniform measure on trace monoids

Theorem (Abbes, Mairesse 2015)

Let $\nu$ be the uniform measure on $\partial \mathcal{M}$. Then the canonical normal decomposition of a trace is a realization of the Markov chain with initial probability measure $h$ which is the Möbius transform of $\nu$. 
Uniform measure on trace monoids

Theorem (Abbes, Mairesse 2015)

Let $\nu$ be the uniform measure on $\partial M$. Then the canonical normal decomposition of a trace is a realization of the Markov chain with initial probability measure $h$ which is the Möbius transform of $\nu$.

- $\#$ paths with length $k$ in the automaton
  $\quad = \#$ traces with height $k$ in a trace monoid
Uniform measure for actions on trace monoids

Theorem (Abbes 2015)

Let $X \times \mathcal{M} \to X$ be an irreducible partial action. Then there exists a uniform Markov measure, satisfying:

$$\forall \alpha \in X \quad \forall x \in \mathcal{M}_\alpha, \quad \nu_\alpha(\uparrow x) = p_0^{|x|} \Gamma(\alpha, \alpha \cdot x).$$

- $G_\alpha(x) = \sum_{u \in \mathcal{M}_\alpha} x^{|u|}$
- $\Gamma(\alpha, \beta) = \lim_{x \to p_0} \frac{G_\beta(x)}{G_\alpha(x)}$
- $\mu_{\alpha,\beta}(x) = \sum_{\gamma \in \mathcal{C}_{\alpha,\beta}} (-1)^{|\gamma|} x^{|\gamma|}$
- Möbius matrix:

$$\mu(x) = (\mu_{\alpha,\beta})(x)$$
Properties of $\Gamma$ function

- Define $\Gamma(\alpha, \beta) = \lim_{x \to p_0} \frac{G_\beta(x)}{G_\alpha(x)}$
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- cocycle relation: $\Gamma(\alpha, \gamma) = \Gamma(\alpha, \beta) \Gamma(\beta, \gamma)$
  $\Gamma(\alpha, \alpha) = 1$
Properties of $\Gamma$ function

- Define $\Gamma(\alpha, \beta) = \lim_{x \to p_0} \frac{G_\beta(x)}{G_\alpha(x)}$
- Cocycle relation: $\Gamma(\alpha, \gamma) = \Gamma(\alpha, \beta) \Gamma(\beta, \gamma)$
- $\Gamma(\alpha, \alpha) = 1$
- Fix a state $\alpha_0$,

  $$(\Gamma(\alpha_0, \beta))_\beta \in \ker \mu(p_0)$$
Calculation of $\Gamma$ function

$\Gamma(M_0, M_0) = \Gamma(M_0, M_1) \in \ker \mu(p_0) - p_0 + (1 - p_0) \cdot \lambda = \lambda = p_0 = 1 - p_0 = \frac{\sqrt{5} - 1}{2}$. 

$M_0 \rightarrow M_0 : \varepsilon, c, \quad M_0 \rightarrow M_1 : a, ac, 
M_1 \rightarrow M_0 : b, \quad M_1 \rightarrow M_1 : \varepsilon, c$. 

\[ \mu(x) = M_0 \rightarrow M_1 \left( 1 - x - x + x^2 - x^3 - x^4 \right) \]
Calculation of $\Gamma$ function

Let $\left(\Gamma\left(\mathcal{M}_0, \mathcal{M}_0\right), \Gamma\left(\mathcal{M}_0, \mathcal{M}_1\right)\right) \in \ker \mu(p_0) - p_0 + (1 - p_0) \cdot \lambda = 0 \Rightarrow \lambda = p_0 = 1 - p_0 = \frac{\sqrt{5} + 1}{2}$.

$\mathcal{M}_0 \to \mathcal{M}_0 : \varepsilon, c, \quad \mathcal{M}_0 \to \mathcal{M}_1 : a, ac,$
$\mathcal{M}_1 \to \mathcal{M}_0 : b, \quad \mathcal{M}_1 \to \mathcal{M}_1 : \varepsilon, c.$

$\mu(x) = \begin{pmatrix} M_0 & \begin{pmatrix} 1 - x & -x + x^2 \\ -x & 1 - x \end{pmatrix} \\ M_1 & \end{pmatrix}.$
Calculation of $\Gamma$ function

Let $(\Gamma(M_0, M_0), \Gamma(M_0, M_1))^T = (1 - x - x + x^2, -x) 
\in \ker\mu(p_0)$.
Calculation of $\Gamma$ function

Let $(\Gamma(M_0, M_0) \quad \Gamma(M_0, M_1))^T = \begin{pmatrix} 1 - x & -x + x^2 \\ -x & 1 - x \end{pmatrix}$. 

$$\mu(x) = M_0 \begin{pmatrix} 1 - x & -x + x^2 \\ -x & 1 - x \end{pmatrix}.$$ 

$$p_0 = \frac{\sqrt{5}+1}{2} \in \ker \mu(p_0)$$

$$-p_0 + (1 - p_0) \cdot \lambda = 0$$
Calculation of $\Gamma$ function

Let \( \left( \Gamma(M_0, M_0) \quad \Gamma(M_0, M_1) \right)^T = \left( 1 \quad \lambda \right)^T \in \ker \mu(p_0) \)

\[-p_0 + (1 - p_0) \cdot \lambda = 0 \implies \lambda = \frac{p_0}{1 - p_0} = \frac{1}{p_0} = \frac{\sqrt{5} - 1}{2}.\]
Uniqueness of uniform measure on trace monoids

New proof from the linear algebra point of view

- Construct the expanded automaton of cliques
  \[ \mathcal{M} = \langle a, b, c \mid a \cdot c = c \cdot a \rangle \]
Uniqueness of uniform measure on trace monoids

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New proof from the linear algebra point of view

- Construct the expanded automaton of cliques
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- Find the Perron eigenvector of the incidence matrix of this automaton
Uniqueness of uniform measure on trace monoids

- Find the Perron eigenvector of the incidence matrix of this automaton

\[
v_{(c,i)} = \frac{1}{p^{i-1}} h(c).
\]

\[
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
\]
Uniqueness of uniform measure on trace monoids

- Find the **Perron eigenvector** of the incidence matrix of this automaton

\[
v_{(c,i)} = \frac{1}{p^{i-1}} h(c).\]

\[
v = \begin{pmatrix}
    h(a) \\
    h(b) \\
    h(c) \\
    h(ac) \\
    \frac{1}{p} h(ac)
\end{pmatrix}
\]
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- Since the incidence matrix is irreducible and aperiodic
- Apply **Perron-Frobenius Theorem**
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\]

- Since the incidence matrix is irreducible and aperiodic
- Apply **Perron-Frobenius** Theorem
- Get the **uniqueness** of the uniform measure
Uniqueness of uniform measure on actions

Example for 1-safe petri net

\[ \nu_{M_0}(C_1 = c) = \nu_{M_0}(\uparrow c) - \nu_{M_0}(\uparrow (ac)) \]
Uniqueness of uniform measure on actions

Example for 1-safe petri net

\[
\nu_{M_0}(C_1 = c) = \nu_{M_0}(\uparrow c) - \nu_{M_0}(\uparrow (ac))
\]
\[
= p_0 \cdot \Gamma(M_0, M_0) - p_0^2 \cdot \Gamma(M_0, M_1)
\]
Uniqueness of uniform measure on actions

Example for 1-safe petri net

\[ \nu_{M_0}(C_1 = c) = \nu_{M_0}(\uparrow c) - \nu_{M_0}(\uparrow (ac)) \]
\[ = p_0 \cdot \Gamma(M_0, M_0) - p_0^2 \cdot \Gamma(M_0, M_1) \]
\[ = p_0 \cdot 1 - p_0^2 \cdot \frac{1}{p_0} = 0 \]
Uniqueness of uniform measure on actions

Example for 1-safe petri net

- The automaton is NOT strongly connected
  → can not apply Perron-Frobenius Theorem
Uniqueness of uniform measure on actions

Example for 1-safe petri net

- The automaton is NOT strongly connected
  \[\rightarrow\text{ can not apply Perron-Frobenius Theorem}\]
- Some state never go through under \(\nu_{M_0}\)
Ongoing directions

- A systematic way to calculate $\Gamma$ function
Ongoing directions

- A systematic way to calculate $\Gamma$ function
- Complete the proof of the uniqueness of uniform measure on actions in a trace monoid
Ongoing directions

- A systematic way to calculate $\Gamma$ function
- Complete the proof of the **uniqueness** of uniform measure on actions in a trace monoid
- **Random generation** for actions on a trace monoid
Ongoing directions

- A systematic way to calculate $\Gamma$ function
- Complete the proof of the uniqueness of uniform measure on actions in a trace monoid
- Random generation for actions on a trace monoid
- For the purpose of model checking (application of this theory)
Thank you!