On universal differential equations

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INTRODUCTION
Picard-Vessiot theory of ordinary differential equation

\((k, \partial)\) a commutative differential ring without zero divisors.
\(\text{Const}(k) = \{ c \in k | \partial c = 0 \}\) is supposed to be a field.

\(\text{(ODE)} \quad (a_n \partial^n + a_{n-1} \partial^{n-1} + \ldots + a_0)y = 0, \quad a_0, \ldots, a_{n-1}, a_n \in k.\)
\(a_n^{-1}\) is supposed to exist.

**Definition 1**

1. Let \(y_1, \ldots, y_n\) be \(\text{Const}(k)\)-linearly independent solutions of \((\text{ODE})\).
   Then \(\{y_1, \ldots, y_n\}\) is called a fundamental set of solutions of \((\text{ODE})\) and it generates a \(\text{Const}(k)\)-vector subspace of dimension at most \(n\).

2. If \(^1 M = k\{y_1, \ldots, y_n\}\) and \(\text{Const}(M) = \text{Const}(k)\) then \(M\) is called a Picard-Vessiot extension related to \((\text{ODE})\).

3. Let \(k \subset K_1\) and \(k \subset K_2\) be differential rings. An isomorphism of rings \(\sigma : K_1 \rightarrow K_2\) is a differential \(k\)-isomorphism if
   \[
   \forall a \in K_1, \quad \partial(\sigma(a)) = \sigma(\partial a) \quad \text{and, if} \quad a \in k, \quad \sigma(a) = a.
   \]
   If \(K_1 = K_2 = K\), the differential galois group of \(K\) over \(k\) is by
   \[
   \text{Gal}_k(K) = \{ \sigma | \sigma \text{ is a differential } k\text{-automorphism of } K \}.
   \]

---

1. Let \(R_1, R_2\) be differential rings s.t. \(R_1 \subset R_2\). Let \(S\) be a subset of \(R_2\).
\(R_1\{S\}\) denotes the smallest differential subring of \(R_2\) containing \(R_1\).
\(R_1\{S\}\) is the ring (over \(R_1\)) generated by \(S\) and their derivatives of all orders.
Linear differential equations and Dyson series

Let \( a_0, \ldots, a_n \in \mathbb{C}(z) \), \( a_n(z)\partial^n y(z) + \ldots + a_1(z)\partial y(z) + a_0(z)y(z) = 0 \).

\[
(ED) \quad \begin{cases} 
\partial q(z) &= A(z)q(z), \quad A(z) \in \mathcal{M}_{n,n}(\mathbb{C}(z)), \\
q(z_0) &= \eta, \\
y(z) &= \lambda q(z), \quad \lambda \in \mathcal{M}_{1,n}(\mathbb{C}), \\
\eta &\in \mathcal{M}_{n,1}(\mathbb{C}).
\end{cases}
\]

By successive Picard iterations, with the initial point \( q(z_0) = \eta \), we get
\( y(z) = \lambda U(z_0; z)\eta \), where \( U(z_0; z) \) is the following functional expansion

\[
U(z_0; z) = \sum_{k \geq 0} \int_{z_0}^{z} A(z)dz_1 \int_{z_0}^{z_1} A(z)dz_2 \ldots \int_{z_0}^{z_{k-1}} A(z)dz_k, \quad \text{(Dyson series)}
\]

and \((z_0, z_1, \ldots, z_k, z)\) is a subdivision of the path of integration \( z_0 \rightsquigarrow z \).

In order to find the matrix \( \Omega(z_0; z) \) s.t.

\[
U(z_0; z) = \exp[\Omega(z_0; z)] = \top \exp \int_{z_0}^{z} A(s)ds, \quad \text{(Feynman's notation)}
\]

Magnus computed \( \Omega(z_0; z) \) as limit of the following Lie-integral-functionals

\[
\Omega_1(z_0; z) = \int_{z_0}^{z} A(z)ds,
\]

\[
\Omega_k(z_0; z) = \int_{z_0}^{z} [A(z) + [A(z), \Omega_{k-1}(z_0; s)]]/2 \\
+ [[A(z), \Omega_{k-1}(z_0; s)], \Omega_{k-1}(z_0; s)]/12 + \ldots ds.
\]

2. Subject to convergence.
Fuchsian linear differential equations

Let $\Omega$ be a simply connected domain and $\mathcal{H}(\Omega)$ be the ring of holomorphic functions over $\Omega$ (with $1_{\mathcal{H}(\Omega)}$ as neutral element). Let us consider, here, $\sigma = \{s_i\}_{i=0,\ldots,m}$, $m \geq 1$, as set of simple poles of $(ED)$ and $\Omega = \overline{\mathbb{C} \setminus \sigma}$.

$$A(z) = \sum_{i=0}^{m} M_i u_i(z),$$

where

$$\begin{cases}
M_i \in \mathcal{M}_{n,n}(\mathbb{C}), \\
u_i(z) = \frac{1}{z - s_i} \in \mathbb{C}(z), \\
\partial q(z) = \left( \sum_{i=0}^{m} M_i u_i(z) \right) q(z), \\
q(z_0) = \eta, \\
y(z) = \lambda q(z).
\end{cases}$$

Let $X^*$ be the set of words over $X = \{x_0, \ldots, x_m\}$ and

$$\alpha_{z_0}^Z \otimes \mathcal{M} : \mathbb{C}\langle X \rangle \otimes \mathbb{C}\langle X \rangle \to \mathcal{M}_{n,n}(\mathcal{H}(\Omega))$$

($z_0 \rightsquigarrow z$ is the path of integration previously introduced) s.t.

$$\begin{align*}
\mathcal{M}(1_{X^*}) &= \text{Id}_n \\
\mathcal{M}(x_{i_1} \cdots x_{i_k}) &= M_{i_1} \cdots M_{i_k}, \\
\alpha_{z_0}^Z (1_{X^*}) &= 1_{\mathcal{H}(\Omega)} \\
\alpha_{z_0}^Z (x_{i_1} \cdots x_{i_k}) &= \int_{z_0}^{z} \frac{dz_1}{z_1 - s_{i_1}} \cdots \int_{z_0}^{z_{k-1}} \frac{dz_k}{z_{k} - s_{i_k}}.
\end{align*}$$

Then $y(z) = \lambda U(z_0; z) \eta$ with

$$U(z_0; z) = \sum_{w \in X^*} \mathcal{M}(w) \alpha_{z_0}^{Z}(w) = (\mathcal{M} \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w.$$ 

3. Subject to convergence.
Example 2 (Hypergeometric equation)

Let \( t_0, t_1, t_2 \) be parameters and
\[ z(1 - z)\ddot{y}(z) + [t_2 - (t_0 + t_1 + 1)z]\dot{y}(z) - t_0 t_1 y(z) = 0. \]

Let \( q_1(z) = -y(z) \) and \( q_2(z) = (1 - z)\dot{y}(z) \). Hence, one has
\[ y(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} \]
and
\[ \begin{pmatrix} \dot{q}_1(z) \\ \dot{q}_2(z) \end{pmatrix} = \begin{pmatrix} M_0 & M_1 \\ z & 1 - z \end{pmatrix} \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix} = (u_0(z)M_0 + u_1(z)M_1) \begin{pmatrix} q_1(z) \\ q_2(z) \end{pmatrix}, \]
where \( u_0(z) = z^{-1}, u_1(z) = (1 - z)^{-1} \) and
\[ M_0 = -\begin{pmatrix} 0 & 0 \\ t_0 t_1 & t_2 \end{pmatrix} \quad \text{and} \quad M_1 = -\begin{pmatrix} 0 & 1 \\ 0 & t_2 - t_0 - t_1 \end{pmatrix}. \]
Nonlinear differential equations

\[
\begin{align*}
\text{(NED)} \quad \left\{ \\
\partial q(z) &= \left( \sum_{i=0}^{m} T_i(q) u_i(z) \right)(q), \\
q(z_0) &= q_0, \\
y(z) &= f(q(z)),
\end{align*}
\]

where

- \( u_i \in (k, \partial) \),
- the state \( q = (q_1, \ldots, q_n) \) belongs the complex analytic manifold \( Q \) of dimension \( n \) and \( q_0 \) is the initial state,
- the observation \( f \in O \), with \( O \) the ring of analytic functions over \( Q \),
- for \( i = 0..1 \), \( T_i = (T_i^1(q) \partial/\partial q_1 + \cdots + T_i^m(q) \partial/\partial q_m) \) is an analytic vector field over \( Q \), with \( T_i^j(q) \in O \), for \( j = 1, \ldots, n \).

With \( X \) and \( \alpha_{z_0}^z \) given as previously, let the morphism \( \tau \) be defined by
\( \tau(1_{X^*}) = \text{Id} \) and \( \tau(x_{i_1} \cdots x_{i_k}) = T_{i_1} \cdots T_{i_k} \). Then
\[ y(z) = T \circ f|_{q_0} \]

with
\[ T = \sum_{w \in X^*} \tau(w) \alpha_{z_0}^z(w) = (\tau \otimes \alpha_{z_0}) \sum_{w \in X^*} w \otimes w. \]

4. Subject to convergence.
Examples of nonlinear dynamical systems (1/2)

Example 3 (Harmonic oscillator)

Let $k_1, k_2$ be parameters and $\partial^2 y(z) + k_1 y(z) + k_2 y^2(z) = u_1(z)$ which can be represented by the following state equations (with $n = 1$)

$$
\begin{align*}
y(z) &= q(z), \\
\partial q(z) &= A_0(q)u_0(z) + A_1(q)u_1(z),
\end{align*}
$$

where $A_0 = -(k_1 q + k_2 q^2) \frac{\partial}{\partial q}$ and $A_1 = \frac{\partial}{\partial q}$.

Example 4 (Duffing equation)

Let $a, b, c$ be parameters and $\partial^2 y(z) + a\partial y(z) + by(z) + cy^3(z) = u_1(z)$ which can be represented by the following state equations (with $n = 2$)

$$
\begin{align*}
y(z) &= q_1(z), \\
\begin{pmatrix}
\partial q_1(z) \\
\partial q_2(z)
\end{pmatrix}
&= \begin{pmatrix}
q_2 \\
-(aq_2 + b^2 q_1 + cq_1^3)
\end{pmatrix} u_0(z) + \begin{pmatrix}
0 \\
1
\end{pmatrix} u_1(z) \\
&= A_0(q)u_0(z) + A_1(q)u_1(z),
\end{align*}
$$

where $A_0 = -(aq_2 + b^2 q_1 + cq_1^3) \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_1}$ and $A_1 = \frac{\partial}{\partial q_2}$. 
Example 5 (Van der Pol oscillator)

Let $\gamma, g$ be parameters and

$$\partial^2 x(z) - \gamma [1 + x(z)^2] \partial x(z) + x(z) = g \cos(\omega z)$$

which can be transformed into (with $C$ is some constant of integration)

$$\partial x(z) = \gamma [1 + x(z)^2/3] x(z) - \int_{z_0}^{z} x(s) ds + \frac{g}{\omega} \sin(\omega z) + C.$$ 

Supposing $x = \partial y$ and $u_1(z) = g \sin(\omega z)/\omega + C$, it leads then to

$$\partial^2 y(z) = \gamma [\partial y(z) + (\partial y(z))^3/3] + y(z) + u_1(z)$$

which can be represented by the following state equations (with $n = 2$)

$$y(z) = q_1(z),$$

$$\begin{pmatrix} \partial q_1(z) \\ \partial q_2(z) \end{pmatrix} = \begin{pmatrix} q_2 \\ \gamma(q_2 + q_2^3/3) + q_1 \end{pmatrix} u_0(z) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_1(z)$$

$$= A_0(q) u_0(z) + A_1(q) u_1(z),$$

where $\begin{align*}
A_0 &= [\gamma(q_2 + q_2^3/3) + q_1] \frac{\partial}{\partial q_2} + q_2 \frac{\partial}{\partial q_1} \\
A_1 &= \frac{\partial}{\partial q_2}.
\end{align*}$
DUAL LAWS AND REPRESENTATIVE SERIES
Dual law in bialgebra

Startting with a $k \rightarrow \text{AAU}$ ($k$ is a ring) $A$. Dualizing $\mu : A \otimes_k A \rightarrow A$, we get the transpose $t_\mu : A^\vee \rightarrow (A \otimes_k A)^\vee$ so that we do not get a co-multiplication in general.

- Remark that when $k$ is a field, the following arrow is into (due to the fact that $A^\vee \otimes_k A^\vee$ is torsionfree)
  $$\Phi : A^\vee \otimes_k A^\vee \rightarrow (A \otimes_k A)^\vee.$$  

- One restricts the codomain of $t_\mu$ to $A^\vee \otimes_k A^\vee$ and then the domain to $(t_\mu)^{-1}\Phi(A^\vee \otimes_k A^\vee) =: A^\circ$.

\[
\begin{array}{ccc}
A^\vee & \xrightarrow{t_\mu} & (A \otimes_k A)^\vee \\
\mathcal{can} & & \phi \\
A^\circ & \xrightarrow{\Delta_\mu} & A^\vee \otimes_k A^\vee \\
\mathcal{can} & & j \otimes j \\
A^{\circ\circ} & \xrightarrow{\Delta_\mu} & A^\circ \otimes_k A^\circ
\end{array}
\]

The descent can stop at first step for a field $k$ and then $A^{\circ\circ} = A^\circ$.
The coalgebra $(A^\circ, \Delta_\mu)$ is called the Sweedler’s dual of $(A, \mu)$. 
Case of algebras noncommutative series

- Denoting the (ordered) alphabets $Y := \{y_k\}_{k \geq 1}$ (with $y_1 \succ y_2 \succ \ldots$) or $X := \{x_0, x_1\}$ (with $x_1 \succ x_0$) by $\mathcal{X}$, we use the correspondence among words of the free monoid $(\mathcal{X}^*, \text{conc}, 1_{\mathcal{X}^*})$:

$$
(s_1, \ldots, s_r) \in \mathbb{N}_+^r \leftrightarrow y_{s_1} \ldots y_{s_r} \in Y^* \xrightarrow{\pi_X} x_0^{s_1-1} x_1 \ldots x_0^{s_r-1} x_1 \in X^* x_1.
$$

- Let $Lyn\mathcal{X}$ denote the set of Lyndon words generated by $\mathcal{X}$.

- Let $(\text{Lie}_A\langle\langle\mathcal{X}\rangle\rangle, [.])$ and $(A\langle\langle\mathcal{X}\rangle\rangle, \text{conc})$ (resp. $\text{Lie}_A\langle\mathcal{X}\rangle, [.]$) and $(A\langle\mathcal{X}\rangle, \text{conc})$ denote the algebras of (Lie) series (resp. polynomials) with coefficients in the ring $A$, over $\mathcal{X}$.

- $\mathcal{H} \uplus (\mathcal{X}) := (A\langle\mathcal{X}\rangle, \text{conc}, 1_{\mathcal{X}^*}, \Delta \uplus, e)$ and $\mathcal{H} \uplus (Y) := (A\langle Y\rangle, \text{conc}, 1_{Y^*}, \Delta \uplus, e)$ with

  $\forall x \in \mathcal{X}, \quad \Delta \uplus x = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x,$

  $\forall y_i \in Y, \quad \Delta \uplus y_i = y_i \otimes 1_{Y^*} + 1_{Y^*} \otimes y_i + \sum_{k+l=i} y_k \otimes y_l.$

- The dual law associated to $\text{conc}$ is defined by

  $\forall w \in \mathcal{X}^*, \quad \Delta_{\text{conc}}(w) = \sum_{u,v \in \mathcal{X}^*, uv=w} u \otimes v.$

5. Or equivalently, for $x, y \in \mathcal{X}, y_i, y_j \in Y$ and $u, v \in \mathcal{X}^*$ (resp. $Y^*$),

  $u \uplus 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} \uplus u = u$ and $xu \uplus yv = x(u \uplus yv) + y(xu \uplus v),$

  $u \uplus 1_{Y^*} = 1_{Y^*} \uplus u = u$ and $x_i u \uplus y_j v = y_i(u \uplus y_j v) + y_j(y_i u \uplus v) + y_{i+j}(u \uplus v).$
Dualizable laws in conc-shuffle bialgebras (1/2)

We can exploit the basis of words as follows

1. Any bilinear law (shuffle, stuffle or any) \( \mu : A\langle \X \rangle \otimes A\langle \X \rangle \to A\langle \X \rangle \) can be described through its structure constants wrt to the basis of words, i.e. for \( u, v, w \in \X^* \), \( \Gamma^{w}_{u,v} := \langle \mu(u \otimes v) | w \rangle \) so that
   \[
   \mu(u \otimes v) = \sum_{w \in \X^*} \Gamma^{w}_{u,v} w.
   \]

2. In the case when \( \Gamma^{w}_{u,v} \) is locally finite in \( w \), we say that the given law is dualizable, the arrow \( ^{t} \mu \) restricts nicely to
   \[A\langle \X \rangle \hookrightarrow A\langle \langle \X \rangle \rangle\]
   and one can define on the polynomials a comultiplication by
   \[
   \Delta_{\mu}(w) := \sum_{u,v \in \X^*} \Gamma^{w}_{u,v} u \otimes v.
   \]

3. When the law \( \mu \) is dualizable, we have
   \[
   \begin{array}{ccc}
   A\langle \langle \X \rangle \rangle & \xrightarrow{^{t} \mu} & A\langle \langle \X^* \otimes \X^* \rangle \rangle \\
   \uparrow{can} & & \uparrow{\Phi|_{A\langle \X \rangle \otimes A\langle \X \rangle}} \\
   A\langle \X \rangle & \xrightarrow{\Delta_{\mu}} & A\langle \X \rangle \otimes A\langle \X \rangle
   \end{array}
   \]

The arrow \( \Delta_{\mu} \) is unique to be able to close the rectangle and
\( \Delta_{\mu}(P) \) is defined as above.
4. Proof that the arrow $A \langle \mathcal{X} \rangle \otimes_A A \langle \mathcal{X} \rangle \longrightarrow A \langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle$ is into:

Let $T = \sum_{i=1}^{n} P_i \otimes_A Q_i$ such that $\Phi(T) = 0$. Rewriting $T$ as a finitely supported sum $T = \sum_{\{u,v\} \in \mathcal{X}^* \times \mathcal{X}^*} c_{u,v} u \otimes v$ (this is indeed the iso between $A \langle \mathcal{X} \rangle \otimes_A A \langle \mathcal{X} \rangle$ and $A[\mathcal{X}^* \times \mathcal{X}^*]$), $\Phi(T)$ is by definition of $\Phi$ the double series (here a polynomial) s.t. $\langle \Phi(T)|u \otimes v \rangle = c_{u,v}$. If $\Phi(T) = 0$, then for all $(u, v) \in \mathcal{X}^* \times \mathcal{X}^*$, $c_{u,v} = 0$ entailing $T = 0$.

We extend by linearity and infinite sums, for $S \in A \langle \mathcal{Y} \rangle$ (resp. $A \langle \mathcal{X} \rangle$), by

$$\Delta_{\shuffle} S = \sum_{w \in \mathcal{Y}^*} \langle S|w \rangle \Delta_{\shuffle} w \in A \langle \mathcal{Y}^* \otimes \mathcal{Y}^* \rangle,$$

$$\Delta_{\text{conc}} S = \sum_{w \in \mathcal{X}^*} \langle S|w \rangle \Delta_{\text{conc}} w \in A \langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle,$$

$$\Delta_{\mathbin{\shuffle\shuffle}} S = \sum_{w \in \mathcal{X}^*} \langle S|w \rangle \Delta_{\mathbin{\shuffle\shuffle}} w \in A \langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle.$$

$A \langle \mathcal{X} \rangle \otimes A \langle \mathcal{X} \rangle$ embeds injectively in $A \langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle \cong [A \langle \mathcal{X} \rangle][\langle \mathcal{X} \rangle]$.

6. $A \langle \mathcal{X} \rangle \otimes A \langle \mathcal{X} \rangle$ contains the elements of the form $\sum_{i \in I} \text{finite } G_i \otimes D_i$, for $(G_i, D_i) \in A \langle \mathcal{X} \rangle \times A \langle \mathcal{X} \rangle$. But since elements of $M \otimes N$ are finite combination of $m_i \otimes n_i, m_i \in M, n_i \in N$ then $\sum_{i \geq 0} u^i \otimes v^i$ belongs to $A \langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle$ and does not belong to $A \langle \mathcal{X} \rangle \otimes A \langle \mathcal{X} \rangle$, for $u, v \in \mathcal{X}^1$. 
Extended Ree’s theorem

Let $S \in A\langle \langle Y \rangle \rangle$ (resp. $A\langle \langle X \rangle \rangle$), $A$ is a commutative ring containing $\mathbb{Q}$.

The series $S$ is said to be

1. a $\sqcup$ (resp. conc, $\sqcup$)-character iff, for any $w, v \in Y^*$ (resp. $X^*$),
   $\langle S|w\rangle\langle S|v\rangle = \langle S|w \sqcup v\rangle$ (resp. $\langle S|wv\rangle, \langle S|w \sqcup v\rangle$) and $\langle S|1\rangle = 1$.

2. an infinitesimal $\sqcup$ (resp. conc, $\sqcup$)-character iff, for any $w, v \in Y^*$ (resp. $X^*$),
   $\langle S|w\rangle\langle S|v\rangle = \langle S|w\rangle\langle v|1_{Y^*}\rangle + \langle w|1_{Y^*}\rangle\langle S|v\rangle$
   (resp. $\langle S|wv\rangle = \langle S|w\rangle\langle v|1_{X^*}\rangle + \langle w|1_{X^*}\rangle\langle S|v\rangle$),
   $\langle S|w \sqcup v\rangle = \langle S|w\rangle\langle v|1_{X^*}\rangle + \langle w|1_{X^*}\rangle\langle S|v\rangle$).

3. a group-like series iff $\langle S|1_{X^*}\rangle = 1$ and $\Delta_{\sqcup} S = \Phi(S \otimes S)$ (resp. $\Delta_{\text{conc}} S = \Phi(S \otimes S), \Delta_{\sqcup} S = \Phi(S \otimes S))$.

4. a primitive series iff $\Delta_{\sqcup} S = 1_{Y^*} \otimes S + S \otimes 1_{Y^*}$ (resp. $\Delta_{\text{conc}} S = 1_{X^*} \otimes S + S \otimes 1_{X^*}, \Delta_{\sqcup} S = 1_{X^*} \otimes S + S \otimes 1_{X^*}$).

Then the following assertions are equivalent

1. $S$ is a $\sqcup$ (resp. conc and $\sqcup$)-character.
2. log $S$ an infinitesimal $\sqcup$ (resp. conc and $\sqcup$)-character.
3. $S$ is group-like, for $\Delta_{\sqcup}$ (resp. $\Delta_{\text{conc}}$ and $\Delta_{\sqcup}$).
4. log $S$ is primitive, for $\Delta_{\sqcup}$ (resp. $\Delta_{\text{conc}}$ and $\Delta_{\sqcup}$).
Extension by continuity (infinite sums)

Now, suppose that the ring $A$ (containing $\mathbb{Q}$) is a field $k$. Then

$\Delta_\Box : k\langle X \rangle \rightarrow k\langle X \rangle \otimes k\langle X \rangle$ and $\Delta_\uplus : k\langle Y \rangle \rightarrow k\langle Y \rangle \otimes k\langle Y \rangle$

are graded for the multidegree. Then $\Delta_\uplus$ is graded for the length. Their extension to the completions (i.e. $k\langle X \rangle$ and $k\langle X^* \otimes X^* \rangle$) are continuous and then, when exist, commute with infinite sums. Hence $7, 8$,

$\forall c \in k, \quad \Delta_\Box (cx)^* = \sum_{n \geq 0} c^n \Delta_\Box x^n = \sum_{n \geq 0} c^n \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}$.

For $c \in \mathbb{N}_{\geq 2}$ which is neither a field nor a ring (containing $\mathbb{Q}$), we also get

$(cx)^* = (c - 1)^{-1} \sum_{a,b \in \mathbb{N}_{\geq 1}, a+b=c} (ax)^*_\Box (bx)^* \in \mathbb{N}_{\geq 2} \langle X \rangle$,

$\Delta_\Box (cx)^* \neq (c - 1)^{-1} \sum_{a,b \in \mathbb{N}_{\geq 1}, a+b=c} (ax)^* \otimes (bx)^* \in \mathbb{Q} \langle X \rangle \otimes \mathbb{Q} \langle X \rangle$,

because

$\langle \text{LHS} | x \otimes 1_X^* \rangle = c$ and $\langle \text{RHS} | x \otimes 1_X^* \rangle = (c - 1)^{-1} \sum_{a=1}^{c-1} a = \frac{c}{2}$.

For $c \in \mathbb{Z}$ (or even $\mathbb{Q}, \mathbb{R}, \mathbb{C}$), the such decomposition is not finite.

7. For $S \in A\langle X \rangle$ s.t. $\langle S | 1_X^* \rangle = 0$, $S^* = \sum_{n \geq 0} S^n$ is called Kleene star of $S$.

8. $\Delta_\Box x^n = (\Delta_\Box x)^n = (1_{X^*} \otimes x + x \otimes 1_{X^*})^n = \sum_{j=0}^{n} \binom{n}{j} x^j \otimes x^{n-j}$.
Case of rational series and of $\Delta_{\text{conc}}$

$A^\text{rat} \langle \mathcal{X} \rangle$ denotes the algebraic closure by $^9 \{ \text{conc}, +, * \}$ of $\widehat{A.\mathcal{X}}$ in $A\langle \mathcal{X} \rangle$.

\[
A\langle \mathcal{X} \rangle \xrightarrow{t_{\text{conc}}} A\langle \mathcal{X}^* \otimes \mathcal{X}^* \rangle
\]
\[
can \uparrow \quad \Phi|_{A^\text{rat} \langle \mathcal{X} \rangle \otimes_A A^\text{rat} \langle \mathcal{X} \rangle}
\]

$A^\text{rat} \langle \mathcal{X} \rangle \longrightarrow A^\text{rat} \langle \mathcal{X} \rangle \otimes_A A^\text{rat} \langle \mathcal{X} \rangle$

The dashed arrow may not exist in general, but for any $R \in A^\text{rat} \langle \mathcal{X} \rangle$ admitting $(\lambda, \mu, \eta)$ as linear representation of dimension $n$, we can get $t_{\text{conc}}(R) = \Phi(\sum_{i=1}^{n} G_i \otimes D_i)$.

Indeed, since $\langle R|xy \rangle = \lambda \mu(xy)\eta = \lambda \mu(x)\mu(y)\eta$ ($x, y \in \mathcal{X}$) then, letting $e_i$ is the vector such that $^t e_i = (0 \ldots 0 1 0 \ldots 0)$, one has

\[
\langle R|xy \rangle = \sum_{i=1}^{n} \lambda \mu(x)e_i^t e_i \mu(y)\eta = \sum_{i=1}^{n} \langle G_i|x \rangle \langle D_i|y \rangle = \sum_{i=1}^{n} \langle G_i \otimes D_i|x \otimes y \rangle.
\]

$G_i$ (resp. $D_i$) admits then $(\lambda, \mu, e_i)$ (resp. $(^t e_i, \mu, \eta)$) as linear representation.

If $A = k$ being a field then, due to the injectivity of $\Phi$, all expressions of the type $\sum_{i=1}^{n} G_i \otimes D_i$, of course, coincide. Hence, the dashed arrow (a restriction of $\Delta_{\text{conc}}$) in the above diagram is well-defined.

9. $A^\text{rat} \langle \mathcal{X} \rangle$ is closed under $\sqcup$ . $A^\text{rat} \langle \mathcal{Y} \rangle$ is also closed under $\sqcup$.
Representative series and Sweedler’s dual

Theorem 6 (representative series)

Let $S \in A \llangle \mathcal{X} \rrangle$. The following assertions are equivalent

1. The series $S$ belongs to $A^{\text{rat}} \llangle \mathcal{X} \rrangle$.

2. There exists a linear representation $(\nu, \mu, \eta)$, of rank $n$, for $S$ with $\nu \in M_{1,n}(A), \eta \in M_{n,1}(A)$ and a morphism of monoids $\mu : \mathcal{X}^* \to M_{n,n}(A)$ s.t., for any $w \in \mathcal{X}^*$, $\langle S | w \rangle = \nu \mu(w) \eta$.

3. The shifts $^{10}\{S \triangleleft w\}_{w \in \mathcal{X}^*}$ (resp. $\{w \triangleright S\}_{w \in \mathcal{X}^*}$) lie within a finitely generated shift-invariant $A$-module.

Moreover, if $A$ is a field $k$, the previous assertions are equivalent to

4. There exist $(G_i, D_i)_{i \in F_{\text{finite}}}$ s.t. $\Delta_{\text{conc}}(S) = \sum_{i \in F_{\text{finite}}} G_i \otimes D_i$.

Hence, $\mathcal{H}_{\shuffle}^\circ(\mathcal{X}) = (k^{\text{rat}} \llangle \mathcal{X} \rrangle, \shuffle, 1_{\mathcal{X}^*}, \Delta_{\text{conc}}, e)$ and $\mathcal{H}_{\shuffle}^\circ(Y) = (k^{\text{rat}} \llangle Y \rrangle, \shuffle, 1_{\mathcal{X}^*}, \Delta_{\text{conc}}, e)$.

Now, let $A_{\text{exc}} \llangle \mathcal{X} \rrangle$ (resp. $A_{\text{exc}}^{\text{rat}} \llangle \mathcal{X} \rrangle$) be the set of exchangeable $^{11}$ series (resp. series admitting a linear representation with commuting matrices).

10. The left (resp. right) shift of $S$ by $P$ is $P \triangleright S$ (resp. $S \triangleleft P$) defined by, for $w \in \mathcal{X}^*$, $\langle P \triangleright S | w \rangle = \langle S | wP \rangle$ (resp. $\langle S \triangleleft P | w \rangle = \langle S | PW \rangle$).

11. i.e. if $S \in A_{\text{exc}} \llangle \mathcal{X} \rrangle$ then $(\forall u, v \in \mathcal{X}^*)(\forall x \in \mathcal{X})(|u|_x \equiv |v|_x) \Rightarrow \langle S | u \rangle = \langle S | v \rangle$. 

Kleene stars of the plane and conc-characters

For any $S \in A\langle\mathcal{X}\rangle$, let $\nabla S$ denotes $S - 1\chi^*$. 

Theorem 7 (rational exchangeable series)

1. $A_{\text{exc}}^\text{rat}\langle\mathcal{X}\rangle \subset A^\text{rat}\langle\mathcal{X}\rangle \cap A_{\text{exc}}^\text{rat}\langle\mathcal{X}\rangle$. If $A$ is a field then the equality holds and $A_{\text{exc}}^\text{rat}\langle\mathcal{X}\rangle = A^\text{rat}\langle x_0 \rangle \sqcup A^\text{rat}\langle x_1 \rangle$ and, for the algebra of series over subalphabets $A_{\text{fin}}^\text{rat}\langle\mathcal{Y}\rangle := \bigcup_{F \subset \text{finite}} \forall A^\text{rat}\langle F \rangle$, we get

\[
A_{\text{exc}}^\text{rat}\langle\mathcal{Y}\rangle \cap A_{\text{fin}}^\text{rat}\langle\mathcal{Y}\rangle = \bigcup_{k \geq 0} A^\text{rat}\langle y_1 \rangle \sqcup \ldots \sqcup A^\text{rat}\langle y_k \rangle \subset A_{\text{exc}}^\text{rat}\langle\mathcal{Y}\rangle.
\]

2. $\forall x \in \mathcal{X}, A^\text{rat}\langle x \rangle = \{ P(1 - xQ)^{-1} \}_{P,Q \in A[x]}$. If $k$ is an algebraically closed field then $k^\text{rat}\langle x \rangle = \text{span}_k \{(ax)^* \sqcup k\langle x \rangle | a \in K\}$. 

3. If $A$ is a $\mathbb{Q}$-algebra without zero divisors, $\{x^*\}_{x \in \mathcal{X}}$ (resp. $\{y^*\}_{y \in \mathcal{Y}}$) are conc-character and algebraically independent over $(A\langle\mathcal{X}\rangle, \sqcup)$ (resp. $(A\langle\mathcal{Y}\rangle, \sqcup)$) within $(A^\text{rat}\langle\mathcal{X}\rangle, \sqcup)$ (resp. $(A^\text{rat}\langle\mathcal{Y}\rangle, \sqcup)$).

4. Let $S \in A\langle\mathcal{X}\rangle$. If $A = k$, a field, then t.f.a.e.

\begin{enumerate}
\item $S$ is groupe-like, for $\Delta_{\text{conc}}$.
\item There exists $M := \sum_{x \in \mathcal{X}} c_x x \in \overline{k.\mathcal{X}}$ s.t. $S = M^*$.
\item There exists $M := \sum_{x \in \mathcal{X}} c_x x \in \overline{k.\mathcal{X}}$ s.t. $\nabla S = MS = SM$.
\end{enumerate}

12. The following identity lives in $A_{\text{exc}}^\text{rat}\langle\mathcal{Y}\rangle$ but not in $A_{\text{exc}}^\text{rat}\langle\mathcal{Y}\rangle \cap A_{\text{fin}}^\text{rat}\langle\mathcal{Y}\rangle$, 

\[
(y_1 + \ldots)^* = \lim_{k \to +\infty} (y_1 + \ldots + y_k)^* = \lim_{k \to +\infty} y_1^* \sqcup \ldots \sqcup y_k^* = \sqcup_{k \geq 1} y_k^*.
\]
CONTINUITY OVER CHEN SERIES
Continuity, indiscernability and growth condition

For \( i = 0, 2 \), let \((k_i, \| . \|_i)\) be a semi-normed space and \( g_i \in \mathbb{Z} \).

Definition 8

1. Let \( Cl \) be a class of \( k_1 \langle \langle \mathcal{X} \rangle \rangle \). Let \( S \in k_2 \langle \langle \mathcal{X} \rangle \rangle \) and it is said to be
   a) \textit{continuous} over \( Cl \) if, for \( \Phi \in Cl \), the following sum is convergent
      \[ \sum_{w \in \mathcal{X}^*} \| \langle S|w \rangle \|_2 \| \langle \Phi|w \rangle \|_1. \]
      We will denote \( \langle S|\Phi \rangle \) the sum \( \sum_{w \in \mathcal{X}^*} \langle S|w \rangle \langle \Phi|w \rangle \) and \( k_2 \langle \langle \mathcal{X} \rangle \rangle \text{cont} \) the set of continuous power series over \( Cl \).
   b) \textit{indiscernable} over \( Cl \) iff, for any \( \Phi \in Cl \), \( \langle S|\Phi \rangle = 0 \).

2. Let \( \chi_1 \) and \( \chi_2 \) be real positive functions over \( \mathcal{X}^* \). Let \( S \in k_1 \langle \langle \mathcal{X} \rangle \rangle \).
   a) \( S \) satisfies the \( \chi_1 \)–\textit{growth condition} of order \( g_1 \) if it satisfies
      \[ \exists K \in \mathbb{R}_+, \exists n \in \mathbb{N}, \forall w \in \mathcal{X}^\geq n, \quad \| \langle S|w \rangle \|_1 \leq K \chi_1(w) |w|^{g_1}. \]
      We denote by \( k_1^{(\chi_1, g_1)} \langle \langle \mathcal{X} \rangle \rangle \) the set of formal power series in \( k_1 \langle \langle \mathcal{X} \rangle \rangle \) satisfying the \( \chi_1 \)–growth condition of order \( g_1 \).
   b) If \( S \) is continuous over \( k_2^{(\chi_2, g_2)} \langle \langle \mathcal{X} \rangle \rangle \) then it will be said to be \( (\chi_2, g_2) \)-\textit{continuous}. The set of formal power series which are \( (\chi_2, g_2) \)-continuous is denoted by \( k_2^{(\chi_2, g_2)} \langle \langle \mathcal{X} \rangle \rangle \text{cont} \).
Convergence condition

Proposition 1
Let $\chi_1$ and $\chi_2$ be real positive functions over $\mathcal{X}^*$. Let $g_1$ and $g_2 \in \mathbb{Z}$ such that $g_1 + g_2 \leq 0$.

1. Let $k_1^{(\chi_1,g_1)}\langle \mathcal{X} \rangle$ and let $P \in k_1\langle \mathcal{X} \rangle$. The right residual of $S$ by $P$ belongs to $k_1^{(\chi_1,g_1)}\langle \mathcal{X} \rangle$.

2. Let $R \in k_2^{(\chi_2,g_2)}\langle \mathcal{X} \rangle$ and let $Q \in k_2\langle \mathcal{X} \rangle$. The concatenation $QR$ belongs to $k_2^{(\chi_2,g_2)}\langle \mathcal{X} \rangle$.

3. $\chi_1, \chi_2$ are morphisms over $\mathcal{X}^*$ satisfying $\sum_{x \in \mathcal{X}} \chi_1(x)\chi_2(x) < 1$. If $F_1 \in k_1^{(\chi_1,g_1)}\langle \mathcal{X} \rangle$ (resp. $F_2 \in k_2^{(\chi_2,g_2)}\langle \mathcal{X} \rangle$) then $F_1$ (resp. $F_2$) is continuous over $k_2^{(\chi_2,g_2)}\langle \mathcal{X} \rangle$ (resp. $k_1^{(\chi_1,g_1)}\langle \mathcal{X} \rangle$).

Proposition 2
Let $\mathcal{C} \subseteq k_1\langle \mathcal{X} \rangle$ be a monoid containing $\{e^{tx} \}_{x \in \mathcal{X}}$. Let $S \in k_2\langle \mathcal{X} \rangle^{cont}$.

1. If $S$ is indiscernable over $\mathcal{C}$ then for any $x \in \mathcal{X}$, $x \triangleright S$ and $S \triangleright x$ belong to $k_2\langle \mathcal{X} \rangle^{cont}$ and they are indiscernable over $\mathcal{C}$.

2. $S$ is indiscernable over $\mathcal{C}$ iff $S = 0$. 
Iterated integrals over $\omega_i(z) = u_{x_i}(z)dz$ and along $z_0 \rightsquigarrow z$

Recall that $\Omega$ is a simply connected domain with $1_{\mathcal{H}(\Omega)}$ as neutral element, $\mathcal{A} := \mathcal{H}(\Omega)$ and $\mathcal{C}_0$ is a differential subring of $\mathcal{A}$ ($\partial(\mathcal{C}_0) \subset \mathcal{C}_0$). $\mathbb{C}\{(g_i)_{i \in I}\}$ denotes the differential subalgebra of $\mathcal{A}$ generated by $(g_i)_{i \in I}$, i.e. the $\mathbb{C}$-algebra generated by $g_i$’s and their derivatives $\{u_{x_i}\}_{i \in \mathcal{X}} :$ elements in $\mathcal{C}_0 \cap \mathcal{A}^{-1}$ in correspondence with $\{\theta_{x_i}\}_{x \in \mathcal{X}}$ ($\theta_{x_i} = u_{x_i}^{-1}\partial$).

Let $\Theta$ be defined by $\Theta(w) = \theta_{x_i}\Theta(u)$, for $w = xu \in \mathcal{X} \mathcal{X}^*$, and $\Theta(1_{\mathcal{X}^*}) = \text{Id}$.

The iterated integral over $\omega_i(z) = u_{x_i}(z)dz$ and along $z_0 \rightsquigarrow z$ is defined by

$$\alpha^z_{z_0}(1_{\mathcal{X}^*}) = 1_\Omega,$$

$$\alpha^z_{z_0}(x_{i_1} \ldots x_{i_k}) = \int_{z_0}^z \omega_{i_1}(z_{1}) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_{k}).$$

$$\partial \alpha^z_{z_0}(x_{i_1} \ldots x_{i_k}) = u_{x_{i_1}}(z) \int_{z_0}^z \omega_{i_2}(z_{2}) \ldots \int_{z_0}^{z_{k-1}} \omega_{i_k}(z_{k}).$$

$$\forall w \in \mathcal{X}^* , \quad \Theta(\tilde{w}) \alpha^z_{z_0}(w) = 1_\Omega.$$

$$\text{span}_\mathbb{C}\{\partial^l \alpha^z_{z_0}(w)\}_{w \in \mathcal{X}^*, l \geq 0} \subset \text{span}_\mathbb{C}\{(u_x)_{x \in \mathcal{X}}\}\{\alpha^z_{z_0}(w)\}_{w \in \mathcal{X}^*}$$

$$\subset \text{span}_\mathbb{C}\{(u_{x}^{\pm1})_{x \in \mathcal{X}}\}\{\alpha^z_{z_0}(w)\}_{w \in \mathcal{X}^*}$$

$$\cong \mathbb{C}\{(u_{x}^{\pm1})_{x \in \mathcal{X}}\} \otimes \mathbb{C}\{\alpha^z_{z_0}(w)\}_{w \in \mathcal{X}^*}?$$
Examples of linear differential equation

Let us consider the following examples, with $k = \mathbb{C}(z)$

$$(\partial - z)y = y' - zy = 0. \quad (1)$$

1. $e^{z^2/2}$ is solution of $(1)$.

2. $ce^{z^2/2} = e^{z^2/2}e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$).

3. $\{e^{z^2/2}\}$ is a fundamental set of solutions of $(1)$.

4. $k\{e^{z^2/2}\}$ is a Picard-Vessiot extension related to $(1)$.

For $\theta_0 = z\partial, \theta_1 = (1 - z)\partial$, since $\partial\theta_1\theta_0 \in k[\partial]$ then let us consider

$$(\partial\theta_1\theta_0)y = \left(z(1 - z)\partial^3 + (2 - 3z)\partial^2 - 1\right)y = z(1 - z)y^{(3)} + (2 - 3z)y'' - y' = 0. \quad (3)$$

1. $(\partial\theta_1\theta_0)\text{Li}_2 = 0$ meaning that $\text{Li}_2$ is solution of $(3)$.

2. $c \text{Li}_2 = \text{Li}_2 e^{\log c}$ is an other solution ($c \in \mathbb{R} \setminus \{0\}$) but it is not independent to $\text{Li}_2$.

3. $\{\text{Li}_2, \log, 1_{\Omega}\}$ is a fundamental set of solutions of $(3)$.

4. $k\{\text{Li}_2, \log, 1_{\Omega}\}$ is a Picard-Vessiot extension $^{13}$ related to $(3)$.

13. $k\{\text{Li}_2(z)\} = k \otimes \mathbb{C}[\text{Li}_2(z), \log(1 - z), \log(z)]$. 
Chen series of $\{\omega_i\}_{i \geq 1}$ and along $z_0 \rightsquigarrow z$

Since iterated integrals satisfy the Chen’s lemma (or Friedrichs criterion), i.e. $\alpha_{z_0}^z (u \boxplus v) = \alpha_{z_0}^z (u) \alpha_{z_0}^z (v)$ $(u, v \in \mathcal{X}^*)$, then the Chen series is given by

$$C_{z_0 \rightsquigarrow z} := \sum_{w \in \mathcal{X}^*} \alpha_{z_0}^z (w) w = (\alpha_{z_0}^z \otimes \text{Id}) \mathcal{D} \mathcal{X} = \prod_{l \in \text{Lyn} \mathcal{X}} e^{\alpha_{z_0}^z (S_l) P_l} \in \mathcal{H}(\Omega) \langle \langle \mathcal{X} \rangle \rangle.$$

**Theorem 9**

If $R \in \mathbb{C}^{\text{rat}} \langle \langle \mathcal{X} \rangle \rangle$ with minimal representation of dimension $n$ then

$$y(z_0, z) = \alpha_{z_0}^z (R) = \langle R \| C_{z_0 \rightsquigarrow z} \rangle$$

and there exists $l = 0, \ldots, n - 1$ s.t. $\{\partial^k y\}_{0 \leq k \leq l}$ is $C_0$-linearly independent and $a_l, \ldots, a_1, a_0 \in C_0$ s.t. $$(a_l \partial^l + a_{l-1} \partial^{l-1} + \ldots + a_1 \partial + a_0) y = 0.$$
Chen series and differential equations

For any \( n \geq 0 \), one has \( d^n C_{z_0 \rightsquigarrow z} = p_n C_{z_0 \rightsquigarrow z} \) with

\[
p_n = \sum_{\text{wgtr}=n} \sum_{w \in \mathcal{X}^n} \prod_{i=1}^{\deg r} \left( \sum_{j=1}^i r_j + j - 1 \right)^{r_i} \tau_r(w) \in C_0 \langle \mathcal{X} \rangle,
\]

where, for any word \( w = x_{i_1} \ldots x_{i_k} \in \mathcal{X}^* \) associated to the derivation multi-index \( r = (r_1, \ldots, r_k) \in \mathbb{N}^k \) of degree \( \deg r = |w| \) and of weight \( \text{wgtr} = |w| + \sum_{i=1}^k r_i \)

\( \tau_r(w) := \tau_{r_1}(x_{i_1}) \ldots \tau_{r_k}(x_{i_k}) \).

Proposition 3

Let \( K \) be a compact on \( \Omega \). There is \( c_K \in \mathbb{R}_{\geq 0} \) and a morphism \( M_K \) s.t.

\[
\forall w \in \mathcal{X}^*, \quad \| \langle C_{z_0 \rightsquigarrow z} | w \rangle \|_K \leq c_K M_K(w) |w|^{-1}.
\]

Let \( R \in \mathcal{C}^{\text{rat}} \langle \mathcal{X} \rangle \) s.t. \( \langle R | C_{z_0 \rightsquigarrow z} \rangle \) exists and \( \alpha_{z_0}^z(R) = \langle R | C_{z_0 \rightsquigarrow z} \rangle \). Thus,

\[
\forall y \in \mathcal{X}, \quad \theta_y \alpha_{z_0}^z(R) = \sum_{x \in \mathcal{X}} u_x(z) u_y^{-1}(z) \alpha_{z_0}^z(R \triangleright x).
\]

The following assertions are equivalent:

1. \( \alpha_{z_0}^z(R) \) satisfies a ODE with coefficients in \( (C_0, \partial) \).

2. There is \( p \in C_0 \langle \mathcal{X} \rangle \) s.t. \( \langle R | p C_{z_0 \rightsquigarrow z} \rangle = \langle R \triangleright p | C_{z_0 \rightsquigarrow z} \rangle = 0 \).

16. Considering \( A = (\mathcal{H}(\Omega), \partial) \) as the differential ring of holomorphic functions on \( \Omega \), equipped \( 1_{\Omega} \) as the neutral element, the differential ring \( (\mathcal{H}(\Omega) \langle \mathcal{X} \rangle, d) \) is defined, for any \( S \in \mathcal{H}(\Omega) \langle \mathcal{X} \rangle \), by \( dS = \sum_{w \in \mathcal{X}^*} (\partial(S|w)) w \in \mathcal{H}(\Omega) \langle \mathcal{X} \rangle \).
More about Chen series

Chen series $C_{z_0 \leadsto z}$ of $\{\omega_i\}_{i \geq 1}$ satisfies the following Freidrichs criterion
\[ \forall u, v \in \mathcal{X}^*, \quad \langle C_{z_0 \leadsto z} | u \uplus v \rangle = \langle C_{z_0 \leadsto z} | u \rangle \langle C_{z_0 \leadsto z} | v \rangle. \]

On the other hand, for any $u$ and $v \in \mathcal{X}^*$,
\[ \langle C_{z_0 \leadsto z} | u \rangle \langle C_{z_0 \leadsto z} | v \rangle = \langle \Delta \uplus C_{z_0 \leadsto z} | u \otimes v \rangle. \]

Hence, $\Delta \uplus C_{z_0 \leadsto z} = C_{z_0 \leadsto z} \otimes C_{z_0 \leadsto z}$ and $\langle C_{z_0 \leadsto z} | 1_{\mathcal{X}^*} \rangle = 1$.

Note that $C_{z_0 \leadsto z}$ only depends on the homotopy class of $z_0 \leadsto z$ and the endpoints $z_0, z$. One has $C_{z_0 \leadsto z} C_{z_1 \leadsto z_0} = C_{z_1 \leadsto z}$. Or equivalently\(^{17}\),
\[ \forall w \in \mathcal{X}^*, \quad \langle C_{z_1 \leadsto z} | w \rangle = \sum_{u, v \in \mathcal{X}^*, uv = w} \langle C_{z_0 \leadsto z} | u \rangle \langle C_{z_1 \leadsto z_0} | v \rangle. \]

Note also that, for $g \in \mathcal{H}(\Omega)$, one has $C_{g(z_0) \leadsto g(z)} = g^* C_{z_0 \leadsto z}$, i.e. the Chen series of $\{g^* \omega_i\}_{i \geq 1}$ along the path $g^*(z_0 \leadsto z)$.

**Example 10 (with $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1 - z)^{-1}dz$)**

<table>
<thead>
<tr>
<th>$g(z)$</th>
<th>$z$</th>
<th>$z^{-1}$</th>
<th>$(z - 1)z^{-1}$</th>
<th>$z(z - 1)^{-1}$</th>
<th>$(1 - z)^{-1}$</th>
<th>$1 - z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g^*\omega_0$</td>
<td>$\omega_0$</td>
<td>$-\omega_0$</td>
<td>$-\omega_1 - \omega_0$</td>
<td>$\omega_1 + \omega_0$</td>
<td>$\omega_1$</td>
<td>$-\omega_1$</td>
</tr>
<tr>
<td>$g^*\omega_1$</td>
<td>$\omega_1$</td>
<td>$\omega_1 + \omega_0$</td>
<td>$-\omega_0$</td>
<td>$-\omega_1$</td>
<td>$-\omega_1 - \omega_0$</td>
<td>$-\omega_0$</td>
</tr>
</tbody>
</table>

\(^{17}\) Although $\Delta_{\text{conc}} w = \sum_{u, v \in \mathcal{X}^*, uv = w} u \otimes v$ but $\Delta_{\text{conc}} C_{z_1 \leadsto z} \neq C_{z_0 \leadsto z} \otimes C_{z_1 \leadsto z}$. 
NONCOMMUTATIVE PV THEORY
AND INDEPENDENCE VIA WORDS
Noncommutative differential equations

Considering \( A = (\mathcal{H}(\Omega), \partial) \) as the differential ring of holomorphic functions on \( \Omega \), the differential ring \( (\mathcal{H}(\Omega)\langle \langle X \rangle \rangle, \mathbf{d}) \) is defined, for any \( S \in \mathcal{H}(\Omega)\langle \langle X \rangle \rangle \), by

\[
\mathbf{d}S = \sum_{w \in X^*} (\partial \langle S \mid w \rangle) w \in \mathcal{H}(\Omega)\langle \langle X \rangle \rangle.
\]

The Chen series \( C_{z_0 \rightsquigarrow z} \) satisfies the following differential equation

\[
(\text{NCDE}) \quad \mathbf{d}S = MS, \quad \text{with} \quad M = \sum_{x \in \mathcal{X}} u_x x.
\]

\[
\Delta \uplus M = \sum_{x \in \mathcal{X}} u_x (1 \mathcal{X}^* \otimes x + x \otimes 1 \mathcal{X}^*) = 1 \mathcal{X}^* \otimes M + M \otimes 1 \mathcal{X}^*.
\]

More generally, for any \( k \geq 1 \), \( C_{z_0 \rightsquigarrow z} \) satisfies \( \mathbf{d}^k S = Q_k S \) with \( Q_k \in \mathbb{C}\{\{(u_x^{\pm 1})_{x \in \mathcal{X}}\}\langle \mathcal{X} \rangle\} \) satisfying the recursion

\[
Q_0 = 1 \quad \text{and} \quad Q_k = Q_{k-1} M + \mathbf{d} Q_{k-1}.
\]

\( Q_k \) can be computed explicitly by (summing over words \( w = x_{i_1} \ldots x_{i_k} \) and derivation multiindices \( r = (r_1, \ldots, r_k) \) of degree \( \text{deg} \ r = |w| = k \) and of weight \( \text{wgt} \ r = k + r_1 + \ldots + r_k \))

\[
Q_k = \sum_{\text{wgt} \ r = k, w \in \mathcal{X}^{\text{deg} \ r}} \prod_{j=1}^{\text{deg} \ r} \left( \sum_{j=1}^{r_j} \binom{r_j + j - 1}{r_k} \right) \tau_r(w), \quad \text{where}
\]

\[
\tau_r(w) = \tau_{r_1}(x_{i_1}) \ldots \tau_{r_k}(x_{i_k}) = (\partial^{r_1} u_{x_{i_1}}) x_{i_1} \ldots (\partial^{r_k} u_{x_{i_k}}) x_{i_k} \in \mathbb{C}\{\{(u_x^{\pm 1})_{x \in \mathcal{X}}\}\langle \mathcal{X} \rangle\}.
\]
First step of noncommutative PV theory

1. The space of solutions of

$$(\text{NCDE}) \quad dS = MS, \quad \text{with} \quad M = \sum_{x \in \mathcal{X}} u_x x.$$

is a right free $\mathbb{C}\langle \langle \mathcal{X} \rangle \rangle$-module of rank 1.

2. By a theorem of Ree, $C_{z_0 \sim z}$ is a group-like solution of (NCDE) and it can be obtained as the limit of a convergent Picard iteration, initialized at $\langle C_{z_0 \sim z} | 1_{\mathcal{X}^*} \rangle = 1_{\mathcal{H}(\Omega)} 1_{\mathcal{X}^*}$, for ultrametric distance.

3. If $G$ and $H$ are group-like solutions (NCDE) there is a constant Lie series $C$ such that $G = He^C$ (and conversely).

From this, it follows that

- the differential Galois group of (NCDE) + group-like is the group $\{ e^C \}_{C \in \mathcal{L}ie_{\mathcal{C}.1_{\Omega}} \langle \langle \mathcal{X} \rangle \rangle}$.

Which leads us to the following definition

- the PV extension related to (NCDE) is $\mathcal{C}_{0}.\mathcal{X} \{ C_{z_0 \sim z} \}$.

It, of course, is such that $\text{Const}(\mathcal{C}_0 \langle \langle \mathcal{X} \rangle \rangle) = \ker d = \mathbb{C}.1_{\Omega} \langle \langle \mathcal{X} \rangle \rangle$.

18. In fact, the Hausdorff group (group of characters) of $\mathcal{H} \langle \langle \mathcal{X} \rangle \rangle$. 
Basic triangular theorem over a differential ring

Suppose that the \( \mathbb{C} \)-commutative ring \( A \) is without zero divisors and equipped with a differential operator \( \partial \) such that \( \mathbb{C} = \ker \partial \).

Let \( S \in \mathcal{A}\langle \mathcal{X} \rangle \) be a group-like solution of \( (NCDE) \) in the following form

\[
S = \sum_{w \in \mathcal{X}^*} \langle S | w \rangle w = \sum_{w \in \mathcal{X}^*} \langle S | S_w \rangle P_w = \prod_{l \in \mathcal{L} \mathcal{Y} \mathcal{N} \mathcal{X}} e^{\langle S | S_l \rangle P_l}.
\]

Then

1. If \( H \in \mathcal{A}\langle \mathcal{X} \rangle \) is another group-like solution then there exists \( C \in \mathcal{L} \mathcal{I} \mathcal{E}_A \langle \mathcal{X} \rangle \) such that \( S = He^C \) (and conversely).

2. The following assertions are equivalent
   a) \( \{\langle S | w \rangle\}_{w \in \mathcal{X}^*} \) is \( C_0 \)-linearly independent,
   b) \( \{\langle S | l \rangle\}_{l \in \mathcal{L} \mathcal{Y} \mathcal{N} \mathcal{X}} \) is \( C_0 \)-algebraically independent,
   c) \( \{\langle S | x \rangle\}_{x \in \mathcal{X}} \) is \( C_0 \)-algebraically independent,
   d) \( \{\langle S | x \rangle\}_{x \in \mathcal{X} \cup \{1_{\mathcal{X}^*}\}} \) is \( C_0 \)-linearly independent,
   e) \( \{u_x\}_{x \in \mathcal{X}} \) is such that, for \( f \in \text{Frac}(C_0) \) and \( (c_x)_{x \in \mathcal{X}} \in \mathbb{C}(\mathcal{X}) \),
   \[
   \sum_{x \in \mathcal{X}} c_x u_x = \partial f \implies (\forall x \in \mathcal{X})(c_x = 0).
   \]
   f) \( (u_x)_{x \in \mathcal{X}} \) is free over \( \mathbb{C} \) and \( \partial \text{Frac}(C_0) \cap \text{span}_{\mathbb{C}} \{u_x\}_{x \in \mathcal{X}} = \{0\} \).
Examples of positive cases over $\mathcal{X} = \{x\}$, $\mathcal{A} = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}$, $u_x(z) = 1_\Omega$, $C_0 = \mathbb{C}\{u_x^{\pm1}\} = \mathbb{C}$.
   
   $\alpha_0^z(x^n) = z^n/n!$, for $n \geq 1$. Thus, $dS = xS$ and
   
   $$ S = \sum_{n \geq 0} \alpha_0^z(x^n)x^n = \sum_{n \geq 0} \frac{z^n}{n!}x^n = e^{zx}. $$

   Moreover, $\alpha_0^z(x) = z$ which is transcendent over $C_0$ and the family $\{\alpha_0^z(x^n)\}_{n \geq 0}$ is $C_0$-free. Let $f \in C_0$ then $\partial f = 0$. Thus, if $\partial f = cu_x$ then $c = 0$.

2. $\Omega = \mathbb{C}\setminus(-\infty, 0]$, $u_x(z) = z^{-1}$, $C_0 = \mathbb{C}\{z^{\pm1}\} = \mathbb{C}[z^{\pm1}] \subset \mathbb{C}(z)$.
   
   $\alpha_1^z(x^n) = \log^n(z)/n!$, for $n \geq 1$. Thus $dS = z^{-1}xS$ and
   
   $$ S = \sum_{n \geq 0} \alpha_1^z(x^n)x^n = \sum_{n \geq 0} \frac{\log^n(z)}{n!}x^n = z^x. $$

   Moreover, $\alpha_1^z(x) = \log(z)$ which is transcendent over $\mathbb{C}(z)$ then over $\mathbb{C}[z^{\pm1}]$. The family the family $\{\alpha_1^z(x^n)\}_{n \geq 0}$ is $\mathbb{C}(z)$-free and then $C_0$-free. Let $f \in C_0$ then $\partial f \in \text{span}_{\mathbb{C}}\{z^{\pm n}\}_{n \neq 1}$. Thus, if $\partial f = cu_x$ then $c = 0$. 
Examples of negative cases over $X = \{x\}$, $A = (\mathcal{H}(\Omega), \partial)$

1. $\Omega = \mathbb{C}, u_x(z) = e^z, \mathcal{C}_0 = \mathbb{C}\{\{e^{\pm z}\}\} = \mathbb{C}[e^{\pm z}]$.
   
   $\alpha_0^z(x^n) = (e^z - 1)^n/n!$, for $n \geq 1$. Thus, $dS = e^z x S$ and
   
   $S = \sum_{n \geq 0} \alpha_0^z(x^n)x^n = \sum_{n \geq 0} \frac{(e^z - 1)^n}{n!}x^n = e^{(e^z - 1)x}.$

   Moreover, $\alpha_0^z(x) = e^z - 1$ which is not transcendent over $\mathcal{C}_0$ and
   and $\{\alpha_0^z(x^n)\}_{n \geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = ce^z \in \mathcal{C}_0$ ($c \neq 0$) then
   $\partial f(z) = ce^z = cu_x(z)$.

2. $\Omega = \mathbb{C}\setminus[-\infty, 0], u_x(z) = z^a (a \notin \mathbb{Q}),$
   
   $\mathcal{C}_0 = \mathbb{C}\{\{z, z^{\pm a}\}\} = \operatorname{span}_\mathbb{C} \{z^{ka+l}\}_{k,l \in \mathbb{Z}}.$
   
   $\alpha_0^z(x^n) = (a + 1)^{-n}z^{n(a+1)/n!}$, for $n \geq 1$. Thus, $dS = z^a x S$ and
   
   $S = \sum_{n \geq 0} \alpha_0^z(x^n)x^n = \sum_{n \geq 0} \frac{z^{n(a+1)}}{(a + 1)^nn!}x^n = e^{(a+1)^{-1}z^{(a+1)x}}.$

   Moreover, $\alpha_0^z(x) = z^{a+1}/(a + 1)$ which is not transcendent over $\mathcal{C}_0$
   and $\{\alpha_0^z(x^n)\}_{n \geq 0}$ is not $\mathcal{C}_0$-free. If $f(z) = cz^{a+1}/(a + 1) \in \mathcal{C}_0$
   ($c \neq 0$) then $\partial f(z) = cz^a = cu_x(z)$.
Chen series of $\omega_0(z) = z^{-1}dz$ and $\omega_1(z) = (1 - z)^{-1}dz$

Let $\gamma_0(\varepsilon)$ and $\gamma_1(\varepsilon)$ be the circular paths of radius $\varepsilon$ encircling 0 and 1 clockwise, respectively. In particular, letting $\beta = \beta_1 - \beta_0$, one considers

$$
\gamma_0(\varepsilon, \beta) = \varepsilon e^{i\beta_0} \mapsto \varepsilon e^{i\beta_1} \subset \gamma_0(\varepsilon), \\
\gamma_1(\varepsilon, \beta) = 1 - \varepsilon e^{i\beta_0} \mapsto 1 - \varepsilon e^{i\beta_1} \subset \gamma_1(\varepsilon).
$$

On the one hand, one has, for any $i = 0$ or $1$ and $w \in X^+$,

$$
|\langle C_{\gamma_i(\varepsilon, \beta)}|w\rangle| \leq \varepsilon^{\|w\|_{\mathcal{M}} |\beta|_{\mathcal{M}}} |w|^{-1}.
$$

It follows then

$$
C_{\gamma_i(\varepsilon, \beta)} = e^{i\beta x_i} + o(\varepsilon) \quad \text{and} \quad C_{\gamma_i(\varepsilon)} = e^{2i\pi x_i} + o(\varepsilon).
$$

On the other hand, for $R \in \mathcal{C}^{\text{rat}} \langle \langle X \rangle \rangle$ of minimal representation $(\lambda, \mu, \eta)$ of dimension $n$, one has, for any $w \in X^*$,

$$
|\langle R|w\rangle| \leq \|\lambda\|_{\infty}^{1,n} \|\mu(w)\|_{\infty}^{n,n} \|\eta\|_{\infty}^{n,1}.
$$

Hence,

$$
\alpha^z_{z_0}(R) := \langle R \parallel C_{z_0 \mapsto z} \rangle = \lambda((\alpha^z_{z_0} \otimes \mu) \mathcal{D} X) \eta = \lambda\left( \prod_{l \in \mathcal{L} X^*} e^{\alpha^z_{z_0}(S_l)\mu(P_l)} \right) \eta.
$$

Note that the map $\alpha^z_{z_0} : \mathcal{C}^{\text{rat}} \langle \langle X \rangle \rangle \to \mathcal{H}(\Omega)$ is not injective. For example,

$$
\alpha^z_{z_0}(z_0 x_0^* + (1 - z_0)(-x_1)^* - 1 x^*) = 0.
$$
Let $C_C := \mathbb{C}[z^a, (1 - z)^b]_{a,b \in \mathbb{C}}$ be equipped with $\partial$.

**Proposition 4**

Let $\text{Dom}(\text{Li}_\bullet)$ be the set of $S = \sum_{n \geq 0} S_n$ with $S_n = \sum_{|w| = n} \langle S| w \rangle w$ s.t. $\sum_{n \geq 0} \text{Li}_n S_n$ converges uniformly on any compact of $\Omega$. Then $\text{Dom}(\text{Li}_\bullet)$, containing $\mathbb{C}^{\text{rat exc}} (\langle X \rangle) \shuffle \mathbb{C} \langle X \rangle$, is closed by shuffle. Moreover, $\forall S, T \in \text{Dom}(\text{Li}_\bullet)$, $\text{Li}_S \shuffle T = \text{Li}_S \text{Li}_T$.

For $R \in \text{Dom}(\text{Li}_\bullet)$, let $\rho := \langle R||L \rangle$. Then, $\forall n \geq 0$, $\partial^n \rho = \langle R||d^n L \rangle$ and

$$d^n L = p_n L \quad \text{with} \quad p_n = \sum \sum \prod_{w \in X^n} \left( \sum_{j=1}^{i} r_j + j - 1 \right)^{r_i} \tau_r(w) \in C \langle X \rangle,$$

where, for any word $w = x_{i_1} \ldots x_{i_k} \in X^*$ associated to the derivation multi-index $r = (r_1, \ldots, r_k) \in \mathbb{N}^k$ of degree $\deg r = |w|$ and of weight $\text{wgt } r = |w| + \sum_{i=1}^{k} r_i$, $\tau_r(w) := \tau_{r_1}(x_{i_1}) \ldots \tau_{r_k}(x_{i_k})$ and, for any $r \geq 0$, $\tau_r(x_0) = -r!(-z)^{-(r+1)} x_0$ and $\tau_r(x_1) = r!(1 - z)^{-(r+1)} x_1$.

**Proposition 5**

The following assertions are equivalent:

1. $\rho$ satisfies a differential equation with coefficients in $(C_C, \partial)$.
2. There exists $P \in C_C \langle X \rangle$ such that $\langle R||PL \rangle = \langle R \triangleright P||L \rangle = 0$. 

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