CCRT2 : Universal Problems, heteromorphisms and adjunctions.

From combinatorics of universal problems
to usual applications.

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Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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CIP seminar, 27 November 2020

Goal of this talk

The goal of this talk is threefold

A bit of category theory: How to construct free objects w.r.t. a functor and two routes to reach the free algebra.

Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients

MRS factorisation: A local system of coordinates for Hausdorff groups

Bits and pieces of representation theory

and how bialgebras arise

Wikipedia says

Representation theory is a branch of mathematics that studies abstract algebraic structures by representing their elements as linear transformations of vector spaces, .../...
The success of representation theory has led to numerous generalizations. One of the most general is in category theory.

As our track is based on Combinatorial Physics and Experimental/Computational Mathematics, we will have a practical approach of the three main points of view

- Algebraic
- Geometric
- Combinatorial
- Categorical

Universal problem w.r.t. a functor

Free structures and objects

1. Let $\mathcal{C}_{left}$, $\mathcal{C}_{right}$ be two categories and $F : \mathcal{C}_{right} \to \mathcal{C}_{left}$ a (covariant) functor between them

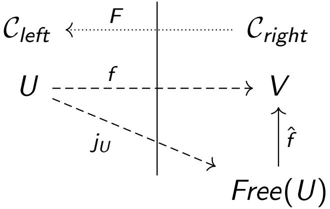


Figure: A solution of the universal problem w.r.t. the functor F is the datum, for each $U \in \mathcal{C}_{left}$, of a pair $(j_U, Free(U))$ (with $j_U \in Hom(U, F[Free(U)])$, $Free(U) \in \mathcal{C}_{right}$) such that, for all $f \in Hom(U, F[V])$ it exists a unique $\hat{f} \in Hom(Free(U), V)$ with $F[\hat{f}] \circ j_U = f$. Elements in $Hom(U, F[V])$ are called heteromorphisms their set is noted $Het_F(U, V)$.

$$\Big(\forall f \in Hom(U, F[V])\Big)\Big(\exists!\ \hat{f} \in Hom(Free(U), V)\Big)\Big(F(\hat{f}) \circ j_U = f\Big)$$

The pair $U \to Free(U)$ is, in fact, a functor.

Which, in turn, will prove to be left-adjoint to F

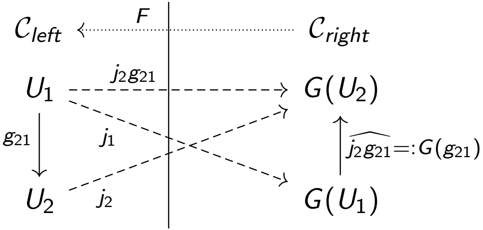


Figure: Making a free functor G ($= Free$) from F: for any morphism $g_{21} \in Hom(U_1, U_2)$, $G(g_{21})$ is the unique morphism in $Hom(U_1, U_2)$ such that

$$F[G(g_{21})] \circ j_1 = j_2 g_{21} \qquad (**)$$

We now prove that G is a functor.

- If $U_1 = U_2$ and $g_{21} = Id_{U_1}$, then $j_1 = j_2 = j_2 g_{21}$ and $F[Id_{G(U_1)}] \circ j_1 = j_1 = j_2 g_{21}$ hence $G[Id_{U_1}] = Id_{G(U_1)}$
- **A remark**: $Het(?,?)$ is intended to give a symmetric middle term/step to the adjunction chain $Hom(U, F[V]) =: Het_F(U,V) \simeq Het^G(U,V) := Hom(G(U), V)$ $\simeq$ being constructed by a set of bijections.

Functor G from $Free/2$

- Let now $U_1 \overset{g_{21}}{\hookrightarrow} U_2 \overset{g_{32}}{\hookrightarrow} U_3$ be a chain of C_{left}-morphisms.
  We have

$$F[G(g_{21})] \circ j_1 = j_2 \circ g_{21} \text{ and } F[G(g_{32})] \circ j_2 = j_3 \circ g_{32}$$

then

$$F[G(g_{32}) \circ G(g_{21})] \circ j_1 \overset{(1)}{=} F[G(g_{32})] \circ F[G(g_{21})] \circ j_1 \overset{(2)}{=}$$
$$F[G(g_{32})] \circ j_2 \circ g_{21} \overset{(3)}{=} j_3 \circ g_{32} \circ g_{21}$$

(1) because F is a functor, (2) is Eq. (**) applied to indices 21,
(3) is Eq. (**) applied to indices 32.
Now, we know that $g \in Hom(U, U')$ being given, the solution
$X \in Hom(G(U), G(U'))$ of

$$F[X] \circ j_1 = j_2 \circ g$$

is unique. Then $G(g_{32}) \circ G(g_{21}) = G(g_{32} \circ g_{21})$ □

Composition of functors F and G

Piling free structures.

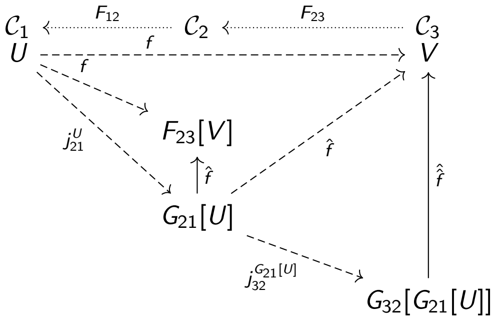
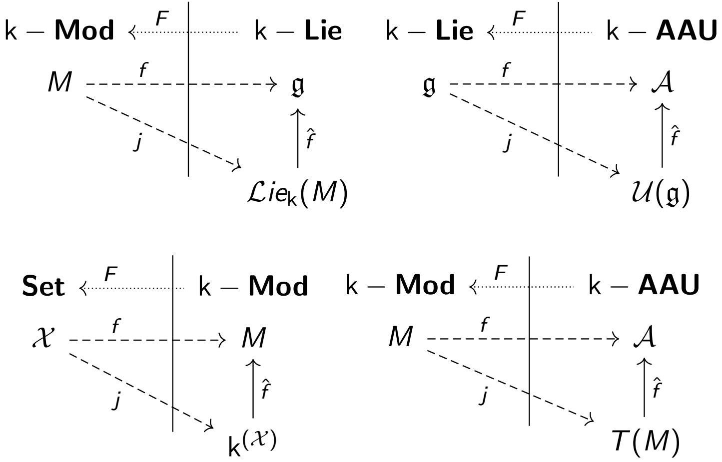


Figure: $\left[F_{12}[j_{32}^{G_{21}[U]}], G_{32}[G_{21}[U]]\right]$ is a solution of the universal problem for $F_{12}F_{23}$.

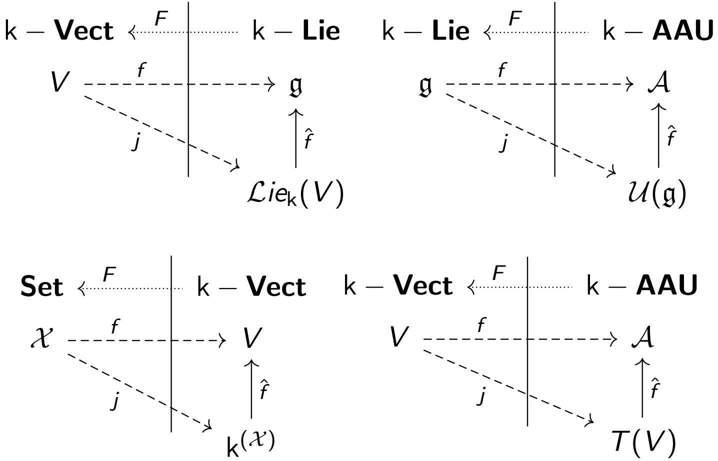
Proof: In fact, $Het_{F_{12}F_{23}}(U,V) = Hom(U, F_{12}F_{23}[V]) = Het_{F_{12}}(U, F_{23}[V])$, hence existence of $\hat{f} \in Hom(G_{21}[U], F_{23}[V]) = Het_{F_{23}}(G_{21}[U], V)$, hence again existence of $\hat{\hat{f}}$. Uniqueness of $\hat{\hat{f}}$ is left to the reader.

First example: $T = UL$.



$$T(M) = \mathcal{U}(\mathcal{L}ie_{\mathrm{k}}(M)) \qquad \mathrm{k}\langle\mathcal{X}\rangle = T(\mathrm{k}^{(\mathcal{X})})$$

First example: $T = UL$, k field.



$$T(V) = \mathcal{U}(\mathcal{L}ie_{\mathrm{k}}(V)) \qquad \mathrm{k}\langle\mathcal{X}\rangle = T(\mathrm{k}^{(\mathcal{X})})$$

An immediate (and although rich) example/2

Piling free structures/2

2. First, $\mathcal{C}_1 = \mathbf{Set}$ (sets and maps) and $\mathcal{C}_2 = \mathbf{Mon}$ (monoids and morphisms) gives you the triple $(\mathcal{X}, j_{21}, \mathcal{X}^*)$

Usually $\mathcal{X}$, a set, is seen as an *alphabet* that is to say a *set of non commuting variables.* Let us introduce the ring k of coefficients

3. With $\mathcal{C}_2 = \mathbf{Mon}$ (monoids and morphisms) and $\mathcal{C}_3 = \mathrm{k} - \mathbf{AAU}$ (k-associative algebras with unit), one gets $\mathrm{k}[M]$ the algebra of a monoid M, we get the triple $(M, j_{32}, \mathrm{k}[M])$ and,
4. by transitivity of free objects with $\mathcal{C}_1 = \mathbf{Set}$ (sets and maps) and $\mathcal{C}_3$ as above, we get the triple $(\mathcal{X}, j_{31}, \mathrm{k}\langle\mathcal{X}\rangle)$, $\mathrm{k}\langle\mathcal{X}\rangle = \mathrm{k}[\mathcal{X}^*]$ being the algebra of noncommutative polynomials.
5. we immediately obtain that $\mathrm{k}\langle\mathcal{X}\rangle = \mathrm{k}[\mathcal{X}^*]$ is free with $\{w\}_{w\in\mathcal{X}^*}$ (this will be useful for the principal pairing)
6. Let us observe here that $\mathrm{k}\langle\mathcal{X}\rangle$ can be reached, instead of (Sets→Mon→k-AAU) by another path, and this will provide a host of other very interesting (combinatorial) bases.

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9. by transitivity of free objects with $\mathcal{C}_1 = \mathbf{Set}$ (sets and maps) and $\mathcal{C}_3$ as above, we get the triple $(\mathcal{X}, j_{31}, \mathrm{k}\langle\mathcal{X}\rangle)$, $\mathrm{k}\langle\mathcal{X}\rangle = \mathrm{k}[\mathcal{X}^*]$ being the algebra of noncommutative polynomials.
10. we immediately obtain that $\mathrm{k}\langle\mathcal{X}\rangle = \mathrm{k}[\mathcal{X}^*]$ is free with $\{w\}_{w\in\mathcal{X}^*}$ (this will be useful for the principal pairing)
11. Let us observe here that $\mathrm{k}\langle\mathcal{X}\rangle$ can be reached, instead of

$$[\mathbf{Set}] \longrightarrow [\mathbf{Mon}] \longrightarrow [\mathrm{k} - \mathbf{AAU}]$$

An immediate (and although rich) example/2

Piling free structures and dual bases

12. The preceding route amounts to the formula $\mathrm{k}\langle\mathcal{X}\rangle = \mathrm{k}[\mathcal{X}^*]$, but it can be also proved that $\mathrm{k}\langle\mathcal{X}\rangle = \mathcal{U}(\mathcal{L}ie_{\mathrm{k}}\langle X\rangle)$

$$[\mathbf{Set}] \longrightarrow [\mathrm{k}-\mathbf{Lie}] \longrightarrow [\mathrm{k}-\mathbf{AAU}]$$

13. From the first (obvious) way (sets to monoids to k-AAU) we got the basis $\{w\}_{w\in\mathcal{X}^*}$ which provides the fine grading of $\mathrm{k}\langle\mathcal{X}\rangle$. indeed to each word $w \in \mathcal{X}^*$, we can associate the family

$$\beta(w) = (|w|_x)_{x\in\mathcal{X}} \in \mathbb{N}^{(\mathcal{X})}$$

14. Therefore, due to this partitioning of the basis (of words), we get

$$\mathrm{k}\langle\mathcal{X}\rangle = \bigoplus_{\alpha\in\mathbb{N}^{(\mathcal{X})}} \mathrm{k}_\alpha\langle\mathcal{X}\rangle \qquad (1)$$

where $\mathrm{k}_\alpha\langle\mathcal{X}\rangle := span_{\mathrm{k}}\{w|\beta(w) = \alpha\}$.

An immediate (and although rich) example/3

Graded bases through free Lie algebra

15 Each $\mathrm{k}_\alpha\langle\mathcal{X}\rangle$ is free of dimension $\dfrac{|\alpha|\,!}{\alpha\,!}$; for example with two letters a, b, we have

$$\mathrm{k}\langle\mathcal{X}\rangle = \bigoplus_{(p,q)\in\mathbb{N}^2} \mathrm{k}_{(p,q)}\langle\mathcal{X}\rangle$$

and $\dim(\mathrm{k}_{(p,q)}\langle\mathcal{X}\rangle) = \dfrac{(p+q)\,!}{p\,!q\,!} = \begin{pmatrix} p \\ q \end{pmatrix}$.

16 This fine grading is a grading of algebra as

$$\mathrm{k}_\alpha\langle\mathcal{X}\rangle \mathrm{k}_\beta\langle\mathcal{X}\rangle \subset \mathrm{k}_{\alpha+\beta}\langle\mathcal{X}\rangle \ ; \ 1_{\mathcal{X}^*} \in \mathrm{k}_0\langle\mathcal{X}\rangle \qquad (2)$$

17 Now through the second route (sets-Lie-AAU), we can construct many finely homogeneous bases of $\mathrm{k}\langle\mathcal{X}\rangle$ using the following scheme

- Pick any finely homogeneous basis of $\mathcal{L}ie_{\mathrm{k}}\langle X\rangle$, $(P_i)_{i\in I}$ (we will construct at least one)
- (Totally) order I and form the PBW basis (of $\mathrm{k}\langle\mathcal{X}\rangle$). it is finely homogeneous (due to eq. 2).

Next steps

1. Semi-simple categories of modules
2. Link with non-degenerate bilinear forms + examples

An immediate (and although rich) example/4

Words and Lyndon words, details.

Algebraic structure

- Concatenation: This law is noted *conc*
- With the empty word as neutral, the set of words is the free monoid $(X^*, conc, 1_{X^*})$
- The pairing between series and polynomials is defined by

$$\langle S|P\rangle = \sum_{w\in\mathcal{X}^*} \langle S|w\rangle\langle P|w\rangle$$

Coding by words gives access to a welter of structures, studies, relations and results (algebra, geometry, topology, probability, combinatorics on words, on polynomials and series). We will use in particular their complete factorisation by Lyndon words.

An immediate (and although rich) example/5

Words and classes

Example with $\mathcal{X}=\{a,b\},\ a<b$, in red Lyndon words ($=\mathcal{L}yn\mathcal{X}$).

Length	words
0	1_{X^*}
1	$a,\ b$
2	$aa,\ ab,\ ba,\ bb$
3	$aaa,\ aab,\ aba,\ abb,\ baa,\ bab,\ bba,\ bbb$
4	$a^4,\ a^3b,\ a^2ba,\ a^2b^2,\ aba^2,\ abab,\ ab^2a,\ ab^3$
	$ba^3,\ ba^2b,\ baba,\ babb,\ b^2a^2,\ b^2ab,\ b^3a,\ b^4$

Two properties of Lyndon words

1. All $\ell\in\mathcal{L}yn\mathcal{X}\setminus\mathcal{X}$ factorises (not uniquely in general) as $\ell=\ell_1\ell_2,\ \ell_1\prec\ell_2,\ \ell_i\in\mathcal{L}yn\mathcal{X}$ (ex. $a^3ba^2bab=a^3b|a^2bab=a^3ba^2b|ab$), the one with the longest right factor will be called standard $\sigma(\ell)=(\ell_1,\ell_2)$.
2. Every word $w\in\mathcal{X}^*$ factorises uniquely $w=\ell_1^{i_1}\ldots\ell_k^{i_k}$ with $\ell_1\succ\ldots\succ\ell_k,(\ell_i\in\mathcal{L}ynX)$

An immediate (and although rich) example/6

Shuffle product(s)

Non deformed case

Coming from the route where $k\langle\mathcal{X}\rangle = \mathcal{U}(\mathcal{L}ie_k\langle X\rangle)$, we have a structure of coalgebra on $k\langle\mathcal{X}\rangle$ its comultiplication is given by its value on letters

$$\Delta_{ш}(x) = x \otimes 1_{\mathcal{X}^*} + 1_{\mathcal{X}^*} \otimes x \tag{3}$$

Then shuffle product is defined as a dual law, for each $w \in \mathcal{X}^*$ by

$$\langle P ш Q | w\rangle = \langle P \otimes Q | \Delta_{ш}(w)\rangle \tag{4}$$

We get the following recursion for shuffle products

$$w ш 1_{\mathcal{X}^*} = 1_{\mathcal{X}^*} ш w = w \qquad \text{for any word } w \in \mathcal{X}^*; \tag{5}$$

$$au ш bv = a(u ш bv) + b(au ш v) \tag{6}$$

An immediate (and although rich) example/7

Two bases in duality/1: Combinatorial constructions

Lyndon basis

$$\begin{array}{rcl} P_x = & x & \text{for } x \in X, \\ P_\ell = & [P_s, P_r] & \text{for } \ell \in \mathcal{L}yn\mathcal{X} \setminus \mathcal{X} \text{ and } \sigma(\ell) = (s, r), \\ P_w = & P_{\ell_1}^{i_1} \ldots P_{\ell_k}^{i_k} & \text{for } w = \ell_1^{i_1} \ldots \ell_k^{i_k}, \ell_1 \succ \ldots \succ \ell_k,\ (\ell_i \in \mathcal{L}yn\mathcal{X}). \end{array}$$

where $\succ$ stands for the lexicographic (strict) ordering defined from $x_0 \prec x_1$.

Triangular property

Indeed $\{P_w\}_{w \in X^*}$ is lower unitriangular w.r.t. words (this property, joined with the fact that this family is finely homogeneous, implies that $\{P_w\}_{w \in X^*}$ is a basis of $k\langle\mathcal{X}\rangle$)

$$P_w = w + \sum_{v \succ w, \beta(v) = \beta(w)} c_v v \text{ with } c_v \in \mathbb{Z} \tag{7}$$

Dual basis

Construction of $(S_w)_{w\in\mathcal{X}^\alpha}$

For each multidegree α, let $\mathcal{X}^\alpha$ be the (finite) set of words with multidegree α and T_α be the lower unitriangular matrix of $\{P_w\}_{\beta(w)=\alpha}$ w.r.t. words of $\mathcal{X}^\alpha$ then, the matrix *transpose*(T^{-1}) defines a family $(S_w)_{w\in\mathcal{X}^\alpha}$ such that

1. $S_w = w + \sum_{v\prec w,\beta(v)=\beta(w)} d_v v$ with $d_v \in \mathbb{Z}$
2. For all $u, v \in \mathcal{X}^\alpha$, $\langle S_u|P_v\rangle = \delta_{u,v}$.
3. The quantification of the preceding property can be extended to all $u, v \in \mathcal{X}^*$ due to the fact that the decomposition (1) is, in fact, orthogonal.

Schützenberger's basis (k is a $\mathbb{Q}$-algebra)

M. -P. Schützenberger proved that, when k is a $\mathbb{Q}$-algebra, the basis $(S_w)_{w\in\mathcal{X}^*}$ can be computed recursively as follows

$$\begin{aligned} S_x &= x && \text{for } x \in \mathcal{X}, \\ S_l &= xS_u, && \text{for } l = xu \in \mathcal{L}yn\mathcal{X} \setminus \mathcal{X}, \\ S_w &= \frac{S_{l_1}^{\,\sqcup\!\sqcup\, i_1} \sqcup\!\sqcup \ldots \sqcup\!\sqcup S_{l_k}^{\,\sqcup\!\sqcup\, i_k}}{i_1! \ldots i_k!} && \text{for } w = l_1^{i_1} \ldots l_k^{i_k}, l_1 \succ \ldots \succ l_k. \end{aligned}$$

Triangular properties (recall)

$$P_w = w + \sum_{v \succ w, \beta(v)=\beta(w)} c_v v \quad \text{and} \quad S_w = w + \sum_{v \prec w, \beta(v)=\beta(w)} d_v v. \quad (8)$$

We recall that the bases $\{S_w\}_{w\in X^*}$ and $\{P_w\}_{w\in X^*}$ are lower and upper triangular respectively and that they are (finely) graded (all the monomials have the same partial degrees).

Table of these bases

Example (First values)

Let $X = \{x_0, x_1\}$ with $x_0 \prec x_1$.

l	P_l	S_l
x_0	x_0	x_0
x_1	x_1	x_1
x_0x_1	$[x_0, x_1]$	x_0x_1
$x_0^2x_1$	$[x_0, [x_0, x_1]]$	$x_0^2x_1$
$x_0x_1^2$	$[[x_0, x_1], x_1]$	$x_0x_1^2$
$x_0^3x_1$	$[x_0, [x_0, [x_0, x_1]]]$	$x_0^3x_1$
$x_0^2x_1^2$	$[x_0, [[x_0, x_1], x_1]]$	$x_0^2x_1^2$
$x_0x_1^3$	$[[[x_0, x_1], x_1], x_1]$	$x_0x_1^3$
$x_0^4x_1$	$[x_0, [x_0, [x_0, [x_0, x_1]]]]$	$x_0^4x_1$
$x_0^3x_1^2$	$[x_0, [x_0, [[x_0, x_1], x_1]]]$	$x_0^3x_1^2$
$x_0^2x_1x_0x_1$	$[[x_0, [x_0, x_1]], [x_0, x_1]]$	$2x_0^3x_1^2 + x_0^2x_1x_0x_1$
$x_0^2x_1^3$	$[x_0, [[[x_0, x_1], x_1], x_1]]$	$x_0^2x_1^3$
$x_0x_1x_0x_1^2$	$[[x_0, x_1], [[x_0, x_1], x_1]]$	$3x_0^2x_1^3 + x_0x_1x_0x_1^2$
$x_0x_1^4$	$[[[[x_0, x_1], x_1], x_1], x_1]$	$x_0x_1^4$

Factorisation of the diagonal as a resolution of identity.

Resolution of identity as an infinite product

Now we are in the position of writing the principal factorisation of the diagonal series. In here, series multiply by shuffle on the left and concatenation on the right.

$$\mathcal{D}_X := \sum_{w\in X^*} w\otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l\otimes P_l) \tag{9}$$

Application to factorisation of characters

If we have a shuffle-character $\chi : \ (\mathrm{k}\langle\mathcal{X}\rangle, ш, 1_{\mathcal{X}^*}) \to \mathcal{A}$, we act on the left

$$\chi = \sum_{w\in X^*} \chi(w)\otimes w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(\chi(S_l)\otimes P_l) \tag{10}$$

But with a conc-character $\chi : \ (\mathrm{k}\langle\mathcal{X}\rangle, conc, 1_{\mathcal{X}^*}) \to \mathcal{A}$, we act on the right

$$\chi = \sum_{w\in X^*} w\otimes \chi(w) = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l\otimes \chi(P_l)) \tag{11}$$

Conclusion

1. The values of iterated integrals (standard of regularized) are shuffle-characters, then we have factorisations and they constitute multiplicative regularizations.
2. The values of matrix representations of the free monoid (as the transitions of rational series for instance) are conc-characters and we get useful factorizations of them.
3. In the next talk (friday morning ?), we will see the deformed case through CQMM and applications to harmonic sums.

Thank you for your attention.

Links

1. Categorical framework(s)

   https://ncatlab.org/nlab/show/category
   https://en.wikipedia.org/wiki/Category_(mathematics)

2. Universal problems

   https://ncatlab.org/nlab/show/universal+construction
   https://en.wikipedia.org/wiki/Universal_property

3. Paolo Perrone, *Notes on Category Theory with examples from basic mathematics*, 181p (2020)
   arXiv:1912.10642 [math.CT]

   https://en.wikipedia.org/wiki/Abstract_nonsense

4. Heteromorphism

   https://ncatlab.org/nlab/show/heteromorphism

5. D. Ellerman, *MacLane, Bourbaki, and Adjoints: A Heteromorphic Retrospective*, David EllermanPhilosophy Department, University of California at Riverside

Links/2

6. https://en.wikipedia.org/wiki/Category_of_modules
7. https://ncatlab.org/nlab/show/Grothendieck+group