On universal differential equations

V. Hoang Ngoc Minh
Université Lille, 1 Place Déliot, 59024 Lille, France.

Séminaire Combinatoire, Informatique et Physique
23 Février, 2, 16 & 23 Mars 2021, Villetaneuse
Outline

1. Introduction
   1.1 Picard-Vessiot theory of ordinary differential equation
   1.2 Fuchsian linear differential equations
   1.3 Nonlinear differential equations

2. Dual laws and representative series
   2.1 conc-shuffle and conc-stuffle bialgebras
   2.2 Dualizable laws in conc-shuffle bialgebras
   2.3 Representative series and Sweedler’s dual

3. Continuity over Chen series
   3.1 Continuity, indiscernability and growth condition
   3.2 Iterated integrals and Chen series
   3.3 Chen series and differential equations

4. Noncommutative PV theory and independences via words
   4.1 First step of noncommutative PV theory
   4.2 Independences over differential ring
   4.3 Dom(Li●)
INTRODUCTION
Picard-Vessiot theory of ordinary differential equation

\((k, \partial)\) a commutative differential ring without zero divisors.
\(\text{Const}(k) = \{ c \in k | \partial c = 0 \}\) is supposed to be a field.

\((ODE)\quad (a_n \partial^n + a_{n-1} \partial^{n-1} + \ldots + a_0)y = 0, \quad a_0, \ldots, a_{n-1}, a_n \in k.\)
a\(^{-1}\n is supposed to exist.

**Definition 1**

1. Let \(y_1, \ldots, y_n\) be \(\text{Const}(k)\)-linearly independent solutions of \((ODE)\). Then \(\{y_1, \ldots, y_n\}\) is called a fundamental set of solutions of \((ODE)\) and it generates a \(\text{Const}(k)\)-vector subspace of dimension at most \(n\).

2. If \(M = k\{y_1, \ldots, y_n\}\) and \(\text{Const}(M) = \text{Const}(k)\) then \(M\) is called a Picard-Vessiot extension related to \((ODE)\).

3. Let \(k \subset K_1\) and \(k \subset K_2\) be differential rings. An isomorphism of rings \(\sigma : K_1 \rightarrow K_2\) is a differential \(k\)-isomorphism if
\[\forall a \in K_1, \quad \partial(\sigma(a)) = \sigma(\partial a)\text{ and, if } a \in k, \ \sigma(a) = a.\]
If \(K_1 = K_2 = K\), the differential galois group of \(K\) over \(k\) is by
\[\text{Gal}_k(K) = \{\sigma | \sigma\text{ is a differential }k\text{-automorphism of }K\}.\]

1. Let \(R_1, R_2\) be differential rings s.t. \(R_1 \subset R_2\). Let \(S\) be a subset of \(R_2\). \(R_1\{S\}\) denotes the smallest differential subring of \(R_2\) containing \(R_1\). \(R_1\{S\}\) is the ring (over \(R_1\)) generated by \(S\) and their derivatives of all orders.
Linear differential equations and Dyson series

Let \( a_0, \ldots, a_n \in \mathbb{C}(z) \),

\[
\begin{align*}
  a_n(z) \partial^n y(z) + \ldots + a_1(z) \partial y(z) + a_0(z)y(z) &= 0. \\
  \text{(ED)} \quad \left\{ \begin{array}{l}
    \partial q(z) = A(z) q(z), \quad A(z) \in \mathcal{M}_{n,n}(\mathbb{C}(z)), \\
    q(z_0) = \eta, \quad \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \\
    y(z) = \lambda q(z), \quad \eta \in \mathcal{M}_{n,1}(\mathbb{C}).
  \end{array} \right.
\end{align*}
\]

By successive Picard iterations, with the initial point \( q(z_0) = \eta \), we get

\[
y(z) = \lambda U(z_0; z) \eta,
\]

where \( U(z_0; z) \) is the following functional expansion

\[
U(z_0; z) = \sum_{k \geq 0} \int_{z_0}^{z} A(z_1)dz_1 \int_{z_0}^{z_1} A(z_2)dz_2 \ldots \int_{z_0}^{z_{k-1}} A(z_k)dz_k, \text{(Dyson series)}
\]

and \((z_0, z_1 \ldots, z_k, z)\) is a subdivision of the path of integration \( z_0 \leadsto z \).

In order to find the matrix \( \Omega(z_0; z) \) s.t.

\[
U(z_0; z) = \exp[\Omega(z_0; z)] = \top \exp \int_{z_0}^{z} A(s)ds, \quad \text{(Feynman's notation)}
\]

Magnus computed \( \Omega(z_0; z) \) as limit of the following Lie-integral-functionals

\[
\begin{align*}
  \Omega_1(z_0; z) &= \int_{z_0}^{z} A(z)ds, \\
  \Omega_k(z_0; z) &= \int_{z_0}^{z} [A(z) + [A(z), \Omega_{k-1}(z_0; s)]]/2 \\
  &+ [[A(z), \Omega_{k-1}(z_0; s)], \Omega_{k-1}(z_0; s)]/12 + \ldots)ds.
\end{align*}
\]

2. Subject to convergence.
Fuchsian linear differential equations

Let $\Omega$ be a simply connected domain and $\mathcal{H}(\Omega)$ be the ring of holomorphic functions over $\Omega$ (with $1_{\mathcal{H}(\Omega)}$ as neutral element). Let us consider, here, $
abla = \{s_i\}_{i=0,\ldots,m}$, $m \geq 1$, as set of simple poles of $(ED)$ and $\Omega = \mathbb{C} \setminus \sigma$.

$$A(z) = \sum_{i=0}^{m} M_i u_i(z),$$

where

$$\begin{cases} 
M_i \in \mathcal{M}_{n,n}(\mathbb{C}), \\
u_i(z) = (z - s_i)^{-1} \in \mathbb{C}(z), \\
\partial q(z) = \left(\sum_{i=0}^{m} M_i u_i(z)\right) q(z), \\
q(z_0) = \eta, \\
y(z) = \lambda q(z).
\end{cases}$$

$(ED)$

Let $X^*$ be the set of words over $X = \{x_0, \ldots, x_m\}$ and

$$\alpha^{z}_{z_0} \otimes \mathcal{M} : \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \rightarrow \mathcal{M}_{n,n}(\mathcal{H}(\Omega))$$

$(z_0 \rightsquigarrow z$ is the path of integration previously introduced) s.t.

$$\mathcal{M}(1_{X^*}) = \text{Id}_n \quad \text{and} \quad \mathcal{M}(x_{i_1} \cdots x_{i_k}) = M_{i_1} \cdots M_{i_k},$$

$$\alpha^{z}_{z_0}(1_{X^*}) = 1_{\mathcal{H}(\Omega)} \quad \text{and} \quad \alpha^{z}_{z_0}(x_{i_1} \cdots x_{i_k}) = \int_{z_0}^{z} \frac{dz_1}{z_1 - s_{i_1}} \cdots \int_{z_0}^{z_{k-1}} \frac{dz_k}{z_k - s_{i_k}}.$$

Then

$$\begin{align*}
y(z) &= \lambda U(z_0; z) \eta \\
U(z_0; z) &= \sum_{w \in X^*} \mathcal{M}(w) \alpha^{z}_{z_0}(w) = (\mathcal{M} \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.
\end{align*}$$

3. Subject to convergence.
Example 2 (Hypergeometric equation)

Let \( t_0, t_1, t_2 \) be parameters and
\[
z(1 - z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0.
\]
Let \( q_1(z) = -y(z) \) and \( q_2(z) = (1 - z)\dot{y}(z) \). Hence, one has
\[
y(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}
\]
and
\[
\begin{pmatrix} \dot{q}_1(z) \\ \dot{q}_2(z) \end{pmatrix} = \begin{pmatrix} M_0 \\ M_1 \end{pmatrix} \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}
\]
= \( (u_0(z)M_0 + u_1(z)M_1) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} \),

where \( u_0(z) = z^{-1} \), \( u_1(z) = (1 - z)^{-1} \) and
\[
M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} \quad \text{and} \quad M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}.
\]
Nonlinear differential equations

\[
\begin{align*}
\partial q(z) &= \left( \sum_{i=0}^{m} T_i(q) u_i(z) \right)(q), \\
q(z_0) &= q_0, \\
y(z) &= f(q(z)),
\end{align*}
\]

where

- \( u_i \in (k, \partial) \),
- the state \( q = (q_1, \ldots, q_n) \) belongs the complex analytic manifold \( Q \) of dimension \( n \) and \( q_0 \) is the initial state,
- the observation \( f \in \mathcal{O} \), with \( \mathcal{O} \) the ring of analytic functions over \( Q \),
- for \( i = 0..1 \), \( T_i = (T_i^1(q) \partial/\partial q_1 + \cdots + T_i^m(q) \partial/\partial q_m) \) is an analytic vector field over \( Q \), with \( T_i^j(q) \in \mathcal{O} \), for \( j = 1, \ldots, n \).

With \( X \) and \( \alpha^z_{z_0} \) given as previously, let the morphism \( \tau \) be defined by \( \tau(1_{X^*}) = \text{Id} \) and \( \tau(x_{i_1} \cdots x_{i_k}) = T_{i_1} \cdots T_{i_k} \). Then \( y(z) = T \circ f_{|q_0} \) with

\[
T = \sum_{w \in X^*} \tau(w) \alpha^z_{z_0}(w) = (\tau \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.
\]

4. Subject to convergence.
Examples of nonlinear dynamical systems (1/2)

Example 3 (Harmonic oscillator)

Let $k_1, k_2$ be parameters and $\partial^2 y(z) + k_1 y(z) + k_2 y^2(z) = u_1(z)$ which can be represented by the following state equations (with $n = 1$)

\[
\begin{align*}
y(z) &= q(z), \\
\partial q(z) &= A_0(q)u_0(z) + A_1(q)u_1(z),
\end{align*}
\]

where $A_0 = -(k_1 q + k_2 q^2) \frac{\partial}{\partial q}$ and $A_1 = \frac{\partial}{\partial q}$.

Example 4 (Duffing equation)

Let $a, b, c$ be parameters and $\partial^2 y(z) + a \partial y(z) + by(z) + cy^3(z) = u_1(z)$ which can be represented by the following state equations (with $n = 2$)

\[
\begin{align*}
y(z) &= q_1(z), \\
\begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} &= \begin{pmatrix} q_2 \\ -(aq_2 + b^2 q_1 + cq_1^3) \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z) \\
&= A_0(q)u_0(z) + A_1(q)u_1(z),
\end{align*}
\]

where $A_0 = -(aq_2 + b^2 q_1 + cq_1^3) \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_1}$ and $A_1 = \frac{\partial}{\partial q_2}$. 
Example 5 (Van der Pol oscillator)

Let $\gamma, g$ be parameters and

$$\partial^2 x(z) - \gamma [1 + x(z)^2] \partial x(z) + x(z) = g \cos(\omega z)$$

which can be transformed into (with $C$ is some constant of integration)

$$\partial x(z) = \gamma [1 + x(z)^2 / 3] x(z) - \int_{z_0}^{z} x(s) ds + \frac{g}{\omega} \sin(\omega z) + C.$$

Supposing $x = \partial y$ and $u_1(z) = g \sin(\omega z)/\omega + C$, it leads then to

$$\partial^2 y(z) = \gamma [\partial y(z) + (\partial y(z))^3 / 3] + y(z) + u_1(z)$$

which can be represented by the following state equations (with $n = 2$)

$$y(z) = q_1(z),$$

$$\begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} = \begin{pmatrix} q_2 \\ \gamma (q_2 + q_2^3 / 3) + q_1 \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z)$$

$$= A_0(q) u_0(z) + A_1(q) u_1(z),$$

where $A_0 = [\gamma (q_2 + q_2^3 / 3) + q_1] \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_1}$ and $A_1 = \frac{\partial}{\partial q_2}$. 


DUAL LAWS AND REPRESENTATIVE SERIES
Dual law in bialgebra

Startting with a $k$-bialgebra $(A, \mu)$. Dualizing $\mu : A \otimes_k A \to A$, we get the transpose $t \mu : A^\vee \to (A \otimes_k A)^\vee$ so that we do not get a co-multiplication in general.

- Remark that when $k$ is a field, the following arrow is into (due to the fact that $A^\vee \otimes_k A^\vee$ is torsionfree)

$$\Phi : A^\vee \otimes_k A^\vee \to (A \otimes_k A)^\vee.$$  

- One restricts the codomain of $t \mu$ to $A^\vee \otimes_k A^\vee$ and then the domain to $(t \mu)^{-1}\Phi(A^\vee \otimes_k A^\vee) =: A^\circ$.

\[\begin{align*}
A^\vee & \xrightarrow{t \mu} (A \otimes_k A)^\vee \\
A^\circ & \xrightarrow{\Delta \mu} A^\vee \otimes_k A^\vee \\
A^{\circ \circ} & \xrightarrow{\Delta \mu} A^\circ \otimes_k A^\circ \\
\end{align*}\]

The descent can stop at first step for a field $k$ and then $A^{\circ \circ} = A^\circ$. The coalgebra $(A^\circ, \Delta \mu)$ is called the Sweedler’s dual of $(A, \mu)$. 
Case of algebras noncommutative series

\( \mathcal{X} \) denotes the ordered alphabets \( Y := \{y_k\}_{k \geq 1} \) or \( X := \{x_0, x_1\} \).

On the free monoid \((\mathcal{X}^*, \text{conc}, 1_{\mathcal{X}^*})\), we use the correspondences

\[
x_0^{s_1-1} x_1 \ldots x_0^{s_r-1} x_1 \in X^* x_1 \quad \pi^Y y_{s_1} \ldots y_{s_r} \in Y^* \leftrightarrow (s_1, \ldots, s_r) \in \mathbb{N}_+^r.
\]

Let \( \text{Lyn} \mathcal{X} \) denote the set of Lyndon words generated by \( \mathcal{X} \).

Let \((\text{Lie}_A \langle \langle \mathcal{X} \rangle \rangle, [\cdot])\) and \((A \langle \langle \mathcal{X} \rangle \rangle, \text{conc})\) (resp. \((\text{Lie}_A \langle \mathcal{X} \rangle, [\cdot])\) and \((A \langle \mathcal{X} \rangle, \text{conc})\)) denote the algebras of (Lie) series (resp. polynomials) with coefficients in the ring \(A\), over \(\mathcal{X}\).

\{\(P_l\)\}_{l \in \text{Lyn} \mathcal{X}}\) (resp. \{\(\Pi_l\)\}_{l \in \text{Lyn} Y}\) is a basis of Lie algebra of primitive elements and \{\(S_l\)\}_{l \in \text{Lyn} \mathcal{X}}\) (resp. \{\(\Sigma_l\)\}_{l \in \text{Lyn} Y}\) is a transcendence basis of \((A \langle \mathcal{X} \rangle, \uplus, 1_{\mathcal{X}^*})\) (resp. \((A \langle Y \rangle, \uplus, 1_{Y^*})\)).

\(\mathcal{H}_{\uplus} (\mathcal{X}) := (A \langle \mathcal{X} \rangle, \text{conc}, 1_{\mathcal{X}^*}, \Delta_{\uplus}, e)\) and

\(\mathcal{H}_{\uplus} (Y) := (A \langle Y \rangle, \text{conc}, 1_{Y^*}, \Delta_{\uplus}, e)\) with\(^5\) (for \(x \in \mathcal{X}, y_i \in Y\))

\[
\Delta_{\uplus} x = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x,
\]

\[
\Delta_{\uplus} y_i = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_l.
\]

The dual law associated to \(\text{conc}\) is defined, for \(w \in \mathcal{X}^*\), by

\[
\Delta_{\text{conc}}(w) = \sum_{u,v \in \mathcal{X}^*, u v = w} u \otimes v.
\]

5. Or equivalently, for \(x, y \in \mathcal{X}, y_i, y_j \in Y\) and \(u, v \in \mathcal{X}^*\) (resp. \(Y^*\)),

\[
u \uplus 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \uplus u = u \quad \text{and} \quad xu \uplus yv = x(u \uplus yv) + y(xu \uplus v),
\]

\[
u u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u \quad \text{and} \quad x_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v).\]
Dualizable laws in conc-shuffle bialgebras (1/2)

We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) \( \mu : A\langle X \rangle \otimes A\langle X \rangle \rightarrow A\langle X \rangle \) can be described through its structure constants wrt to the basis of words, i.e. for \( u, v, w \in X^* \), \( \Gamma^w_{u,v} := \langle \mu(u \otimes v) | w \rangle \) so that
   \[
   \mu(u \otimes v) = \sum_{w \in X^*} \Gamma^w_{u,v} \cdot w.
   \]

2. In the case when \( \Gamma^w_{u,v} \) is locally finite in \( w \), we say that the given law is dualizable, the arrow \( ^t \mu \) restricts nicely to \( A\langle X \rangle \hookrightarrow A\langle \langle X \rangle \rangle \) and one can define on the polynomials a comultiplication by
   \[
   \Delta_\mu(w) := \sum_{u,v \in X^*} \Gamma^w_{u,v} \cdot u \otimes v.
   \]

3. When the law \( \mu \) is dualizable, we have

\[
\begin{array}{ccc}
A\langle \langle X \rangle \rangle & \xrightarrow{^t \mu} & A\langle X^* \otimes X^* \rangle \\
\uparrow \text{can} & & \downarrow \Phi|_{A\langle X \rangle \otimes_A A\langle X \rangle} \\
A\langle X \rangle & \xrightarrow{\Delta_\mu} & A\langle X \rangle \otimes_A A\langle X \rangle
\end{array}
\]

The arrow \( \Delta_\mu \) is unique to be able to close the rectangle and \( \Delta_\mu(P) \) is defined as above.
Dualizable laws in conc-shuffle bialgebras (2/2)

4. Proof that the arrow $A\langle X\rangle \otimes_A A\langle X\rangle \rightarrow A\langle\langle X^* \otimes X^*\rangle\rangle$ is into:

Let $T = \sum_{i=1}^{n} P_i \otimes_A Q_i$ such that $\Phi(T) = 0$. Rewriting $T$ as a finitely supported sum $T = \sum_{u,v \in X^*} c_{u,v} u \otimes v$ (this is indeed the iso between $A\langle X\rangle \otimes_A A\langle X\rangle$ and $A[X^* \times X^*]$), $\Phi(T)$ is by definition of $\Phi$ the double series (here a polynomial) s.t. $\langle\Phi(T)|u \otimes v\rangle = c_{u,v}$. If $\Phi(T) = 0$, then for all $(u, v) \in X^* \times X^*$, $c_{u,v} = 0$ entailing $T = 0$.

We extend by linearity and infinite sums, for $S \in A\langle Y\rangle$ (resp. $A\langle X\rangle$), by

$$\Delta_{\uplus\uplus} S = \sum_{w \in Y^*} \langle S|w\rangle \Delta_{\uplus\uplus} w \in A\langle Y^* \otimes Y^*\rangle,$$

$$\Delta_{\text{conc}} S = \sum_{w \in X^*} \langle S|w\rangle \Delta_{\text{conc}} w \in A\langle X^* \otimes X^*\rangle,$$

$$\Delta_{\uplus\uplus} S = \sum_{w \in X^*} \langle S|w\rangle \Delta_{\uplus\uplus} w \in A\langle X^* \otimes X^*\rangle.$$

$A\langle X\rangle \otimes A\langle X\rangle$ embeds injectively in $A\langle\langle X^* \otimes X^*\rangle\rangle \cong [A\langle X\rangle]\langle X\rangle$.

6. $A\langle X\rangle \otimes A\langle X\rangle$ contains the elements of the form $\sum_{i \in I} \text{finite } G_i \otimes D_i$, for $(G_i, D_i) \in A\langle X\rangle \times A\langle X\rangle$. But since elements of $M \otimes N$ are finite combination of $m_i \otimes n_i$, $m_i \in M$, $n_i \in N$ then $\sum_{i \geq 0} u^i \otimes v^i$ belongs to $A\langle\langle X^* \otimes X^*\rangle\rangle$ and does not belong to $A\langle X\rangle \otimes A\langle X\rangle$, for $u, v \in X^\geq 1$. 
Extended Ree’s theorem

Let $S \in A\langle\langle Y\rangle\rangle$ (resp. $A\langle\langle X\rangle\rangle$), $A$ is a commutative ring containing $\mathbb{Q}$. The series $S$ is said to be

1. a $\sqcup\sqcup$ (resp. conc, $\sqcup$)-character iff, for any $w, v \in Y^*$ (resp. $X^*$),
   $$\langle S|w\rangle\langle S|v\rangle = \langle S|w \sqcup v\rangle \text{ (resp. } \langle S|wv\rangle, \langle S|w \sqcup v\rangle) \text{ and } \langle S|1\rangle = 1.$$

2. an infinitesimal $\sqcup\sqcup$ (resp. conc, $\sqcup$)-character iff, for any $w, v \in Y^*$ (resp. $X^*$),
   $$\langle S|w\rangle\langle S|v\rangle = \langle S|w\rangle\langle v|1_{Y^*}\rangle + \langle w|1_{Y^*}\rangle\langle S|v\rangle \text{ (resp. } \langle S|wv\rangle = \langle S|w\rangle\langle v|1_{X^*}\rangle + \langle w|1_{X^*}\rangle\langle S|v\rangle, \langle S|w \sqcup v\rangle = \langle S|w\rangle\langle v|1_{X^*}\rangle + \langle w|1_{X^*}\rangle\langle S|v\rangle).$$

3. a group-like series iff $\langle S|1_{X^*}\rangle = 1$ and $\Delta_{\sqcup\sqcup} S = \Phi(S \otimes S)$ (resp. $\Delta_{\text{conc}} S = \Phi(S \otimes S), \Delta_{\sqcup} S = \Phi(S \otimes S)$).

4. a primitive series iff $\Delta_{\sqcup\sqcup} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\text{conc}} S = 1_{X^*} \otimes S + S \otimes 1_{X^*}, \Delta_{\sqcup} S = 1_{X^*} \otimes S + S \otimes 1_{X^*}$).

Then the following assertions are equivalent

1. $S$ is a $\sqcup\sqcup$ (resp. conc and $\sqcup$)-character.
2. log $S$ an infinitesimal $\sqcup\sqcup$ (resp. conc and $\sqcup$)-character.
3. $S$ is group-like, for $\Delta_{\sqcup\sqcup}$ (resp. $\Delta_{\text{conc}}$ and $\Delta_{\sqcup}$).
4. log $S$ is primitive, for $\Delta_{\sqcup\sqcup}$ (resp. $\Delta_{\text{conc}}$ and $\Delta_{\sqcup}$).
Extension by continuity (infinite sums)

Now, suppose that the ring $A$ (containing $\mathbb{Q}$) is a field $k$. Then

$$\Delta : k\langle X \rangle \to k\langle X \rangle \otimes k\langle X \rangle$$

and $\Delta_+ : k\langle Y \rangle \to k\langle Y \rangle \otimes k\langle Y \rangle$ are graded for the multidegree. Then $\Delta_+$ is graded for the length. Their extension to the completions (i.e. $k\langle\langle X \rangle\rangle$ and $k\langle\langle X^* \otimes X^* \rangle\rangle$) are continuous and then, when exist, commute with infinite sums. Hence $7, 8$,

$$\forall c \in k, \quad \Delta_+ (cx)^* = \sum_{n \geq 0} c^n \Delta_+ x^n = \sum_{n \geq 0} c^n \sum_{j=0}^n \binom{n}{j} x^j \otimes x^{n-j}.$$  

For $c \in \mathbb{N}_{\geq 2}$ which is neither a field nor a ring (containing $\mathbb{Q}$), we also get

$$(cx)^* = (c - 1)^{-1} \sum_{a, b \in \mathbb{N}_{\geq 1}, a+b=c} (ax)^* \otimes (bx)^* \in \mathbb{N}_{\geq 2}\langle\langle X \rangle\rangle,$$

$$\Delta_+ (cx)^* \neq (c - 1)^{-1} \sum_{a, b \in \mathbb{N}_{\geq 1}, a+b=c} (ax)^* \otimes (bx)^* \in \mathbb{Q}\langle\langle X \rangle\rangle \otimes \mathbb{Q}\langle\langle X \rangle\rangle,$$

because

$$\langle \text{LHS}|x \otimes 1_{X^*} \rangle = c \quad \text{and} \quad \langle \text{RHS}|x \otimes 1_{X^*} \rangle = (c - 1)^{-1} \sum_{a=1}^{c-1} a = \frac{c}{2}.$$  

For $c \in \mathbb{Z}$ (or even $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), the such decomposition is not finite.

---

7. For $S \in A\langle\langle X \rangle\rangle$ s.t. $\langle S|1_{X^*} \rangle = 0$, $S^* = \sum_{n \geq 0} S^n$ is called Kleene star of $S$.

8. $\Delta_+ x^n = (\Delta_+ x)^n = (1_{X^*} \otimes x + x \otimes 1_{X^*})^n = \sum_{j=0}^n \binom{n}{j} x^j \otimes x^{n-j}.$
Case of rational series and of $\Delta_{\text{conc}}$

$A^{\text{rat}} \langle \mathcal{X} \rangle$ denotes the algebraic closure by $^9 \{\text{conc}, +, *\}$ of $\widehat{A.\mathcal{X}}$ in $A\langle \mathcal{X} \rangle$.

\[
\begin{array}{ccc}
A\langle \mathcal{X} \rangle & \xrightarrow{t_{\text{conc}}} & A\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle \\
\text{can} & \uparrow & \\
A^{\text{rat}} \langle \mathcal{X} \rangle & \longrightarrow & A^{\text{rat}} \langle \mathcal{X} \rangle \otimes_A A^{\text{rat}} \langle \mathcal{X} \rangle
\end{array}
\]

The dashed arrow may not exist in general, but for any $R \in A^{\text{rat}} \langle \mathcal{X} \rangle$ admitting $(\lambda, \mu, \eta)$ as linear representation of dimension $n$, we can get

$\quad t_{\text{conc}}(R) = \Phi(\sum_{i=1}^n G_i \otimes D_i)$.

Indeed, since $\langle R|xy \rangle = \lambda \mu(xy) \eta = \lambda \mu(x) \mu(y) \eta \ (x, y \in \mathcal{X})$ then, letting $e_i$ is the vector such that $^t e_i = (0 \ldots 0 1 0 \ldots 0)$, one has

$\langle R|xy \rangle = \sum_{i=1}^n \lambda \mu(x) e_i^t e_i \mu(y) \eta = \sum_{i=1}^n \langle G_i|x \rangle \langle D_i|y \rangle = \sum_{i=1}^n \langle G_i \otimes D_i|x \otimes y \rangle$.

$G_i$ (resp. $D_i$) admits then $(\lambda, \mu, e_i)$ (resp. $(^t e_i, \mu, \eta)$) as linear representation.

If $A = k$ being a field then, due to the injectivity of $\Phi$, all expressions of the type $\sum_{i=1}^n G_i \otimes D_i$, of course, coincide. Hence, the dashed arrow (a restriction of $\Delta_{\text{conc}}$) in the above diagram is well-defined.

9. $A^{\text{rat}} \langle \mathcal{X} \rangle$ is closed under $\sqcup$. $A^{\text{rat}} \langle \mathcal{Y} \rangle$ is also closed under $\sqcup$. 
Representative series and Sweedler’s dual

Theorem 6 (representative series)

Let $S \in A\llangle \mathcal{X} \rrangle$. The following assertions are equivalent

1. The series $S$ belongs to $A^{\text{rat}}\llangle \mathcal{X} \rrangle$.

2. There exists a linear representation $(\nu, \mu, \eta)$, of rank $n$, for $S$ with
   $\nu \in M_{1,n}(A)$, $\eta \in M_{n,1}(A)$ and a morphism of monoids
   $\mu : \mathcal{X}^* \rightarrow M_{n,n}(A)$ s.t., for any $w \in \mathcal{X}^*$, $\langle S|w \rangle = \nu \mu(w) \eta$.

3. The shifts$^{10}$ $\{S \triangleleft w\}_w \in \mathcal{X}^*$ (resp. $\{w \triangleright S\}_w \in \mathcal{X}^*$) lie within a finitely
   generated shift-invariant $A$-module.

Moreover, if $A$ is a field $k$, the previous assertions are equivalent to

4. There exist $(G_i, D_i)_{i \in F_{\text{finite}}}$ s.t. $\Delta_{\text{conc}}(S) = \sum_{i \in F_{\text{finite}}} G_i \otimes D_i$.

Hence, $H_{\llangle \mathcal{X} \rrangle}^\circ(\llangle \mathcal{X} \rrangle) = (k^{\text{rat}}\llangle \mathcal{X} \rrangle, \sqcup, 1_{\mathcal{X}^*}, \Delta_{\text{conc}}, e)$ and

$H_{\llangle \mathcal{Y} \rrangle}^\circ(\llangle \mathcal{Y} \rrangle) = (k^{\text{rat}}\llangle \mathcal{Y} \rrangle, \sqcup, 1_{\mathcal{Y}^*}, \Delta_{\text{conc}}, e)$.

Now, let $A_{\text{exc}}\llangle \mathcal{X} \rrangle$ (resp. $A_{\text{exc}}^{\text{rat}}\llangle \mathcal{X} \rrangle$) be the set of exchangeable$^{11}$ series
(resp. series admitting a linear representation with commuting matrices).

10. The left (resp. right) shift of $S$ by $P$ is $P \triangleright S$ (resp. $S \triangleleft P$) defined by, for
    $w \in \mathcal{X}^*$, $\langle P \triangleright S|w \rangle = \langle S|wP \rangle$ (resp. $\langle S \triangleleft P|w \rangle = \langle S|Pw \rangle$).

11. i.e. if $S \in A_{\text{exc}}\llangle \mathcal{X} \rrangle$ then $(\forall u, v \in \mathcal{X}^*)(\forall x \in \mathcal{X}) (|u|_x = |v|_x) \Rightarrow \langle S|u \rangle = \langle S|v \rangle$.
Kleene stars of the plane and conc-characters

For any $S \in A\langle X \rangle$, let $\nabla S$ denotes $S - 1X^*$. 

Theorem 7 (rational exchangeable series)

1. $A_{\text{exc}}\langle\langle X \rangle\rangle \subset A_{\text{rat}}\langle\langle X \rangle\rangle \cap A_{\text{exc}}\langle\langle X \rangle\rangle$. If $A$ is a field then the equality holds and $A_{\text{exc}}\langle\langle X \rangle\rangle = A_{\text{rat}}\langle\langle X_0 \rangle\rangle \uplus A_{\text{rat}}\langle\langle X_1 \rangle\rangle$ and, for the algebra of series over subalphabets $A_{\text{fin}}\langle\langle Y \rangle\rangle := \bigcup_{F \subset \text{finite}} A_{\text{rat}}\langle\langle F \rangle\rangle$, we get $A_{\text{exc}}\langle\langle Y \rangle\rangle \cap A_{\text{fin}}\langle\langle Y \rangle\rangle = \bigcup_{k \geq 0} A_{\text{rat}}\langle\langle y_1 \rangle\rangle \uplus \ldots \uplus A_{\text{rat}}\langle\langle y_k \rangle\rangle \subset A_{\text{exc}}\langle\langle Y \rangle\rangle$.

2. $\forall x \in X, A_{\text{rat}}\langle\langle x \rangle\rangle = \{P(1-xQ)^{-1}\}_{P,Q \in A[x]}$. If $k$ is an algebraically closed field then $k_{\text{rat}}\langle\langle x \rangle\rangle = \text{span}_k \{(ax)^* \uplus k\langle x \rangle | a \in K\}$.

3. If $A$ is a $\mathbb{Q}$-algebra without zero divisors, $\{x^*\}_{x \in X}$ (resp. $\{y^*\}_{y \in Y}$) are conc-character and algebraically independent over $(A\langle X \rangle, \uplus)$ (resp. $(A\langle Y \rangle, \uplus)$) within $(A_{\text{rat}}\langle\langle X \rangle\rangle, \uplus)$ (resp. $(A_{\text{rat}}\langle\langle Y \rangle\rangle, \uplus)$).

4. Let $S \in A\langle\langle X \rangle\rangle$. If $A = k$, a field, then t.f.a.e.
   
   a) $S$ is groupe-like, for $\Delta_{\text{conc}}$.
   
   b) There exists $M := \sum_{x \in X} c_x x \in \widehat{k.X}$ s.t. $S = M^*$.
   
   c) There exists $M := \sum_{x \in X} c_x x \in \widehat{k.X}$ s.t. $\nabla S = MS = SM$.

12. The following identity lives in $A_{\text{exc}}\langle\langle Y \rangle\rangle$ but not in $A_{\text{exc}}\langle\langle Y \rangle\rangle \cap A_{\text{fin}}\langle\langle Y \rangle\rangle$,

   
   $$(y_1 + \ldots)^* = \lim_{k \to +\infty} (y_1 + \ldots + y_k)^* = \lim_{k \to +\infty} y_1^* \uplus \ldots \uplus y_k^* = \uplus_{k \geq 1} y_k^*.$$
Triangular sub bialgebras of \((A^{\text{rat}} \langle \langle X \rangle \rangle, \sqcup \sqcup, 1_{X^*}, \Delta_{\text{conc}}, e)\)

Let \((\nu, \mu, \eta)\) be a linear representation of \(R \in A^{\text{rat}} \langle \langle X \rangle \rangle\) and \(L\) be the Lie algebra generated by \(\{\mu(x)\}_{x \in X}\).

Let \(M(x) := \mu(x)x\), for \(x \in X\). Then \(R = \nu M(X^*)\eta\). If \(\{\mu(x)\}_{x \in X}\) are triangular then let \(D(X)\) (resp. \(N(X)\)) be the diagonal (resp. nilpotent) letter matrix s.t. \(M(X) = D(X) + N(X)\) then
\[M(X^*) = ((D(X^*)T(X))^* D(X^*)).\]
Moreover, if \(X = \{x_0, x_1\}\) then
\[M(X^*) = (M(x_1^*)M(x_0))^* M(x_1^*) = (M(x_0^*)M(x_1))^* M(x_0^*).\]

If \(A\) is an algebraically closed field, the modules generated by the following families are closed by \(\text{conc}, \sqcup \sqcup\) and coproducts:
\[
\begin{align*}
(F_0) & \quad E_1x_1 \ldots E_jx_1E_{j+1}, & \text{where} & \quad E_k \in A^{\text{rat}} \langle \langle x_0 \rangle \rangle, \\
(F_1) & \quad E_1x_0 \ldots E_jx_0E_{j+1}, & \text{where} & \quad E_k \in A^{\text{rat}} \langle \langle x_1 \rangle \rangle, \\
(F_2) & \quad E_1x_i \ldots E_jx_iE_{j+1}, & \text{where} & \quad E_k \in A^{\text{rat}}_{\text{exc}} \langle \langle X \rangle \rangle, x_{i_k} \in X.
\end{align*}
\]

It follows then that

1. \(R\) is a linear combination of expressions in the form \((F_0)\) (resp. \((F_1)\)) iff \(M(x_1^*)M(x_0)\) (resp. \(M(x_0^*)M(x_1)\)) is nilpotent,

2. \(R\) is a linear combination of expressions in the form \((F_2)\) iff \(L\) is solvable. Thus, if \(R \in A^{\text{rat}}_{\text{exc}} \langle \langle X \rangle \rangle \sqcup A\langle X \rangle\) then \(L\) is nilpotent.
CONTINUITY OVER CHEN SERIES
Continuity, indiscernability and growth condition

For \( i = 0, 2 \), let \((k_i, \|\cdot\|_i)\) be a semi-normed space and \( g_i \in \mathbb{Z} \).

**Definition 8**

1. Let \( \mathcal{C}l \) be a class of \( k_1 \langle \langle X \rangle \rangle \). Let \( S \in k_2 \langle \langle X \rangle \rangle \) and it is said to be
   a) **continuous** over \( \mathcal{C}l \) if, for \( \Phi \in \mathcal{C}l \), the following sum is convergent
      \[
      \sum_{w \in X^*} \|\langle S|w\rangle\|_2 \|\langle\Phi|w\rangle\|_1.
      \]
      We will denote \( \langle S\|\Phi\rangle \) the sum \( \sum_{w \in X^*} \langle S|w\rangle \langle\Phi|w\rangle \) and \( k_2 \langle \langle X \rangle \rangle \text{cont} \) the set of continuous power series over \( \mathcal{C}l \).
   b) **indiscernable** over \( \mathcal{C}l \) iff, for any \( \Phi \in \mathcal{C}l \), \( \langle S\|\Phi\rangle = 0 \).

2. Let \( \chi_1 \) and \( \chi_2 \) be real positive functions over \( X^* \). Let \( S \in k_1 \langle \langle X \rangle \rangle \).
   a) \( S \) satisfies the \( \chi_1 \)-growth condition of order \( g_1 \) if it satisfies
      \[
      \exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in X^\geq n, \quad \|\langle S|w\rangle\|_1 \leq K \chi_1(w) \, \|w\|^{g_1}.
      \]
      We denote by \( k_1^{(\chi_1, g_1)} \langle \langle X \rangle \rangle \) the set of formal power series in \( k_1 \langle \langle X \rangle \rangle \) satisfying the \( \chi_1 \)-growth condition of order \( g_1 \).
   b) If \( S \) is continuous over \( k_2^{(\chi_2, g_2)} \langle \langle X \rangle \rangle \) then it will be said to be \( (\chi_2, g_2) \)-continuous. The set of formal power series which are \( (\chi_2, g_2) \)-continuous is denoted by \( k_2^{(\chi_2, g_2)} \langle \langle X \rangle \rangle \text{cont} \).
Convergence condition

Proposition 1
Let \( \chi_1 \) and \( \chi_2 \) be real positive functions over \( \mathcal{X}^* \).
Let \( g_1 \) and \( g_2 \in \mathbb{Z} \) such that \( g_1 + g_2 \leq 0 \).

1. Let \( k_1^{(\chi_1, g_1)} \langle \langle \mathcal{X} \rangle \rangle \) and let \( P \in k_1 \langle \mathcal{X} \rangle \).
The right residual of \( S \) by \( P \) belongs to \( k_1^{(\chi_1, g_1)} \langle \langle \mathcal{X} \rangle \rangle \).

2. Let \( R \in k_2^{(\chi_2, g_2)} \langle \langle \mathcal{X} \rangle \rangle \) and let \( Q \in k_2 \langle \mathcal{X} \rangle \).
The concatenation \( QR \) belongs to \( k_2^{(\chi_2, g_2)} \langle \langle \mathcal{X} \rangle \rangle \).

3. \( \chi_1, \chi_2 \) are morphisms over \( \mathcal{X}^* \) satisfying \( \sum_{x \in \mathcal{X}} \chi_1(x)\chi_2(x) < 1 \).
If \( F_1 \in k_1^{(\chi_1, g_1)} \langle \langle \mathcal{X} \rangle \rangle \) (resp. \( F_2 \in k_2^{(\chi_2, g_2)} \langle \langle \mathcal{X} \rangle \rangle \)) then \( F_1 \) (resp. \( F_2 \)) is continuous over \( k_2^{(\chi_2, g_2)} \langle \langle \mathcal{X} \rangle \rangle \) (resp. \( k_1^{(\chi_1, g_1)} \langle \langle \mathcal{X} \rangle \rangle \)).

Proposition 2
Let \( \mathcal{C}l \subset k_1 \langle \langle \mathcal{X} \rangle \rangle \) be a monoid containing \( \{ e^{tx} \}_{t\in k_1} \). Let \( S \in k_2 \langle \langle \mathcal{X} \rangle \rangle^{cont} \).

1. If \( S \) is indiscernable over \( \mathcal{C}l \) then for any \( x \in \mathcal{X} \), \( x \triangleleft S \) and \( S \triangleright x \) belong to \( k_2 \langle \langle \mathcal{X} \rangle \rangle^{cont} \) and they are indiscernable over \( \mathcal{C}l \).

2. \( S \) is indiscernable over \( \mathcal{C}l \) iff \( S = 0 \).
Iterated integrals over $\omega_i(z) = u_{x_i}(z)dz$ and along $z_0 \rightsquigarrow z$

Let $\Omega$ be a simply connected domain admitting $\mathbf{1}_{\mathcal{H}(\Omega)}$ as neutral element. Let $\mathcal{A} := \mathcal{H}(\Omega)$ and let $\mathcal{C}_0$ be a differential subring of $\mathcal{A}$ ($\partial(\mathcal{C}_0) \subset \mathcal{C}_0$) which is an integral domain containing $\mathbb{C}$.

$\mathbb{C}\{(g_i)_{i \in I}\}$ denotes the differential subalgebra of $\mathcal{A}$ generated by $(g_i)_{i \in I}$, i.e. the $\mathbb{C}$-algebra generated by $g_i$'s and their derivatives

$\{u_x\}_{x \in \mathcal{X}} :$ elements in $\mathcal{C}_0 \cap \mathcal{A}^{-1}$ in correspondence with $\{\theta_x\}_{x \in \mathcal{X}}$ ($\theta_x = u_x^{-1} \partial$).

The iterated integral associated to $x_{i_1} \ldots x_{i_k} \in \mathcal{X}^*$, over the differential forms $\omega_i(z) = u_{x_i}(z)dz$, and along a path $z_0 \rightsquigarrow z$ on $\Omega$, is defined by

$$
\alpha_{z_0}^{z}(1_{\mathcal{X}^*}) = 1_{\Omega}, \\
\alpha_{z_0}^{z}(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^{z} \omega_{i_1}(z_1) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k). \\
\partial\alpha_{z_0}^{z}(x_{i_1} \ldots x_{i_k}) = u_{x_{i_1}}(z) \int_{z_0}^{z} \omega_{i_2}(z_2) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_k).
$$

$$
\text{span}_{\mathbb{C}}\{\partial^l \alpha_{z_0}^{z}(w)\}_{w \in \mathcal{X}^*, l \geq 0} \subset \text{span}_{\mathbb{C}}\{(u_x)_{x \in \mathcal{X}}\}\{\alpha_{z_0}^{z}(w)\}_{w \in \mathcal{X}^*} \\
\subset \text{span}_{\mathbb{C}}\{(u_{x}^{\pm 1})_{x \in \mathcal{X}}\}\{\alpha_{z_0}^{z}(w)\}_{w \in \mathcal{X}^*} \\
\cong \mathbb{C}\{(u_{x}^{\pm 1})_{x \in \mathcal{X}}\} \otimes_{\mathbb{C}} \text{span}_{\mathbb{C}}\{\alpha_{z_0}^{z}(w)\}_{w \in \mathcal{X}^*}.
$$
Let $\mathcal{C} = \mathbb{C}\{(u_x^{\pm 1})_{x \in \mathcal{X}}\}$. One has $\theta_x \in \mathcal{C}[\partial]$, for $x \in \mathcal{X}$, and
\[ \forall x, y \in \mathcal{X}, \forall w \in \mathcal{X}^*, \theta_x \alpha^z_{z_0}(yw) = u_x^{-1}(z)u_y(z)\alpha^z_{z_0}(w). \]
Now, let $\Theta$ be the morphism $\mathbb{C}\langle \mathcal{X} \rangle \longrightarrow \mathcal{C}[\partial]$ defined as follows
\[ \Theta(w) = \begin{cases} 
\text{Id} & \text{if } w = 1_{\mathcal{X}^*}, \\
\Theta(u)\theta_x & \text{if } w = ux \in \mathcal{X}^* \mathcal{X}.
\end{cases} \]
One has, for any $w \in \mathcal{X}^*$,
1. $\Theta(\tilde{w})\alpha^z_{z_0}(w) = 1_{\Omega}$, and then $\partial(\Theta(\tilde{w})\alpha^z_{z_0}(w)) = 0$.
2. $L_w \alpha^z_{z_0}(\tilde{w}) = 0$, where $L_w := \partial \Theta(w) \in \mathcal{C}[\partial]$.

**Proposition 3**

If $\{\alpha^z_{z_0}(w)\}_{w \in \mathcal{X}^*}$ is $\mathcal{C}$-linearly independent then
1. $\mathcal{C}\{\{\alpha^z_{z_0}(w)\}_{w \in \mathcal{X}^*}\}$ forms the universal $\mathcal{C}$-module of solutions of all differential equations $L_y = 0$,
2. $\mathcal{C}\{\alpha^z_{z_0}(w)\}_{w \in \mathcal{X}^*}$ forms the universal Picard-Vessiot extension related to all differential equations $L_y = 0$,

where $L$'s are linear differential operators belonging to $\mathcal{C}[\partial]$. 
Sections of $\{\theta_x\}_{x \in \mathcal{X}}$

For any $x_i \in \mathcal{X}$, let us consider a section of $\theta_{x_i} : \theta_{x_i} \iota_{x_i}^{z_0} = \text{Id}$, i.e.

$$\forall f \in \mathcal{H}(\Omega), \quad \iota_{x_i}^{z_0} f(z) = \int_{z_0}^{z} \omega_i(s)f(s).$$

The operator $\theta_y \iota_x^{z_0}$, for $x \neq y$, admits $u_y u_x^{-1}$ as eigenvalue, i.e.

$$\forall f \in \mathcal{H}(\Omega), \quad (\theta_y \iota_x^{z_0}) f = u_y u_x^{-1} f,$$

in particular, $(\theta_y \iota_x^{z_0})1_{\Omega} = u_y u_x^{-1}$.

Now, let $\mathcal{S}^{z_0}$ be the morphism defined as follows

$$\mathcal{S}^{z_0}(w) = \begin{cases} 
\text{Id} & \text{if } w = 1_{\mathcal{X}^*}, \\
\mathcal{S}^{z_0}(u) \iota_x^{z_0} & \text{if } w = u_x \in \mathcal{X}^* \mathcal{X}.
\end{cases}$$

Hence, for any $w \in \mathcal{X}^*$, $\mathcal{S}^{z_0}(w)1_{\Omega} = \alpha_{z_0}^{z}(w)$.

Example 9 (with $\omega_0(z) = z^{-1} dz$ and $\omega_1(z) = (1 - z)^{-1} dz$)

Let $\mathcal{C} := \mathbb{C}[z, z^{-1}, (1 - z)^{-1}]$. Here, $\theta_{x_0} = z \partial$ and $\theta_{x_1} = (1 - z) \partial$. Then

$$\theta_{x_0} + \theta_{x_1} = [\theta_{x_1}, \theta_{x_0}] = \partial$$

and, for any $L \in \mathcal{C}[\partial]$, there is $P \in \mathcal{C}\langle \mathcal{X} \rangle$ s.t. $13$ $L = \Theta(P)$. One also has

1. $(\theta_{x_0} \iota_{x_1}^{z_0})(\theta_{x_1} \iota_{x_0}^{z_0}) = (\theta_{x_1} \iota_{x_0}^{z_0})(\theta_{x_0} \iota_{x_1}^{z_0}) = \text{Id}$.

2. $\forall w \in \mathcal{X}^* x_1, \mathcal{S}^{0}(w)1_{\Omega} = \alpha_0^{z}(w) = \text{Li}_w(z)$.

3. $(\theta_{x_0} \iota_{x_1}^{z_0})1_{\Omega} = z(1 - z)^{-1}$ and $(\theta_{x_1} \iota_{x_0}^{z_0})1_{\Omega} = z^{-1} - 1$.

13. i.e. $\Theta$ is surjective and non injective. $\ker \Theta$?
Examples of linear differential equation

Example 10 (with $C_0 = \mathbb{C}(z)$)

\[(\partial - z)y = 0. \tag{1}\]

1. $e^{z^2/2}$ is solution of (1).

2. $ce^{z^2/2} = e^{z^2/2}e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$).

3. $\{e^{z^2/2}\}$ is a fundamental set of solutions of (1).

4. $k\{e^{z^2/2}\}$ is a Picard-Vessiot extension related to (1).

For $\theta_{x_0} = z\partial$ and $\theta_{x_1} = (1 - z)\partial$, since $L_{x_1x_0} = \partial\theta_{x_1}\theta_{x_0} \in k[\partial]$ then let

\[L_{x_1x_0}y = (z(1 - z)\partial^3 + (2 - 3z)\partial^2 - 1)y = 0. \tag{2}\]

1. $L_{x_1x_0}Li_2 = 0$ meaning that $Li_2$ is solution of (2).

2. $cLi_2 = Li_2 e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$) but it is not independent to $Li_2$.

3. $\{Li_2, \log, 1_\Omega\}$ is a fundamental set of solutions of (2).

4. $k\{Li_2, \log, 1_\Omega\}$ is a Picard-Vessiot extension\(^{14}\) related to (2).

\[14. k\{Li_2(z)\} = k \otimes \mathbb{C}[Li_2(z), \log(1 - z), \log(z)].\]
Chen series of $\{\omega_i\}_{i \geq 1}$ and along $z_0 \rightsquigarrow z$

For any $A$ containing $Q$, we get, on $\mathcal{H} \left( X \right)$ and $\mathcal{H} \left( Y \right)$,

$$
D_X := \sum_{w \in X^*} w \otimes w = \prod_{l \in \mathcal{L}yn X} e^{S_l \otimes P_l} \quad \text{and} \quad D_Y := \sum_{w \in Y^*} w \otimes w = \prod_{l \in \mathcal{L}yn Y} e^{S_l \otimes P_l}.
$$

Hence, since iterated integrals satisfy $\alpha_{z_0}^z (u \sqcup v) = \alpha_{z_0}^z (u) \alpha_{z_0}^z (v) \quad (u, v \in X^*)$ then the Chen series, $C_{z_0 \rightsquigarrow z} \in \mathcal{H}(\Omega) \langle \langle X \rangle \rangle$, is given by

$$
C_{z_0 \rightsquigarrow z} := \sum_{w \in X^*} \alpha_{z_0}^z (w) w = (\alpha_{z_0}^z \otimes \text{Id}) D_X = \prod_{l \in \mathcal{L}yn X} e^{\alpha_{z_0}^z (S_l) P_l}
$$

and then $\Delta \sqcup \Delta C_{z_0 \rightsquigarrow z} = C_{z_0 \rightsquigarrow z} \otimes C_{z_0 \rightsquigarrow z}$ and $\langle C_{z_0 \rightsquigarrow z} | 1_{X^*} \rangle = 1$.

For any $n \geq 0$, one has $d^n C_{z_0 \rightsquigarrow z} = p_n C_{z_0 \rightsquigarrow z}$, where

$$
p_n = \sum_{\text{wgt} r = n} \sum_{w \in X^n} \prod_{i=1}^{\deg r} \left( \sum_{j=1}^{r_i} r_j + j - 1 \right) \tau_r (w) \in C_0 \langle \langle X \rangle \rangle,
$$

and, for $w = x_{i_1} \ldots x_{i_k} \in X^*$ associated to the derivation multiindex $r = (r_1, \ldots, r_k) \in \mathbb{N}^k$ of weight $\text{wgt} r = |w| + \sum_{i=1}^k r_i$ and of degree $\deg r = |w|$, $\tau_r (w) := \tau_{r_1} (x_{i_1}) \ldots \tau_{r_k} (x_{i_k}) = (\partial_1 u_{x_{i_1}}) x_{i_1} \ldots (\partial_k u_{x_{i_k}}) x_{i_k}$.

15. $\langle C_{z_0 \rightsquigarrow z} | u \sqcup v \rangle = \langle C_{z_0 \rightsquigarrow z} | u \rangle \langle C_{z_0 \rightsquigarrow z} | v \rangle$ and on the other hand,

$$
\langle C_{z_0 \rightsquigarrow z} | u \otimes v \rangle = \langle \Delta \sqcup \Delta C_{z_0 \rightsquigarrow z} | u \otimes v \rangle, \langle C_{z_0 \rightsquigarrow z} | u \rangle \langle C_{z_0 \rightsquigarrow z} | v \rangle = \langle C_{z_0 \rightsquigarrow z} \otimes C_{z_0 \rightsquigarrow z} | u \otimes v \rangle.
$$

16. $\forall S \in \mathcal{H}(\Omega) \langle \langle X \rangle \rangle$, $dS = \sum_{w \in X^*} (\partial \langle S | w \rangle) w \in \mathcal{H}(\Omega) \langle \langle X \rangle \rangle$. 

Chen series and differential equations

Let $K$ be a compact on $\Omega$. There is $c_K \in \mathbb{R}_{\geq 0}$ and a morphism $M_K$ s.t.

$$\forall w \in \mathcal{X}^*, \quad \| \langle C_{z_0 \sim z} | w \rangle \|_K \leq c_K M_K(w) | w |^{-1}.$$  

Let $R \in \mathbb{C}^{\text{rat}} \langle \langle X \rangle \rangle$ of minimal representation $(\lambda, \mu, \eta)$ of dimension $n$. Then

$$\forall w \in \mathcal{X}^*, \quad | \langle R | w \rangle | \leq \| \lambda \|_{1,n}^1 \| \mu(w) \|_{n,\infty}^n \| \eta \|_{n,1}^1.$$  

With these data, we have

**Theorem 11**

If $c_K \| \lambda \|_{1,n}^1 \| \eta \|_{n,1}^n \sum_{x \in \mathcal{X}} M_K(x) \| \mu(x) \|_{n,\infty}^n < 1$ then $\alpha_{z_0}^z(R) = \langle R | C_{z_0 \sim z} \rangle$ and

$$\forall x \in \mathcal{X}, \quad \theta_x \alpha_{z_0}^z(R) = \sum_{x' \in \mathcal{X}} u_{x}^{-1}(z) u_{x'}(z) \alpha_{z_0}^z(R \triangleleft x').$$  

Letting $y(z_0, z) := \langle R | C_{z_0 \sim z} \rangle$, the following assertions are equivalent:

1. There is $p \in C_0 \langle \mathcal{X} \rangle$ s.t. $\langle R | p C_{z_0 \sim z} \rangle = \langle R \triangleleft p | C_{z_0 \sim z} \rangle = 0$.

2. There is $l = 0, \ldots, n-1$ s.t. $\{ \partial^k y \}_{0 \leq k \leq l}$ is $C_0$-linearly independent and $a_l, \ldots, a_1, a_0 \in C_0$ s.t. $(a_l \partial^l + \ldots + a_1 \partial + a_0)y = 0$.

**Proposition 4**

Let $G \in \mathbb{C} \langle \langle X \rangle \rangle$ and $H \in \mathbb{C}_{\text{exc}} \langle \langle X \rangle \rangle$ s.t. $\alpha_{z_0}^z(G) = \langle G | C_{z_0 \sim z} \rangle$ and

$h(\alpha_{z_0}^z(x_0), \alpha_{z_0}^z(x_1)) := \alpha_{z_0}^z(H) = \langle H | C_{z_0 \sim z} \rangle$ exist ($X = \{ x_0, x_1 \}$). Then

$$\alpha_{z_0}^z(HG) = \langle G|1_{\mathcal{X}^*} \rangle \alpha_{z_0}^z(H) + \int_{z_0}^z h(\alpha_{s}^z(x_0), \alpha_{s}^z(x_1)) d\alpha_{z_0}^s(G).$$
NONCOMMUTATIVE PV THEORY AND INDEPENDENCE VIA WORDS
First step of noncommutative PV theory

The Chen series $C_{z_0 \rightsquigarrow z}$ satisfies the following differential equation

$$(NCDE) \quad dS = MS, \quad \text{with} \quad M = \sum_{x \in \mathcal{X}} u_x x.$$

$$\Delta \ _M = \sum_{x \in \mathcal{X}} u_x (1 \mathcal{X}^* \otimes x + x \otimes 1 \mathcal{X}^*) = 1 \mathcal{X}^* \otimes M + M \otimes 1 \mathcal{X}^*.$$

The space of solutions of $(NCDE)$ is a right free $\mathbb{C} \langle \langle X \rangle \rangle$-module of rank 1.

By a theorem of Ree, $C_{z_0 \rightsquigarrow z}$ is a $\ _\mathbb{W}$-group-like solution$^{17}$ of $(NCDE)$. Moreover, if $G$ and $H$ are $\ _\mathbb{W}$-group-like solutions $(NCDE)$ there is a constant Lie series $C$ such that $G = He^C$ (and conversely).

From this, it follows that

- the differential Galois group of $(NCDE) + \ _\mathbb{W}$-group-like is the group$^{18}$ $\{ e^C \}_{C \in \mathcal{L}ie_{\mathbb{C}} 1 \mathcal{X}}$.

Which leads us to the following definition

- the PV extension related to $(NCDE)$ is $\hat{C}_0 \mathcal{X} \{ C_{z_0 \rightsquigarrow z} \}$.

It, of course, is such that $\text{Const}(C_0 \langle \langle X \rangle \rangle) = \ker d = \mathbb{C} 1 \mathcal{X}$.

$^{17}$ It can be obtained as the limit of a convergent Picard iteration, initialized at $\langle C_{z_0 \rightsquigarrow z} \mathcal{X}^* \rangle = 1 \mathcal{H}(\Omega) 1 \mathcal{X}^*$, for ultrametric distance.

$^{18}$ In fact, the Hausdorff group (group of characters) of $\mathcal{H} \ _\mathbb{W} (\mathcal{X})$. 
Basic triangular theorem over a differential ring

Suppose that the \( \mathbb{C} \)-commutative ring \( A \) is without zero divisors and equipped with a differential operator \( \partial \) such that \( \mathbb{C} = \ker \partial \).

Let \( S \in A\langle \langle X \rangle \rangle \) be a group-like solution of \((NCDE)\) in the following form

\[
S = \sum_{w \in \mathcal{X}^*} \langle S | w \rangle w = \sum_{w \in \mathcal{X}^*} \langle S | S_w \rangle P_w = \prod_{l \in \mathcal{L} \text{yn} \mathcal{X}} e^{\langle S | S_l \rangle P_l}.
\]

Then

1. If \( H \in A\langle \langle X \rangle \rangle \) is another grouplike solution then there exists \( C \in \text{Lie}_A\langle \langle X \rangle \rangle \) such that \( S = He^C \) (and conversely).

2. The following assertions are equivalent
   a) \( \{ \langle S | w \rangle \}_{w \in \mathcal{X}^*} \) is \( C_0 \)-linearly independent,
   b) \( \{ \langle S | l \rangle \}_{l \in \mathcal{L} \text{yn} \mathcal{X}} \) is \( C_0 \)-algebraically independent,
   c) \( \{ \langle S | x \rangle \}_{x \in \mathcal{X}} \) is \( C_0 \)-algebraically independent,
   d) \( \{ \langle S | x \rangle \}_{x \in \mathcal{X} \cup \{ 1 \mathcal{X}^* \}} \) is \( C_0 \)-linearly independent,
   e) \( \{ u_x \}_{x \in \mathcal{X}} \) is such that, for \( f \in \text{Frac}(C_0) \) and \( (c_x)_{x \in \mathcal{X}} \in \mathbb{C}^{(\mathcal{X})} \),
      \[
      \sum_{x \in \mathcal{X}} c_x u_x = \partial f \implies (\forall x \in \mathcal{X})(c_x = 0).
      \]
   f) \( (u_x)_{x \in \mathcal{X}} \) is free over \( \mathbb{C} \) and \( \partial \text{Frac}(C_0) \cap \text{span}_\mathbb{C} \{ u_x \}_{x \in \mathcal{X}} = \{ 0 \} \).
Examples of positive cases over $\mathcal{X} = \{x\}$, $\mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}$, $u_x(z) = 1_{\Omega}$, $C_0 = \mathbb{C}\{u_x^{\pm 1}\} = \mathbb{C}$.
   \[ \alpha_0^z(x^n) = z^n/n!, \text{ for } n \geq 1. \]
   Thus, $dS = xS$ and
   \[ S = \sum_{n \geq 0} \alpha_0^z(x^n)x^n = \sum_{n \geq 0} \frac{z^n}{n!}x^n = e^{zx}. \]

   Moreover, $\alpha_0^z(x) = z$ which is transcendent over $C_0$
   and the family $\{\alpha_0^z(x^n)\}_{n \geq 0}$ is $C_0$-free. Let $f \in C_0$ then $\partial f = 0$. Thus,
   if $\partial f = cu_x$ then $c = 0$.

2. $\Omega = \mathbb{C}\ \setminus \ (-\infty, 0]$, $u_x(z) = z^{-1}$, $C_0 = \mathbb{C}\{z^{\pm 1}\} = \mathbb{C}[z^{\pm 1}] \subset \mathbb{C}(z)$.
   \[ \alpha_1^z(x^n) = \log^n(z)/n!, \text{ for } n \geq 1. \]
   Thus $dS = z^{-1}xS$ and
   \[ S = \sum_{n \geq 0} \alpha_1^z(x^n)x^n = \sum_{n \geq 0} \frac{\log^n(z)}{n!}x^n = z^x. \]

   Moreover, $\alpha_1^z(x) = \log(z)$ which is transcendent over $\mathbb{C}(z)$ then
   over $\mathbb{C}[z^{\pm 1}]$. The family $\{\alpha_1^z(x^n)\}_{n \geq 0}$ is $\mathbb{C}(z)$-free and
   then $C_0$-free. Let $f \in C_0$ then $\partial f \in \text{span}_\mathbb{C}\{z^{\pm n}\}_{n \neq 1}$. Thus,
   if $\partial f = cu_x$ then $c = 0$. 
Examples of negative cases over $\mathcal{X} = \{x\}, A = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = e^z, C_0 = \mathbb{C}\{\{e^{\pm z}\}\} = \mathbb{C}[e^{\pm z}]$.
   
   $\alpha^z_0(x^n) = (e^z - 1)^n/n!, \text{ for } n \geq 1$. Thus, $dS = e^z xS$ and
   
   $$S = \sum_{n \geq 0} \alpha^z_0(x^n)x^n = \sum_{n \geq 0} \frac{(e^z - 1)^n}{n!} x^n = e^{(e^z-1)x}.$$

   Moreover, $\alpha^z_0(x) = e^z - 1$ which is not transcendent over $C_0$ and and $\{\alpha^z_0(x^n)\}_{n \geq 0}$ is not $C_0$-free. If $f(z) = ce^z \in C_0 \ (c \neq 0)$ then
   
   $\partial f(z) = ce^z = cu_x(z)$.

2. $\Omega = \mathbb{C}\setminus ]-\infty, 0], u_x(z) = z^a (a \notin \mathbb{Q}),$
   
   $C_0 = \mathbb{C}\{\{z, z^{\pm a}\}\} = \text{span}_\mathbb{C}\{z^{ka+l}\}_{k,l \in \mathbb{Z}}$.

   $\alpha^z_0(x^n) = (a + 1)^{-n} z^{n(a+1)}/n!, \text{ for } n \geq 1$. Thus, $dS = z^a xS$ and
   
   $$S = \sum_{n \geq 0} \alpha^z_0(x^n)x^n = \sum_{n \geq 0} \frac{z^{n(a+1)}}{(a + 1)^n n!} x^n = e^{(a+1)^{-1}z^{(a+1)x}}.$$

   Moreover, $\alpha^z_0(x) = z^{a+1}/(a + 1)$ which is not transcendent over $C_0$ and $\{\alpha^z_0(x^n)\}_{n \geq 0}$ is not $C_0$-free. If $f(z) = cz^{a+1}/(a + 1) \in C_0 \ (c \neq 0)$ then
   
   $\partial f(z) = cz^a = cu_x(z)$. 
Chen series of $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1 - z)^{-1}dz$

Let $\gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon)$ be the circular paths of radius $\varepsilon$ encircling 0 and 1 clockwise, respectively. In particular, letting $\beta = \beta_1 - \beta_0$, one considers

$$
\begin{align*}
\gamma_0(\varepsilon, \beta) &= \varepsilon e^{i\beta_0} \mapsto \varepsilon e^{i\beta_1} \subset \gamma_0(\varepsilon), \\
\gamma_1(\varepsilon, \beta) &= 1 - \varepsilon e^{i\beta_0} \mapsto 1 - \varepsilon e^{i\beta_1} \subset \gamma_1(\varepsilon).
\end{align*}
$$

On the one hand, one has, for any $i = 0$ or 1 and $w \in X^+$,

$$
|\langle C_{\gamma_i(\varepsilon, \beta)}| w \rangle| \leq \varepsilon^{|w_x|}\beta^{|w|} w^{-1}.
$$

It follows then

$$
C_{\gamma_i(\varepsilon, \beta)} = e^{i\beta x_i} + o(\varepsilon) \quad \text{and} \quad C_{\gamma_i(\varepsilon)} = e^{2i\pi x_i} + o(\varepsilon).
$$

Hence $^{19}$, for $R \in \mathbb{C}^{\text{rat}} \langle \langle X \rangle \rangle$ of minimal representation $(\lambda, \mu, \eta)$, one has

$$
\begin{align*}
\langle R \| C_{\gamma_i(\varepsilon, \beta)} \rangle &= \lambda \left( \prod_{l \in \text{Lyn}X} e^{\alpha_{\gamma_i(\varepsilon, \beta)}(S_l)\mu(P_l)} \right) \eta, \\
\langle R \| C_{\gamma_i(\varepsilon)} \rangle &= \lambda \left( \prod_{l \in \text{Lyn}X} e^{\alpha_{\gamma_i(\varepsilon)}(S_l)\mu(P_l)} \right) \eta.
\end{align*}
$$

19. Recall that the map $\alpha^Z_{z_0} : \mathbb{C}^{\text{rat}} \langle \langle X \rangle \rangle \rightarrow \mathcal{H}(\Omega)$ is not injective. For example,

$$
\alpha^Z_{z_0}(z_0 x_0^* + (1 - z_0)(-x_1)^* - 1 x^*) = 0.
$$
Dom(Li.)

**Proposition 5**

Let Dom(Li.) be the set of \( S = \sum_{n \geq 0} S_n \) with \( S_n = \sum_{|w| = n} \langle S|w \rangle w \) s.t. \( \sum_{n \geq 0} Li S_n \) converges uniformly on any compact of \( \Omega \). Then Dom(Li.), containing \( \mathbb{C}^{rat}_{exc} \langle \langle X \rangle \rangle \uplus \mathbb{C} \langle X \rangle \), is closed by shuffle and then \( Li S \uplus T = Li S Li T \), for \( S \) and \( T \in \text{Dom}(Li.) \).

**Proposition 6** \((L(z) = Cz^0 \Rightarrow z L(z) = Cz^0)\)

Let \( C := \mathbb{C}[z^a, (1 - z)^b]_{a, b \in \mathbb{C}} \). For \( R \in \text{Dom}(Li.) \), let \( \rho := \langle R||L \rangle \). Then, for \( n \geq 0 \), \( \partial^n \rho = \langle R||d^n L \rangle \) and \( d^n L = p_n L \), where \( p_n \) is given previously, with \( \tau_r(x_0) = -r!(-z)^{-(r+1)}x_0 \) and \( \tau_r(x_1) = r!(1 - z)^{-(r+1)}x_1 \).

The following assertions are equivalent:

1. \( \rho \) satisfies a differential equation with coefficients in \((C, \partial)\).
2. There exists \( P \in \mathbb{C} \langle X \rangle \) such that \( \langle R||PL \rangle = \langle R \triangleleft P||L \rangle = 0 \).

**Example 12** \((\omega_0(z) = z^{-1}dz, \omega_1(z) = (1 - z)^{-1}dz \& |c| < 1)\)

\[
\text{Li}_{(c x_0)^* x_1}(z) = \alpha^z((c x_0)^* x_1) = \int_0^z e^{c \log(z/s)} \omega_1(s) = z^c \int_0^z \sum_{n \geq 0} s^{n-c} ds
\]

\[
= z^c \sum_{n \geq 0} \frac{z^{n-c+1}}{n-c+1} = \sum_{n \geq 1} \frac{z^n}{n-c}.
\]
Bibliography


G.H.E. Duchamp, V. Hoang Ngoc Minh, Q.H. Ngo.– *Kleene stars of the plane, polylogarithms and symmetries*, Theoretical Computer Science, Volume 800, 31 December 2019, Pages 52-72


V. Hoang Ngoc Minh.– *On the solutions of the universal differential equation with three regular singularities (On solutions of KZ$_3$)*, CONFLUENTES MATHEMATICI (2020).

THANK YOU FOR YOUR ATTENTION