CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems to usual applications.

G.H.E. Duchamp

Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
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C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough.

CIP seminar,

Friday conversations:

For this seminar, please have a look at Slide CCRT[n] & ff.

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
    1. w.r.t. a functor with - at least - two combinatorial applications:
        1. the two routes to reach the free algebra
        2. alphabets interpolating between commutative and non commutative worlds
    2. without functor: sums, tensor and free products
    3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

CCRT[16] Higher order BTT (part 3).

One-parameter groups and identities among series.

Disclaimer. – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

Introduction

1. Today, we will use the same analysis/synthesis method as in CCRT[16] (part one) and use the information gathered to consider solutions of the BTT as paths drawn on closed subgroups on the Magnus group.
2. The mental process for the making of the BTT [9] on various subgroups will be the following

$$\underbrace{\text{Integrating the obervable} \to \ \text{Differentiation} \to}_{\textit{Analysis}}$$
$$\underbrace{\text{Technical condition} \to \ \text{NSC} \to \ \text{Proof \& Boundaries}}_{\textit{Synthesis / Integration}}$$

3. This method is not new, it is that of Archimedes (-287, -212) [1], Liu Hui (220-280) [18] and Cavalieri (1598-1647) [6]. Archimedes work was originally thought to be lost, but in 1906 was rediscovered in the celebrated Archimedes Palimpsest.

Limiting processes and topologies/1

1. We have seen last time some limiting processes (like Riemann integral and Lebesgues y-axis sampling) which are not reducible to sequences, (we will return to this point later on).
2. In order to understand deeply and master our calculations with group-like series (of all sorts not only for the co-shuffle coproduct), we have to deal with closed subgroups of the Magnus group.
3. Let us first examine and analyse some simple limits of sequences of series.
4. We first address the following identity

$$\lim_{n\to+\infty}(1+\frac{z}{n})^n = e^z \tag{1}$$

Which can be considered within the formal realm (i.e. LHS, for each n, within $\mathbb{C}\langle z\rangle = \mathbb{C}[z]$ and RHS within $\mathbb{C}\langle\langle z\rangle\rangle = \mathbb{C}[[z]]$) or in $\mathcal{H}(\mathbb{C})$ with compact convergence.

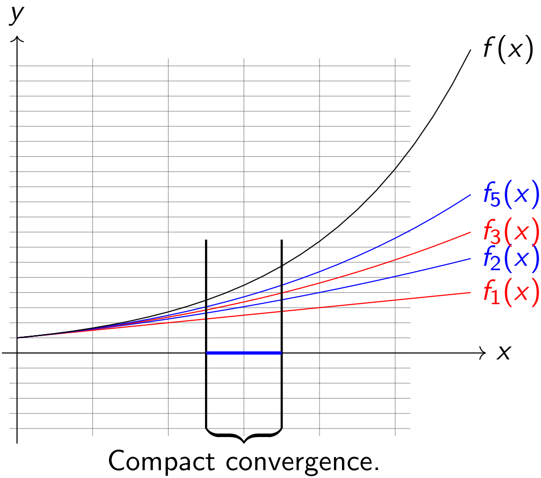

Figure: The one-parameter group $f(x) = \mathrm{e}^{\frac{x}{2}}$ as the limit of $f_n(x) = (1 + x/(2n))^n$.

Limiting processes and topologies/2

5. In fact, a variant of (1)[a] was used by Montgomery and Zippin to solve Hilbert's fifth problem [29].
6. (Informal) definition:[b] A one-parameter group, is a correspondence G to some group such that

$$G(t_1 + t_2) = G(t_1)G(t_2)$$

7. In fact, we are interested in creating a new theory of
   1. Paths drawn on groups of series
   2. One-parameter groups on infinite-dimensional Lie groups of series and their combinatorics.
   3. We use an application to stuffle identity, introducing a "Holomorphic functional calculus" [15] in order to get and prove non-trivial identities within Hausdorff groups.

[a] In fact, the construction of one-parameter groups as limits of this kind.

[b] Informal, means here "at the level of general idea".

Every path drawn on the group is a solution of $y'(t) = m(t)y(t)$

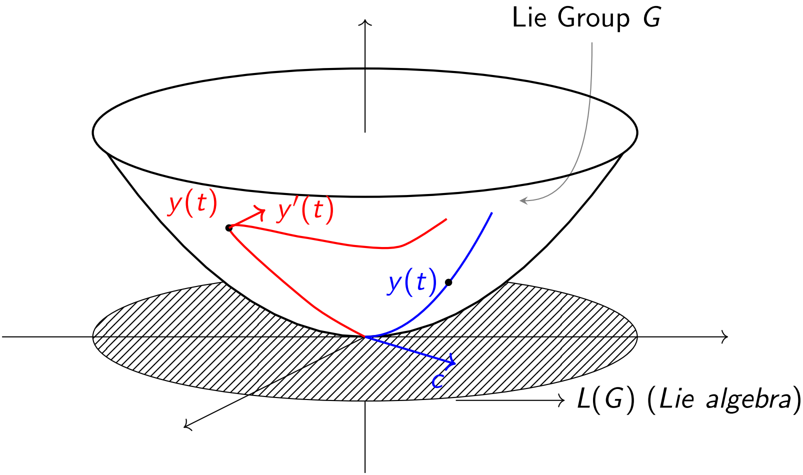

Figure: For one-parameter groups $y'(t)y(t)^{-1} = c$ is constant.

An identity in the stuffle algebra/1

8. We begin by an application on the Hausdorff group of a particular bialgebra. Here, with $Y = \{y_i\}_{i\geq 1}$

$$\mathcal{B} = \mathcal{B}_{\uplus} = (\underbrace{\mathcal{C}\langle Y\rangle, \mathtt{conc}, 1_{Y^*}}_{\text{algebra part}}, \Delta_{\uplus}, \epsilon) \tag{2}$$

and we first establish an identity within the stuffle algebra, taking "stars of the plane" as arguments.

$$(\sum_{i\geq 1} \alpha_i\, y_i)^* \uplus (\sum_{j\geq 1} \beta_j\, y_j)^* = (\sum_{i\geq 1} \alpha_i\, y_i + \sum_{j\geq 1} \beta_j\, y_j + \sum_{i,j\geq 1} \alpha_i\beta_j\, y_{i+j})^* \tag{3}$$

As the alphabet is infinite, we use here homogeneous series of degree one as $\sum_{i\geq 1} \alpha_i\, y_i$. These sums are not necessarily finite (they are, in general, a series) but can be so. Series like this form the vector space $\mathbb{C}^Y$ (called by Pr. Schützenberger "the plane of letters"), noted, in our works, $\widehat{\mathbb{C}.Y}$ as it is the completion of $\mathbb{C}.Y = \mathbb{C}^{(Y)}$ for some topology.

An identity in the stuffle algebra/2: Generalities

9. In fact, identity (3) describes completely the composition of characters (i.e. the composition within $\Xi(\mathcal{B})$). In fact $\mathcal{B}_{\uplus}$ (see its elements in eq. 2) is a conc-bialgebra and conc-characters are exactly "stars of the plane" i.e., for generic $\mathcal{X}$, of the form $(\sum_{x\in\mathcal{X}} \alpha_x\, x)^*$.
10. We recall that $\Delta_{\uplus}(y_n) = y_n \otimes 1 + 1 \otimes y_n + \sum_{\substack{p,q\geq 1\\ p+q=n}} y_p \otimes y_q$.
11. In fact this comultiplication is a particular case of $\Delta_{ш_\varphi}$ comultiplications which read, for each letter $x \in \mathcal{X}$ (see [13]),

$$\Delta_{ш_\varphi}(x) = x \otimes 1 + 1 \otimes x + \sum_{y,z\in\mathcal{X}} \gamma_x^{y,z}\, y \otimes z \tag{4}$$

where the tensor $\gamma_x^{y,z}$ is locally finite in x.

12. For these conc-bialgebras, we have in general

$$(\sum_{y\in\mathcal{X}} \alpha_y\, y)^* ш_\varphi (\sum_{z\in\mathcal{X}} \beta_z\, z)^* = (\sum_{y\in\mathcal{X}} \alpha_y\, y + \sum_{z\in\mathcal{X}} \beta_z\, z + \sum_{x,y,z\in\mathcal{X}} \alpha_y \beta_z\, \gamma_x^{y,z} x)^* \tag{5}$$

An identity in the stuffle algebra/3: Generalities

13. One proof of (5) rests on the fact that the algebra is generated by $\mathcal{X}$ and, then, we have just, knowing the form of the LHS-RHS, to test equality on letters. Let us recall some definitions and properties (**k** is a commutative ring)
    1. Let $\mathcal{B} = (\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \epsilon)$ be a bialgebra.
    2. We call $\Xi(\mathcal{B})$ the set of characters of $(\mathcal{B}, , \mu, 1_{\mathcal{B}})$ (with values in **k**)
    3. When $\mathcal{C}$ is another **k**-algebra, we will note $\Xi(\mathcal{B}; \mathcal{C})$, the set of characters of $\mathcal{B}$ with values in $\mathcal{C}$.[a]
14. One can show that, if $\mathcal{C}$ is commutative, characters compose through convolution. Indeed, the dual $\mathcal{B}^{\vee}$ (now $\mathcal{C} = \mathbf{k}$) is an algebra under ${}^{t}\Delta$ (which will be noted $\circledast$) and $\Xi(\mathcal{B}) \subset \mathcal{B}^{\vee}$ is closed under $\circledast$.

[a] This set is none other than the Hom-set of the algebras, i.e. we have truly

$$\Xi(\mathcal{B}; \mathcal{C}) = Hom_{\mathbf{k}-\mathbf{AAU}}(\mathcal{B}, \mathcal{C})$$

but the point of view is commpletely different.

Some exercise about these generalities

15 Let **k** be a commutative ring and $\mathcal{B} = (\mathcal{B}, \mu, 1_{\mathcal{B}}, \Delta, \epsilon)$ be a **k**-bialgebra. As $\Delta : \ \mathcal{B} \longrightarrow \mathcal{B} \otimes \mathcal{B}$, we have ${}^t\Delta : (\mathcal{B} \otimes \mathcal{B})^{\vee} \longrightarrow \mathcal{B}^{\vee}$

16 (Q1) Explain the arrow

$$can : \ \mathcal{B}^{\vee} \otimes \mathcal{B}^{\vee} \ \longrightarrow \ (\mathcal{B} \otimes \mathcal{B})^{\vee} \tag{6}$$

and prove that ${}^t\Delta \circ can$ is a law of $\mathbf{k} - \mathbf{AAU}$ in $\mathcal{B}^{\vee}$ (we will note this law $\circledast$).

17 (Q2) i) Let $\mathcal{C}$ be a $\mathbf{k} - \mathbf{CAAU}$, prove that $\Xi(\mathcal{B})$ is a submonoid of $(\mathcal{B}^{\vee}, \circledast, \epsilon)$.
ii) Extend these results to $\Xi(\mathcal{B}; \mathcal{C})$ (where $\mathcal{C}$ is an object of $\mathbf{k} - \mathbf{CAAU}$).

18 (Q3) i) For $t \in \mathbb{C}$, compute $(2ty_1 + t^2 y_2)^*$ under the form of an exponential.
ii) Recall that "Stars of the plane" are conc-characters and prove that, for $t \neq 0$, $(y_1^*, (2ty_1 + t^2 y_2)^*, y_3^*)$ are algebraically independent over $(\mathbb{C}\langle Y \rangle, ⧢, 1_{Y^*})$ within $(\mathbb{C}\langle\langle Y \rangle\rangle, ⧢, 1_{Y^*})$.
iii) More generally, prove that, if $Q_i \in \widehat{\mathbb{C}.Y}$ are $\mathbb{Z}$-linearly independent, then $(Q_i^*)_{i \in I}$ are algebraically independent.

Exercise (cont'd)

19. Before proving the (very hard) question (iii) of exercise 18 above let us give a bit of a categorical motivation.
20. $\mathcal{H}(\Omega)$ is a $\mathbb{C}$-vector space, in fact a $\mathbb{C}-$ **CAAU** (and hence all derived substructures: monoid and the like). Then, if one has a correpondence (a set-theoretical map)
$$\Phi_{set} : \mathcal{X} \longrightarrow \mathcal{H}(\Omega) \tag{7}$$
(be it for "inputs" or everything else, arbitrary) one can extend it to $\mathbb{C}\langle\mathcal{X}\rangle$ as we do for $\alpha_{z_0}^{z}$, Θ, ... One gets at once an extension
$$\Phi_{\mathbb{C}-\mathbf{AAU}} : \mathbb{C}\langle\mathcal{X}\rangle \longrightarrow \mathcal{H}(\Omega) \tag{8}$$
21. The question will be addressed next time will be to extend (8) to (certain) series.
22. On the RHS of (8), we have a space with a topology (apparently, the only reasonable one, see [16]). On the LHS, there are several topologies.

An algebraic one-parameter group for stuffles/1

23. (Holomorphic functional calculus [15]) Let $S \in \mathbb{C}_+\langle\langle Y \rangle\rangle$ (sometimes called "a proper series") and $T = \sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$, we first remark that $(a_n S^{\uplus n})_{n\geq 0}$ is "summable" (see definition below, equation (9) and use the weight).

Definition

A family of series $(S_i)_{i\in I}$ in $\mathbf{k}\langle\langle \mathcal{X} \rangle\rangle$ is said *summable* if, for all $w \in \mathcal{X}^*$, the map $i \mapsto \langle S_i | w \rangle$ is finitely supported. In this case the sum of the family is defined by

$$\sum_{i\in I}(S_i) := \sum_{w\in\mathcal{X}^*}\sum_{i\in I}\langle S_i|w\rangle\, w \tag{9}$$

24. For $T \in \mathbb{C}[[z]]$ and $S \in \mathbb{C}_+\langle\langle Y \rangle\rangle$, we note

$$T_{\uplus}(S) := \sum_{n\geq 0}\langle T|z^n\rangle\, S^{\uplus n} \tag{10}$$

An algebraic one-parameter group for stuffles/2

25. For $S \in \mathbb{C}_+\langle\langle Y \rangle\rangle$, we have

$$\log_{\uplus}(1_{Y^*} + S) \;\; \exp_{\uplus}(S) - 1_{Y^*} \text{ belong to } \mathbb{C}_+\langle\langle Y \rangle\rangle \text{ and} \quad (11)$$

$$\exp_{\uplus}(\log_{\uplus}(1_{Y^*} + S)) = 1_{Y^*} + S \;\; \log_{\uplus}(\exp_{\uplus}(S)) = S \quad (12)$$

26. (Commutation and polynomial type coefficients) For $S, T \in \mathbb{C}_+\langle\langle Y \rangle\rangle$ and $P(z) \in \mathbb{C}[z]$, we have

$$\exp_{\uplus}(S + T) = \exp_{\uplus}(S) \uplus \exp_{\uplus}(T) \text{ and} \quad (13)$$

$$\exp_{\uplus}(P(z).S) \in \mathbb{C}[z]\langle\langle Y \rangle\rangle \; ; \quad (14)$$

$$\frac{d}{dz}(\exp_{\uplus}(P(z).S)) = (P'(z).S) \uplus \exp_{\uplus}(P(z).S) \quad (15)$$

An algebraic one-parameter group for stuffles/3

27. Now, we code "the plane" by Umbral calculus.
28. Let x be an auxiliary letter, The map

$$\pi_Y^{Umbra} : \sum_{n\geq 1} \alpha_n x^n \mapsto \sum_{n\geq 1} \alpha_n y_n \tag{16}$$

from $\mathbb{C}_+[[x]]$ to $\widehat{\mathbb{C}.Y}$ is linear and bijective. We will call π_x^{Umbra} its inverse.

29. For $S, T \in \mathbb{C}_+[[x]]$, one can show that

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^* \tag{17}$$

30. Therefore, for $z \in \mathbb{C}$ and $T \in \mathbb{C}_+[[x]]$, one sets

$$G(z) = (\pi_Y^{Umbra}(e^{z.T}-1))^* \tag{18}$$

An algebraic one-parameter group for stuffles/4

31. From (17), (15) and (3) one gets, for $z_1, z_2 \in \mathbb{C}$,

$$G(z_1 + z_2) = G(z_1) \uplus G(z_2) \; ; \; G(0) = 1_{Y^*} \tag{19}$$

(then G can truly be called a "stuffle one parameter group").

32. We check that

$$\frac{d}{dz}(G(z)) = (\pi_Y^{Umbra}(T)) \uplus G(z) \tag{20}$$

and deduce that

$$G(z) = e_{\uplus}^{z.\pi_Y^{Umbra}(T)} \tag{21}$$

33. What precedes shows us that, for each $P = \sum_{i\geq 1} \langle P|y_i\rangle \, y_i \in \widehat{\mathbb{C}.Y}$

$$\log_{\uplus}(P^*) = \pi_Y^{Umbra}(\log(1 + \pi_x^{Umbra}(P))) \tag{22}$$

34 In particular, using (22), we show that

$$(ty_k)^* = \exp_{\uplus}\Big(\sum_{n\geq 1} \frac{(-1)^{n-1} t^n y_{nk}}{n}\Big) \tag{23}$$

Limiting processes and topologies/3

35. Our first examples are taken in $\mathbb{C}[[z]] = \mathbb{C}\langle\langle z \rangle\rangle$.
36. First, we return to S^* (S is without constant term) and $(1+\frac{z}{n})^n$.
37. In the first case, calling $\omega(S)$ the minimal length of *supp*(S) (and still supposing $\langle S|1_{\mathcal{X}^*}\rangle = 0$) we have $\omega(S^n) \geq n$ and then $(S^n)_{n\geq 0}$ is summable.
38. In the second one, one has

$$(1+\frac{z}{n})^n = 1 + z + \frac{(n)(n-1)}{n^2}z^2 + \ldots = 1 + z + \frac{(n-1)}{n}z^2 + \ldots \quad (24)$$

the series of differences $T_n = (1+\frac{z}{n+1})^{n+1} - (1+\frac{z}{n})^n$ is NOT summable as $T_n = \frac{1}{n(n+1)}z^2 + \ldots$ and then for all $n \in \mathbb{N}$, $\omega(T^n) = 2$. What happens in fact is that, for all $N \in \mathbb{N}$,

$$\lim_{n\to\infty}\langle(1+\frac{z}{n})^n|z^N\rangle = \frac{1}{N!}$$

so that, even if the series of differences is not summable, the limit exists. This term-by-term topology (which is the product topology) is called "Treves Topology" in [10] (see [30] Ch10 Example III).

A general theorem

Theorem (GHED,D.Grinberg,HNM [11])

Let $(\mathcal{B}, ., 1_{\mathcal{B}}, \Delta, \epsilon)$ *be a* $\mathbf{k}$*-bialgebra. As usual, let* $\Delta = \Delta_{\mathcal{B}}$ *and* $\epsilon = \epsilon_{\mathcal{B}}$ *be its comultiplication and its counit.*
Let $\mathcal{B}_+ = \ker(\epsilon)$*. For each* $N \geq 0$*, let* $\mathcal{B}_+^N = \underbrace{\mathcal{B}_+ \cdot \mathcal{B}_+ \cdot \cdots \cdot \mathcal{B}_+}_{N \text{ times}}$*, where* $\mathcal{B}_+^0 = \mathcal{B}$*. Note that* $(\mathcal{B}_+^0, \mathcal{B}_+^1, \mathcal{B}_+^2, \ldots)$ *is called the standard decreasing filtration of* $\mathcal{B}$*.*
For each $N \geq -1$*, we define a* $\mathbf{k}$*-submodule* $\mathcal{B}_N^\vee$ *of* $\mathcal{B}^\vee$ *by*

$$\mathcal{B}_N^\vee = (\mathcal{B}_+^{N+1})^\perp = \left\{ f \in \mathcal{B}^\vee \mid f\left(\mathcal{B}_+^{N+1}\right) = 0 \right\}. \tag{25}$$

Thus, $(\mathcal{B}_{-1}^\vee, \mathcal{B}_0^\vee, \mathcal{B}_1^\vee, \ldots)$ *is an increasing filtration of* $\mathcal{B}_\infty^\vee := \bigcup_{N \geq -1} \mathcal{B}_N^\vee$ *with* $\mathcal{B}_{-1}^\vee = 0$*.*

A general theorem cont'd

Theorem (GHED,D.Grinberg,HNM [11])

Then:

(a) *We have* $\mathcal{B}_p^\vee \circledast \mathcal{B}_q^\vee \subseteq \mathcal{B}_{p+q}^\vee$ *for any* $p, q \geq -1$ *(where we set* $\mathcal{B}_{-2}^\vee = 0$*). Hence,* $\mathcal{B}_\infty^\vee$ *is a subalgebra of the convolution algebra* $\mathcal{B}^\vee$.

(b) *Assume that* $\mathbf{k}$ *is an integral domain. Then, the set* $\Xi(\mathcal{B})^\times$ *of invertible characters (i.e., of invertible elements of the monoid* $\Xi(\mathcal{B})$*) is left* $\mathcal{B}_\infty^\vee$*-linearly independent.*

Application to the stuffle algebra

39. $\mathcal{B} = (\mathbb{C}\langle Y\rangle, \uplus, 1_{Y^*}, \Delta_{\texttt{conc}}, \epsilon)$. Then:
40. $(\mathcal{B}_+)^n = \mathbb{C}_{\geq n}\langle Y\rangle$, $\mathcal{B}_n^\vee = \mathbb{C}_{\leq n}\langle\langle Y\rangle\rangle$ (prolynomials)
41. We consider a family $Q_i \in \widehat{\mathbb{C}.Y}$ which is $\mathbb{Z}$-linearly independent and will prove that then $(Q_i^*)_{i\in I}$ are algebraically independent.

Conclusion

- We have explained what are monoids of characters (with a perspective towards $\mathcal{C}$-valued characters where $\mathcal{C}$ is a commutative algebra.
- For $\mathtt{conc}$-bialgebras, we have the form of all characters: they are precisely "Kleene Stars of the Plane" and we can use combinatorics on words to compute non-trivial identities.
- Next time we will see more on topological settings and correspondences.

THANK YOU FOR YOUR ATTENTION !

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