A GENERALIZATION OF SYMANZIK POLYNOMIALS

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Abstract. Symanzik polynomials are defined on Feynman graphs. They are used in quantum field theory to compute Feynman amplitudes. But they also appear in mathematics in various domains. For example, in article [3], first Symanzik is obtained in a dual theorem of the well-known Kirchhoff's matrix tree theorem. This article use the result, see [12] and [11], stating that Symanzik polynomials compute the volume of the tropical Jacobian of a metric graph. Another important example is article [1], where Theorem 1.1 studies the variation of the ratio of two Symanzik polynomials, and this theorem has consequences studied in [2].

In this paper, we generalize Symanzik polynomials to simplicial complexes and study their basic properties and applications. For example, we obtain some geometric invariants which compute interesting data on triangulable surfaces. These invariants do not depend on the chosen triangulation.

Actually, the Symanzik polynomials can even be defined for any matrices on a PID, for different ranks and with more parameters. The duality relation with what we call Kirchhoff polynomials, as well as Theorem 1.1 of [1], extend to this more general case. In order to show that theorem, we will make great use of oriented matroids. We give a complete classification of the connected component of the exchange graph of a matroid, and use that to prove a *boundedness of variation result* for Symanzik rational fractions, extending Theorem 1.1 of [1] to our setting.

1. Introduction

Symanzik polynomials appear naturally in quantum fields theory for computing Feynman amplitudes. They are defined on Feynman graphs. Let $G = (V, E)$ be a graph with vertex set V and edge set R. Let $\mathfrak{p} = (p_v)_{v\in V} \in \mathbb{R}^n$ such that each p_v, called the *external momentum of* $v \in V$, is an element of $\mathbb{R}^D$, for some positive integer D. $\mathbb{R}^D$ is endowed with a Minkowski bilinear form. We suppose that $\sum_{v\in V} p_v = 0$. Such a pair $(G, \mathfrak{p})$ is called a *Feynman graph*. In this paper we will only consider the case $D = 1$, but the results can be extended to the more general setting as in [2].

The first Symanzik polynomial, denoted ψ_G is defined as

$$\psi_G(\underline{x}) := \sum_{T\in\mathcal{T}} \prod_{e\notin T} x_e, \tag{1}$$

where $\mathcal{T}$ denoted the set of spanning subtrees of G, where the product is on all edges of G which are not in T, and where $\underline{x} = (x_e)_{e\in E}$ is a collection of variables.

The second Symanzik polynomial, denoted ϕ_G, is defined as

$$\phi_G(\mathfrak{p}, \underline{x}) := \sum_{F\in\mathcal{SF}_2} q(F) \prod_{e\notin F} x_e, \tag{2}$$

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where $\mathcal{SF}_2$ denoted the set of spanning forests G which have two connected components, and, if $F \in \mathcal{SF}_2$, $q(F) := -\langle \mathfrak{p}_{F_1}, \mathfrak{p}_{F_2} \rangle$, where F_1 and F_2 are the two connected components of F, and where, for $i \in \{1,2\}$, $\mathfrak{p}_{F_i}$ is the sum of the momenta of vertices in F_i. Then, the Feynman amplitude can be computed as an integral of $\exp(-i\phi_G/\psi_G)$.

In this paper, we are interested in Symanzik polynomials because they naturally appear in several other works and the question of generalizing them has been in the air ans should have connections to other branches of mathematics, e.g. asymptotic Hodge theory. Thus, we have tried to gather the different known results, to find new ones, and to generalize them to a bigger set of polynomial that we will naturally call *Symanzik polynomials*. The idea of the generalization comes from the well-known Kirchhoff's matrix-tree theorem (see [10]).

We hope that the reader will not meet any problem in following our notations in this introduction. In any case, all notations are defined in Subsection 2.1.

Let $G = (V, E)$ be an oriented simple connected graph with vertex set V, of size p, and edge set E, of size n. The incident matrix of G is the matrix $Q = (\mathbf{q}_{v,e})_{v \in V, e \in E}$ of dimensions $p \times n$ over $\mathbb{Z}$ defined as

$$\mathbf{q}_{v,e} = \begin{cases} 1 & \text{if } v \text{ is the head of } e, \\ -1 & \text{if } v \text{ is the tail of } e, \\ 0 & \text{otherwise.} \end{cases}$$

Let v be any vertex of G and let $\widetilde{Q}$ be the matrix where we delete the column corresponding to v. We have the following well-known theorem.

Theorem 1.1 (Weighted Kirchhoff's theorem)**.** *With the above notations,*

$$\widetilde{Q} X \widetilde{Q}^{\intercal} = \sum_{T \in \mathcal{T}} \prod_{e \in T} x_e,$$

where $X = (\mathbf{x}_{e,e'})_{e \in E, e' \in E}$ is the diagonal matrix defined by

$$\mathbf{x}_{e,e'} = \begin{cases} x_e & \text{if } e = e', \\ 0 & \text{otherwise.} \end{cases}$$

A possible demonstration of this theorem uses the Cauchy-Binet formula:

$$\widetilde{Q} X \widetilde{Q}^{\intercal} = \sum_{\substack{H \subset E \\ |H| = |V|-1}} \det((\widetilde{Q}^{\intercal})_H)^2 \prod_{e \in H} x_e,$$

where $(\widetilde{Q}^{\intercal})_H$ is the square matrix obtained restricting columns of $\widetilde{Q}^{\intercal}$ to those whose index is an edge of H. The end of the demonstration consists in showing that $\det(\widetilde{Q}^{\intercal}_H)^2$ equals 1 if H is a subtree of G, and 0 otherwise.

Notice that the formula in the above theorem is very similar to the definition of the first Symanzik polynomial, except that the product is on T instead of T^c. Thus, the idea to generalize Symanzik polynomials to any matrix is to use the Cauchy-Binet formula. However, we will choose another method different to deleting some row of Q.

Let A be a PID, n and p be two positive integers, $R \in \mathcal{M}_{n,p}(A)$ be a matrix, r be its rank and f be a free family of size r in A^p such that $\mathrm{Im}(R^{\intercal})$ is included into the A-submodule generated by f. Let $F \in \mathcal{M}_{p,r}(A)$ be the matrix associated to the family f. There exists a unique matrix $\widetilde{R} \in \mathcal{M}_{n,r}(A)$ such that $F\widetilde{R}^{\intercal} = R^{\intercal}$. If I is a subset of $[1 \,..\, n] := \{1, \ldots, n\}$, then $\widetilde{R}_I$ denotes the matrix $\widetilde{R}$ restricted to columns whose index is in i. Then, we define the *Symanzik polynomial of R with basis f and order k* by

$$\mathrm{Sym}_k(R, f; \underline{x}) := \sum_{\substack{I \subset [1 \cdot\cdot n] \\ |I| = r}} \big(\sigma(I) \det(\widetilde{R}_I) \big)^k \underline{x}^{I^c},$$

where $\underline{x}^J := \prod_{j \in J} x_j$, and where $\sigma(I) := \prod_{i \in I} (-1)^i$ (see Definition 2.10). The first Symanzik polynomial is only defined for the order 2. Actually, in this paper, most of the results only stand for even positive k. Studying the other orders will be useful, but the Symanzik polynomial of odd order are less natural objects: for example, we have to add arbitrary signs $\sigma(I)$ is their definition.

The fact that our definition generalizes first Symanzik polynomials is not obvious. We will see it in Example 2.14. Actually, if R is the transpose of the incident matrix of a graph, one can choose f such that $\widetilde{R}$ corresponds to the above $\widetilde{Q}$.

We will also define Kirchhoff polynomials in Section 2. All polynomials which can be obtained by the weighted Kirchhoff's theorem are Kirchhoff polynomials. Symanzik and Kirchhoff polynomials are dual, as we will see in Theorem 2.15. Article [3] speaks about "a dual version of Kirchhoff's celebrated Matrix-Tree Theorem". Actually, without that the name "Symanzik" appears in the article, this dual version provides Symanzik polynomials instead of Kirchhoff ones. With above notations, suppose that f is a basis of $\mathrm{Im}(R^\intercal)$. Let q be a positive integer, let $S \in \mathcal{M}_{n,q}(A)$ be a matrix of rank $s := n - r$ such that $R^\intercal S = 0$, and let g be a basis of $\mathrm{Im}(S^\intercal)$. Then, there exists an $a \in A^*$ such that

$$\mathrm{Sym}_k(R, f; \underline{x}) = a^k \,\mathrm{Kir}_k(S, g; \underline{x}).$$

This duality is deeply linked to an important property of Kirchhoff and Symanzik polynomials. They respect similar determinantal formulæ. The determinantal formula for Kirchhoff polynomials is natural because of the statement of the Kirchhoff's theorem. But the existence of a formula for Symanzik polynomials is not obvious. It has been enlighten in Subsection 1.1 of [1] and un [2]. Namely, if Q, seen as a matrix over A, is an incident matrix of some Feynman graph $(G, \mathfrak{p})$, and if H is a matrix whose columns, seen as elements of A^n, form a basis of $\ker(Q)$, then there exists an $a \in A^*$ such that

$$\psi_G(\underline{x}) = a^2 H^\intercal X H,$$

In this paper, a similar statement will be obtained (see Proposition 2.21) replacing ψ_G by any Symanzik polynomial. Actually, the determinantal formulæ only hold for the order 2. Since this formulæ will be important for the last theorems of this paper, we generalize it to all positive even order. This is possible thanks to multidimensional matrices we define in the Appendix. These objects were already studied by Arthur Cayley in 1843 (see [5]).

But we have not yet generalized the second Symanzik polynomials (2). In [1] and in [2], the authors state a second determinantal formula which computes the second Symanzik polynomials. The hypothesis that the total external momenta is 0, i.e.,

$$\sum_{v \in V} \mathfrak{p}_v = 0.$$

This hypothesis implies that there exists a column matrix $\mathbf{v}$ such that $Q\mathbf{v} = \mathfrak{p}$. The determinantal formula states that there exists an $a \in A^*$ such that

$$\phi_G(\mathfrak{p}, \underline{x}) = a^2 (H \star \mathbf{v})^\intercal X (H \star \mathbf{v}),$$

where $H \star \mathbf{v}$ is the matrix H where we add the column $\mathbf{v}$ on the right.

This time, we will use the idea of the determinantal formula to define Symanzik polynomials with parameters (actually, one can add more than one parameter). With the notations of the paragraph about duality, suppose that $q = s$ (i.e., columns of S are free), that g is the standard basis on A^s, and that

$$\mathrm{Sym}_k(R, f; \underline{x}) = \mathrm{Kir}_k(S, g; \underline{x}).$$

One can always find such a matrix S. Thus, if l is a positive integer and if $u_1, \ldots, u_l \in \mathrm{Im}(R^\intercal)$, then the *Symanzik polynomials of R of order k with parameters* $u_1, \ldots, u_l$ is defined as

$$\mathrm{Sym}_k(R, f, u_1, \ldots, u_l; \underline{x}) := \mathrm{Kir}_k(S \star \mathbf{v}_1 \star \ldots \star \mathbf{v}_n, g; \underline{x}),$$

where, for $i \in [1 \cdot\cdot l]$, $\mathbf{v}_i$ is such that $R^\intercal \mathbf{v}_i = \mathbf{u}_i$, with $\mathbf{u}_i$ the column matrix associated to u_i.

Until now, we have already used graphs and linear algebra. The intersection, or maybe the union, of this two theories contains the theory of matroids. A subsection will be devoted to defining Symanzik polynomials of a matroid. An interesting question, about covering finite linear spaces with some specific hyperplanes, will arises. But, more importantly, matroids are a powerful tool which we will use in every sections. Finally, the matroid point of view will be useful to have another understanding of Symanzik polynomials on simplicial complexes in Section 3.

Symanzik polynomials compute important data on graphs, as one can see in the important Example 3.14 about the volume of the Jacobian torus of a metric graph (this example comes from [12] and [11], and [3] uses it). Moreover, Theorem 1.1 of article [1], which we will generalize in Section 5, states an interesting property of the ratio of two polynomials, and so of the geometry of metric graphs. Thus, it is natural to try to generalize that example and this theorem to greater dimensions.

Moreover, many recent articles (see [6], [9]) deal with forests on simplicial complexes (one could see simplicial complexes as a generalization of graphs). Since original Symanzik polynomials are defined thanks to forests, it is natural to extend the definition of Symanzik polynomials to simplicial complexes. We will do it taking the Symanzik polynomial of the transpose of the d-th incident matrix of a simplicial complex of dimension d. Here, the l-th incident matrix is the matrix associated to the l-th reduced boundary map, choosing some standard bases on l-chains and on $(l-1)$-chains. The important Examples 3.19 and 3.44 confirm that a such extension is interesting.

Let us quickly speak of these two examples. Take $\mathcal{S}$ a compact orientable surface endowed with a finite measure π. It is well-known that $\mathcal{S}$ is triangulable, i.e., that one can find a simplicial complex homeomorphic to $\mathcal{S}$. Thus, we will be able to associate a Symanzik polynomial to $\mathcal{S}$. It happens that, under some good conditions, the Symanzik polynomial computes the total measure $\pi(\mathcal{S})$. Notice that this value does not depend on the chosen triangulation. This fact is more general: Proposition 3.18 states that Symanzik polynomials of a simplicial complex are invariant (in some way we will precise) under subdivisions. By the way, modulo this kind of invariance, Example 3.20 explains that for all matrix R over $\mathbb{Z}$, one can find a simplicial complex whose Symanzik polynomial equals the Symanzik polynomial of R.

The second example explains what happens when we add a parameter which is a *simple* boundary. For example, a simple closed loop which is a boundary is a simple boundary. Adding such a parameter is equivalent to contract the loop into a point. On $\mathcal{S}$, a simple boundary cuts $\mathcal{S}$ into two surfaces. Then, the Symanzik polynomial will compute the product of the measure of both surfaces.

Moreover, we will introduce orientations of the bases of a matroid, known in the literature as *chirotopes*. They will naturally appear in Theorem 3.32 and in Corollary 3.37, and so in the computation of the Symanzik polynomials. These results are needed in order to understand why Symanzik polynomials with parameters generalize the second Symanzik polynomials.

Section 4 studies the connected components of the exchange graph of a matroid. Corollary 4.13 generalizes Theorem 2.12 of [1]. In order to give an idea of what the exchange graph is, we will explain a well-known property of the set of the spanning subtrees of a graph. We say that two spanning subtrees are linked if they differ only by one edge (all subtrees have the same number of edges). It happens that, from any spanning subtree, we can obtain any other spanning subtree by a path of linked subtrees. The exchange graph is constructed with the same idea, but replacing spanning subtrees with ordered pairs of forests (or more exactly with ordered pairs of independents; see Definition 4.1).

The last section generalizes Theorem 1.1 of [1]. The variation of the ratio of two Symanzik polynomials are bounded, independently of the value of the variables, under bounded perturbations. It has some important consequences: see Theorem 1.2 of [1] and [2] for more details.

2. Kirchhoff and Symanzik polynomials and duality

2.1. **Notations.** We begin this section with few notations that will be useful all along the article.

In the whole article, A will always be a PID. Moreover, k will always be any nonnegative integer.

If p, q are integers with $p \leqslant q$, then the set $\{p, p+1, \dots, q\}$ will be written $[p \cdot\cdot q]$. The function *sign* will be useful: for p in $\mathbb{Z}$ or $\mathbb{R}$,

$$\operatorname{sgn}(p) := \begin{cases} -1 & \text{if } p < 0, \\ 0 & \text{if } p = 0, \\ 1 & \text{if } p > 0. \end{cases}$$

Let I be a finite set. Then $|I|$ is its cardinality and $\mathcal{P}(I)$ is its power set. If $J \in \mathcal{P}(I)$ is a subset of I and if there is no ambiguity, then $J^c := I \setminus J$ denotes the complementary of J. Moreover, if I is a finite set of integers, the *signature of* I denoted by $\sigma(I)$ is defined by

$$\sigma(I) := \prod_{i \in I} (-1)^i.$$

If $i \in I^c$, then $I + i := I \cup \{i\}$, and if $i \in I$, then $I - i := I \setminus \{i\}$ (using these notations means respectively that $i \in I^c$ or that $i \in I$).

In the whole article, if n is a positive integer, $(x_1, \dots, x_n)$ will be a family of variables. We will often use the notation $\underline{x}$ to denotes this family. $A[\underline{x}]$ is the set of polynomials over A with variables $x_1, \dots, x_n$. Following a usual notation, if $I \subset [1 \cdot\cdot n]$,

$$\underline{x}^I := \prod_{i \in I} x_i.$$

We will use permutations. $\mathfrak{S}_n$ will be the set of permutations of $[1 \cdot\cdot n]$. If τ is such a permutation, then $\operatorname{sgn}(\tau)$ will be its signature.

If U is a set, $A\langle U\rangle$ is the free A-module on U. The set of units of A is denoted by A^*, and we set

$$A^{*k} := \{a^k | a \in A^*\}.$$

By convention, if $a \in A$, then

$$a^0 = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Let n, p, q be positive integers. $\mathcal{M}_{p,q}(A)$ will denote the set of matrices with p rows and q columns over A and $\mathcal{M}_n(A) := \mathcal{M}_{n,n}(A)$. If u is an element of A^n, $\mathbf{u}$ will denote the column matrix in $\mathcal{M}_{n,1}(A)$ associated to u for the standard basis of A^n. Reciprocally, to $\mathbf{u} \in \mathcal{M}_{n,1}(A)$ one can naturally associate $u \in A^n$. If $P \in \mathcal{M}_{n,p}(A)$, then $P^\intercal \in \mathcal{M}_{p,n}(A)$ denotes the transpose of P. Let I be a subset of $[1\mathinner{..}n]$ and J be a subset of $[1\mathinner{..}p]$. Then $P_{I,J} \in \mathcal{M}_{|I|,|J|}(A)$ denotes the submatrix of P restricted to entries whose indices are in $I \times J$. In order to simplify notations, we set $P_{*,J} := P_{[1\mathinner{..}n],J}$ and $P_I := P_{I,[1\mathinner{..}p]}$.

Let $Q \in \mathcal{M}_{n,q}(A)$ a second matrix. The *horizontal concatenation operator* $\star$ is such that $P \star Q$ is the only matrix $R \in \mathcal{M}_{n,p+q}(A)$ with $R_{*,[1\mathinner{..}p]} = P$ and $R_{*,[p+1\mathinner{..}p+q]} = Q$.

If p is a positive integer and $f := (f_1, \dots, f_p)$ is a family of elements of A^n, then the same letter in uppercase will always denote the associated matrix

$$F := \mathbf{f}_1 \star \mathbf{f}_2 \star \cdots \star \mathbf{f}_p \in \mathcal{M}_{n,p}(A).$$

If U is an A-submodule of A^n, then we say that f *overgenerates* U if U is included in the A-submodule generated by the elements of f.

Let $(u_1, \dots, u_p)$ be the family of elements of A^n associated to columns of P (such that $P = \mathbf{u}_1 \star \cdots \star \mathbf{u}_p$). Then $\mathrm{Im}(P)$ is the A-submodule of A^n generated by $u_1, \dots, u_p$, $\ker(P)$ is the A-submodule of $v \in A^p$ such that $P\mathbf{v} = 0$, and $\mathrm{rk}(P) = \mathrm{rk}(\mathrm{Im}(P))$ is the rank of P and of $\mathrm{Im}(P)$. We also define

$$\overline{\mathrm{Im}}(P) = \{v \in A^n | \exists a \in A, av \in \mathrm{Im}(P)\}.$$

Let $R \in \mathcal{M}_{p,q}(A)$ be of rank q. Let f be a free family of size q in A^p overgenerating $\mathrm{Im}(R)$. Then there exists a unique matrix $\widetilde{R} \in \mathcal{M}_q(A)$ such that $R = F\widetilde{R}$.

Definition 2.1. With the above notations, we define the *determinant of R relative to f* denotes by $\det_f(R)$ as the determinant of $\widetilde{R}$.

This definition verifies the following useful lemmas.

Lemma 2.2. *Let $\mathfrak{A}$ be the field of fraction of A. Let $R \in \mathcal{M}_{p,q}(\mathfrak{A})$ be of rank q. Let f and f' be two bases of $\mathrm{Im}(R) \otimes_A \mathfrak{A}$. Then,*

$$\det_{f'}(R) = \det_{f'}(F) \det_f(R).$$

Proof. As f and f' are bases of the same subspace of $\mathfrak{A}^q$, there exists $P \in \mathcal{M}_q(\mathfrak{A})$ such that $F = F'P$. Let $\widetilde{R} \in \mathcal{M}_q(\mathfrak{A})$ be such that $R = F\widetilde{R}$. We have $R = F'P\widetilde{R}$. All this shows that

$$\det_f(R) = \det(\widetilde{R}),$$
$$\det_{f'}(R) = \det(P\widetilde{R}) \text{ and}$$
$$\det_{f'}(F) = \det(P).$$

Combining these three equations gives us the lemma. ∎

Lemma 2.3. *If f and f' are free families of size p in A^q and if f overgenerates* $\mathrm{Im}(F') \subset A^q$*, then*

$$\det_f(F') = |\,\mathrm{Im}(F)/\,\mathrm{Im}(F')| \pmod{A^*}.$$

Proof. This is a consequence of the stacked bases theorem. The theorem gives the existence of a basis $\widetilde{f} = (\widetilde{f}_1, \ldots, \widetilde{f}_p)$ of $\mathrm{Im}(F)$ and of invariant factors $(d_1, \ldots, d_p)$ such that

$$\widetilde{f}' := (d_1\widetilde{f}_1, \ldots, d_p\widetilde{f}_p) \tag{3}$$

is a basis of $\mathrm{Im}(F')$. Clearly, $\det_f(\widetilde{F})$ and $\det_{\widetilde{f}'}(F')$ are invertible elements of A. Thus, using Lemma 2.2,

$$\begin{aligned}\det_f(F') &= \det_f(\widetilde{F})\det_{\widetilde{f}}(\widetilde{F}')\det_{\widetilde{f}'}(F')\\ &= \det_{\widetilde{f}}(\widetilde{F}') \pmod{A^*}\\ &= \det(\mathrm{diag}(d_1, \ldots, d_p)) \pmod{A^*}\\ &= d_1 \cdots d_p \pmod{A^*}.\end{aligned}$$

But Equation (3) implies

$$\mathrm{Im}(F)/\,\mathrm{Im}(F') \simeq A/d_1A \times \cdots \times A/d_pA.$$

Finally,

$$\begin{aligned}\det_f(F') &= d_1 \cdots d_p \pmod{A^*}\\ &= |\,\mathrm{Im}(F)/\,\mathrm{Im}(F')| \pmod{A^*}.\end{aligned}$$

□

Some specific notations will be introduced later but we can already deal with the heart of the subject.

2.2. **Kirchhoff and Symanzik polynomials.** In this article, Kirchhoff polynomials are a generalization of polynomials appearing in the weighted Kirchhoff's matrix tree theorem whereas Symanzik polynomials generalize first and second Symanzik polynomials better known in physics (look at Examples 2.9, 2.14, 3.38, at Theorem 3.9 and at the introduction for more details). They are dual in a way we will see at the end of this subsection (Theorem 2.15).

We will fix some objects for the rest of the section. Let p, n be two positive integers, $R \in \mathcal{M}_{n,p}(A)$ be a matrix, $r := \mathrm{rk}(R)$ its rank and $s := n - r$.

Definition 2.4. Let f be a family of size r in A^p overgenerating $\mathrm{Im}(R^\intercal)$. The *Kirchhoff polynomial of order k of R associated to f* denoted by $\mathrm{Kir}_k(R, f; \underline{x})$ is defined as

$$\mathrm{Kir}_k(R, f; \underline{x}) := \sum_{\substack{I \subset [1 \cdot\cdot n]\\ |I| = r}} \det_f(R_I^\intercal)^k \underline{x}^I.$$

Remark 2.5. It would be useful to notice that if R' is another matrix and f' is a free family of size $\mathrm{rk}(R')$ overgenerating $\mathrm{Im}(R'^\intercal)$, and if for some nonnegative integer k

$$\mathrm{Kir}_k(R, f; \underline{x}) = \mathrm{Kir}_k(R', f'; \underline{x}),$$

then this equation holds for any order which is a multiple of k. The same will be true for Symanzik polynomials below.

Kirchhoff polynomials of order 2 have a more computable definition under some conditions. It is not difficult to obtain a case verifying these conditions, as we will see in many proofs.

Proposition 2.6. *If columns of R are free (or equivalently if $\mathrm{rk}(R) = p$), and if e is the standard basis of A^p, then*

$$\mathrm{Kir}_2(R, e; \underline{x}) = \det\left(R^\intercal X R\right)$$

where $X \in \mathcal{M}_n(A[\underline{x}])$ is the diagonal matrix $\mathrm{diag}(x_1, x_2, \dots, x_n)$.

Proof. This is essentially the Cauchy-Binet formula:

$$\begin{aligned}\det(R^\intercal X R) &= \sum_{\substack{I \subset [1\mathinner{\ldotp\ldotp} n]\\ |I|=r}} \det(R^\intercal_{*,I}) \det((XR)_I)\\ &= \sum_{\substack{I \subset [1\mathinner{\ldotp\ldotp} n]\\ |I|=r}} \det(R_I) \det(X_I R)\\ &= \sum_{\substack{I \subset [1\mathinner{\ldotp\ldotp} n]\\ |I|=r}} \det(R_I) \sum_{\substack{J \subset [1\mathinner{\ldotp\ldotp} n]\\ |J|=r}} \det(X_{I,J}) \det(R_J).\end{aligned}$$

But, as X is a diagonal matrix,

$$\det(X_{I,J}) = \begin{cases} \underline{x}^I & \text{if } I = J,\\ 0 & \text{otherwise,}\end{cases}$$

thus,

$$\begin{aligned}\det(R^\intercal X R) &= \sum_{\substack{I \subset [1\mathinner{\ldotp\ldotp} n]\\ |I|=r}} \det(R_I)^2 \underline{x}^I\\ &= \mathrm{Kir}_2(R, e; \underline{x}).\end{aligned}$$

□

For orders other than 2, one can obtain similar formulæ using matrices of any dimension which can be easily defined, as the determinant can. These definitions and some basic properties are detailed in the Appendix. The results and the notations of the appendix are only used in the following Proposition, the corresponding Proposition 2.22 for Symanzik polynomials, and Theorem 5.2 which generalizes Theorem 5.1.

Proposition 2.7. *If columns of R are free (or equivalently $\mathrm{rk}(R) = p$), if e is the standard basis of A^p, and if k is an even positive integer, then*

$$\mathrm{Kir}_k(R, e; \underline{x}) = \det\left(X \cdot_1 R \cdot_2 \cdots \cdot_k R\right),$$

where $X \in \mathfrak{C}^k_n(A[\underline{x}])$ is equal to $\mathrm{diag}^k(x_1, \dots, x_n)$ defined by, for every k-tuple $[u] \leqslant [n, \dots, n]$,

$$\mathrm{diag}^k(x_1, \dots, x_n)[u] = \begin{cases} x_l & \textit{if } u_1 = \cdots = u_k = l,\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. Once again, this is essentially the generalized Cauchy-Binet formula (Proposition A.5).

$\det\left(X \cdot_1 R \cdot_2 \cdots \cdot_k R\right)$

$$= \sum_{\substack{I_k \subset [1 \cdot\cdot n] \\ |I_k| = r}} \det\left((X \cdot_1 R \cdot_2 \cdots \cdot_{k-1} R)_{k:I_k} \right) \det(R_{I_k}),$$

$$= \sum_{\substack{I_k, I_{k-1} \subset [1 \cdot\cdot n] \\ |I_k| = |I_{k-1}| = r}} \det\left((X \cdot_1 R \cdot_2 \cdots \cdot_{k-2} R)_{k-1:I_{k-1}, k:I_k} \right) \det(R_{I_{k-1}}) \det(R_{I_k}),$$

$$\vdots$$

$$= \sum_{\substack{I_k, \dots, I_1 \subset [1 \cdot\cdot n] \\ |I_k| = \cdots = |I_1| = r}} \det\left(X_{1:I_1, \dots, k:I_k} \right) \det(R_{I_1}) \cdots \det(R_{I_k}).$$

Yet,

$$\det\left(X_{1:I_1, \dots, k:I_k} \right) = \begin{cases} \underline{x}^I & \text{if } I_1 = \cdots = I_k = I, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\begin{aligned} \det\left(X \cdot_1 R \cdot_2 R \cdots \cdot_k R \right) &= \sum_{\substack{I \subset [1 \cdot\cdot n] \\ |I| = r}} \det(R_I)^k \underline{x}^I \\ &= \mathrm{Kir}_k(R, e; \underline{x}). \end{aligned}$$

□

Example 2.8. Let us study what happens if $k = 0$. Let f be a family of size r in A^p overgenerating $\mathrm{Im}(R^\intercal)$. We have

$$\mathrm{Kir}_0(R, f; \underline{x}) = \sum_{\substack{I \subset [1 \cdot\cdot n] \\ |I| = r}} \det_f(R_I^\intercal)^0 \underline{x}^I.$$

We have to know when $\det_f(R_I^\intercal)$ is nonzero. It is clearly when R_I is of maximal rank, i.e., when $\mathrm{rk}(R_I) = r$. Thus,

$$\mathrm{Kir}_0(R, f; \underline{x}) = \sum_{\substack{I \subset [1 \cdot\cdot n] \\ |I| = r,\ \mathrm{rk}(R_I) = r}} \underline{x}^I.$$

Previous example implies that Kir_0 does not depend on the choice of the basis. Thus, from now we will not precise the basis in this case writing $\mathrm{Kir}_0(R; \underline{x})$.

Example 2.9. Now let us explain more precisely the link between Kirchhoff as mathematician and Kirchhoff polynomials. Let $\mathcal{G}$ be a graph with vertex set V of size p and edge set E of size n. Let $Q = (\mathbf{q}_{v,e}) \in \mathcal{M}_{p,n}(\mathbb{Z})$ be an incidence matrix of $\mathcal{G}$, i.e., suppose that elements of V and of E are enumerated from 1 to, respectively, p and n, put an orientation on edges of $\mathcal{G}$ and set, for all $v \in [1 \cdot\cdot p]$ and all $e \in [1 \cdot\cdot n]$,

$$\mathbf{q}_{v,e} := \begin{cases} 0 & \text{if } e \text{ is a loop,} \\ 1 & \text{if the vertex numbered } v \text{ is the head of the edge numbered } e, \\ -1 & \text{if the vertex numbered } v \text{ is the tail of the edge numbered } e, \\ 0 & \text{if the vertex numbered } v \text{ and the edge numbered } e \text{ are not incident.} \end{cases}$$

Let $J \subset [1 \mathinner{\ldotp\ldotp} n]$ be a subset of size $n-1$ and $R := Q_J^\intercal$. The well-known Kirchhoff's matrix-tree theorem (the simple form is proven in [10]), in its weighted form, states that

$$\det(R^\intercal X R) = \sum_{I \in \mathcal{T}} x^I,$$

where $J \subset [1 \mathinner{\ldotp\ldotp} n]$ is any subset of size $n-1$, $X = \operatorname{diag}(x_1, \dots, x_n)$ and $\mathcal{T} \subset \mathcal{P}([1 \mathinner{\ldotp\ldotp} n])$ is the family of all subsets $I \in \mathcal{T}$ which verify that the subgraph of $\mathcal{G}$ with vertex set V and edges whose number is in I is a subtree of $\mathcal{G}$ ($\mathcal{T} = \emptyset$ if $\mathcal{G}$ is not connected). Then Proposition 2.6 implies that

$$\operatorname{Kir}_2(R, e; \underline{x}) = \sum_{I \in \mathcal{T}} x^I.$$

We even have in this very special case that $\operatorname{Kir}_{2k}(R, e; \underline{x})$ does not depend on k.

In fact, there are other formulæ which do not require to delete a column which gives the same sum. We will see them in Theroem 3.9 which generalizes Kirchhoff's theorem to the case of finite simplicial complexes.

Symanzik polynomials have a similar, but little more complicated, definition.

Definition 2.10. Let f be a family of size r in A^p overgenerating $\operatorname{Im}(R^\intercal)$. The *Symanzik polynomial of order k of R associated to f* denoted by $\operatorname{Sym}_k(R, f; \underline{x})$ is defined as

$$\operatorname{Sym}_k(R, f; \underline{x}) := \sum_{\substack{I \subset [1 \mathinner{\ldotp\ldotp} n] \\ |I| = r}} \Big(\sigma(I) \det{}_f(R_I^\intercal)\Big)^k \underline{x}^{I^c}.$$

Notice that signatures of sets disappear when the order is even. Actually, signatures are in the definition in order to make the duality true for odd orders. But one can equally define Kirchhoff polynomials with the signatures instead of Symanzik ones. Moreover, most of our results will only be true for even order, beginning with determinantal formulæ (Proposition 2.7).

Example 2.11. When $k = 0$, Symanzik polynomials behave exactly as Kirchhoff ones (see Example 2.8) except that the exponents are the complementaries:

$$\operatorname{Sym}_0(R, f; \underline{x}) := \sum_{\substack{I \subset [1 \mathinner{\ldotp\ldotp} n] \\ |I| = r,\ \operatorname{rk}(R_I) = r}} \underline{x}^{I^c}.$$

Following lemma explains how a change of basis affects Kirchhoff and Symanzik polynomials.

Lemma 2.12. *If f and f' are two families of size r in A^p overgenerating $\operatorname{Im}(R^\intercal)$, then, in the field of fractions of A,*

$$\operatorname{Kir}_k(R, f'; \underline{x}) = \det{}_{f'}(F)^k \operatorname{Kir}_k(R, f; \underline{x}) \text{ and}$$
$$\operatorname{Sym}_k(R, f'; \underline{x}) = \det{}_{f'}(F)^k \operatorname{Kir}_k(R, f; \underline{x}).$$

Proof. This is a direct consequence of Lemma 2.2 about changes of basis:

$$\operatorname{Kir}_k(R, f'; \underline{x}) := \sum_{\substack{I \subset [1 \mathinner{\ldotp\ldotp} n] \\ |I| = r}} \det{}_{f'}(R_I^\intercal)^k \underline{x}^I$$

$$= \sum_{\substack{I \subset [1 \,..\, n] \\ |I| = r}} (\det_{f'}(F) \det_f(R_I^\intercal))^k \underline{x}^I$$
$$= \det_{f'}(F)^k \operatorname{Kir}_k(R, f; \underline{x}).$$

The case of Symanzik polynomials is similar. □

Remark 2.13. Kirchhoff and Symanzik polynomials are very similar. In fact, one can define ones from the others thanks to following formulæ.

$$\operatorname{Sym}_k(R, f; \underline{x}) = x_1 \cdots x_n \operatorname{Kir}_k(R, f; (-1)^k x_1^{-1}, (-1)^{2k} x_2^{-1}, \ldots, (-1)^{nk} x_n^{-1}),$$
$$\operatorname{Kir}_k(R, f; \underline{x}) = (-1)^{\frac{kn(n+1)}{2}} x_1 \cdots x_n \operatorname{Sym}_k(R, f; (-1)^k x_1^{-1}, (-1)^{2k} x_2^{-1}, \ldots, (-1)^{nk} x_n^{-1}).$$

Proof. Developing the first right hand-side member we obtain

$$x_1 \cdots x_n \operatorname{Kir}_k(R, f; (-1)^k x_1^{-1}, (-1)^{2k} x_2^{-1}, \ldots, (-1)^{kn} x_n^{-1})$$
$$= \sum_{\substack{I \subset [1 \,..\, n] \\ |I| = r}} \det_f(R_I^\intercal)^k \left(\prod_{i \in I} (-1)^{ik} \right) \underline{x}^{[1 \,..\, n]} / \underline{x}^I,$$
$$= \sum_{\substack{I \subset [1 \,..\, n] \\ |I| = r}} \det_f(R_I^\intercal)^k \, \sigma(I)^k \underline{x}^{I^c},$$
$$= \operatorname{Sym}_k(R, f; \underline{x}),$$

and, using the first formula,

$$(-1)^{\frac{kn(n+1)}{2}} x_1 \cdots x_n \operatorname{Sym}_k(R, f; (-1)^k x_1^{-1}, (-1)^{2k} x_2^{-1}, \ldots, (-1)^{kn} x_n^{-1})$$
$$= (-1)^{\frac{kn(n+1)}{2}} x_1 \cdots x_n (-1)^k x_1^{-1} \cdots (-1)^{kn} x_n^{-1} \times$$
$$\operatorname{Kir}_k(R, f; (-1)^k ((-1)^k x_1^{-1})^{-1}, \ldots, (-1)^{kn} ((-1)^{kn} x_n^{-1})^{-1}),$$
$$= \operatorname{Kir}_k(R, f; \underline{x}).$$

□

Example 2.14. What is the link between Symanzik polynomials defined in Definition 2.10 and the first Symanzik polynomial of the introduction (1)? Let $\mathcal{G}$ be a graph as defined in Example 2.9. Set $k = 2$. We have seen in Example 2.9 that, using the same notations,

$$\operatorname{Kir}_2(R, e; \underline{x}) = \sum_{I \in \mathcal{T}} x^I.$$

Using Remark 2.13, we obtain

$$\operatorname{Sym}_2(R, e; \underline{x}) = x_1 \cdots x_n \operatorname{Kir}_2(R, f; x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$$
$$= x_1 \cdots x_n \sum_{I \in \mathcal{T}} (x^I)^{-1}$$
$$= \sum_{I \in \mathcal{T}} x^{I^c}.$$

which exactly is the first Symanzik polynomial. Notice that, once more, in this very special case, $\operatorname{Sym}_{2k}(R, e; \underline{x})$ does not depend on k.

Now we can state the duality theorem. Roughly speaking, take a matrix S whose columns span the kernel of $R^{\intercal}$ if A was a field. Then the Kirchhoff polynomial of S is almost the Symanzik polynomial of R (up to a factor in A^{*k}).

Theorem 2.15 (Duality)**.** *Let q be a positive integer and $S \in \mathcal{M}_{n,q}(A)$ be a matrix of rank $s = n - r$ such that $R^{\intercal} S = 0$. Let f be a* basis *of* $\mathrm{Im}(R^{\intercal})$ *and g be a* basis *of* $\mathrm{Im}(S^{\intercal})$*. Then there exists an $a \in A^*$ such that, for all nonnegative integers k,*

$$\mathrm{Kir}_k(S, g; \underline{x}) = a^k \,\mathrm{Sym}_k(R, f; \underline{x}).$$

Proof. Using notations of the proposition, set $\widetilde{R} \in \mathcal{M}_{n,r}(A)$ and $\widetilde{S} \in \mathcal{M}_{n,s}(A)$ the only matrices such that $R^{\intercal} = F\widetilde{R}^{\intercal}$ and $S^{\intercal} = G\widetilde{S}^{\intercal}$. Using the fact that all elements of f are in $\mathrm{Im}(R^{\intercal})$, we know there exists $R' \in \mathcal{M}_{n,r}(A)$ such that $R^{\intercal} R' = F$, thus $F\widetilde{R}^{\intercal} R' = F$ and finally $\widetilde{R}^{\intercal} R' = \mathrm{Id}_r$ because columns of F form a free family. With the same argument, let $S' \in \mathcal{M}_{n,s}(A)$ be such that $\widetilde{S}^{\intercal} S' = \mathrm{Id}_s$. The equality of the proposition is, by definition,

$$\sum_{\substack{I \subset [1\,..\,n] \\ |I| = s}} \det\nolimits_g(S_I^{\intercal})^k \underline{x}^I = a^k \sum_{\substack{I \subset [1\,..\,n] \\ |I| = s}} (\sigma(I) \det\nolimits_f(R_I^{\intercal}))^k \underline{x}^{I^c}.$$

Comparing coefficients of the polynomials, it suffices to show that there exists an $a \in A^*$ such that, for all $I \subset [1\,..\,n]$ of size s,

$$\det\nolimits_g(S_I^{\intercal}) = a\,\sigma(I) \det\nolimits_f(R_{I^c}^{\intercal}),$$

which is exactly (removing the transposition)

$$\det(\widetilde{S}_I) = a\,\sigma(I) \det(\widetilde{R}_{I^c}). \tag{4}$$

The next step will be to explain the presence of $\sigma(I)$.

Lemma 2.16. *Set*

$$a := \sigma([1\,..\,s]) \det \begin{pmatrix} \widetilde{S}^{\intercal} \\ R'^{\intercal} \end{pmatrix}$$

and, if $I \subset [1\,..\,n]$ and $|I| = s$,

$$a_I := \det \begin{pmatrix} \widetilde{S}_I^{\intercal} & \widetilde{S}_{I^c}^{\intercal} \\ R'^{\intercal}_I & R'^{\intercal}_{I^c} \end{pmatrix}.$$

Then $a_I = \sigma(I) a$.

Proof of the lemma. Let $I = \{i_1, \dots, i_s\} \subset [1\,..\,n]$ such that $|I| = s$ and $i_1 < \dots < i_s$ and let a and a_I be as in the lemma. Let $\tau \in \mathfrak{S}_n$ be the only permutation which is increasing from I to $[1\,..\,s]$ and from I^c to $[s+1\,..\,n]$. In order to compute $\sigma(\tau)$, we will count the number of inversions of the permutation. Clearly such an inversion can only be between an element of I and an element of I^c. The set of elements of I^c inverting with i_1 is $[1\,..\,i_1 - 1]$, with i_2 is $[1\,..\,i_2 - 1] - i_1$, etc., with i_s is $[1\,..\,i_s - 1] \setminus \{i_1, \dots, i_s\}$. Thus, the number of inversions is

$$i_1 - 1 + i_2 - 2 + \dots + i_s - s = (i_1 + \dots + i_s) + (1 + \dots + s)$$

and

$$\sigma(\tau) = (-1)^{(i_1 + \dots + i_s) + (1 + \dots + s)},$$

$$\sigma(\tau) = \sigma(I)\,\sigma([1\,..\,s]). \tag{5}$$

Let $T \in \mathcal{M}_n(A)$ be the permutation matrix associated to τ, i.e., the only matrix such that, for some positive q and for all $u_1, \dots, u_n \in A^q$,

$$(\mathbf{u}_1 \star \cdots \star \mathbf{u}_n)T = \mathbf{u}_{\tau^{-1}(1)} \star \cdots \star \mathbf{u}_{\tau^{-1}(n)}.$$

Then,

$$\begin{pmatrix} \boxed{\widetilde{S}^\intercal} \\ \boxed{R'^\intercal} \end{pmatrix} T = \begin{pmatrix} \boxed{\widetilde{S}_I^\intercal} & \boxed{\widetilde{S}_{I^c}^\intercal} \\ \boxed{R_I'^\intercal} & \boxed{R_{I^c}'^\intercal} \end{pmatrix},$$

$$\det \begin{pmatrix} \boxed{\widetilde{S}^\intercal} \\ \boxed{R'^\intercal} \end{pmatrix} \det(T) = \det \begin{pmatrix} \boxed{\widetilde{S}_I^\intercal} & \boxed{\widetilde{S}_{I^c}^\intercal} \\ \boxed{R_I'^\intercal} & \boxed{R_{I^c}'^\intercal} \end{pmatrix},$$

$$\sigma([1 \,..\, s])a\,\sigma(\tau) = a_I.$$

Finally, Equation (5) proofs the lemma. □

Now we finish the proof of the proposition. From

$$\begin{pmatrix} \boxed{\widetilde{S}_I^\intercal} & \boxed{\widetilde{S}_{I^c}^\intercal} \\ \boxed{R_I'^\intercal} & \boxed{R_{I^c}'^\intercal} \end{pmatrix} \begin{pmatrix} \boxed{\mathrm{Id}_s} & \boxed{\widetilde{R}_I} \\ \boxed{0} & \boxed{\widetilde{R}_{I^c}} \end{pmatrix} = \begin{pmatrix} \boxed{\widetilde{S}_I^\intercal} & \boxed{0} \\ \boxed{*} & \boxed{\mathrm{Id}_r} \end{pmatrix}$$

and from

$$\begin{pmatrix} \boxed{S_I'^\intercal} & \boxed{S_{I^c}'^\intercal} \\ \boxed{\widetilde{R}_I^\intercal} & \boxed{\widetilde{R}_{I^c}^\intercal} \end{pmatrix} \begin{pmatrix} \boxed{\widetilde{S}_I} & \boxed{0} \\ \boxed{\widetilde{S}_{I^c}} & \boxed{\mathrm{Id}_r} \end{pmatrix} = \begin{pmatrix} \boxed{\mathrm{Id}_s} & \boxed{*} \\ \boxed{0} & \boxed{\widetilde{R}_{I^c}^\intercal} \end{pmatrix}$$

we deduce

(6) $$a_I \det(\widetilde{R}_{I^c}) = \det(\widetilde{S}_I)$$

and

(7) $$b_I \det(\widetilde{S}_I) = \det(\widetilde{R}_{I^c}),$$

where a_I is defined as in Lemma 2.16 and

$$b_I := \det \begin{pmatrix} \boxed{S_I'^\intercal} & \boxed{S_{I^c}'^\intercal} \\ \boxed{\widetilde{R}_I^\intercal} & \boxed{\widetilde{R}_{I^c}^\intercal} \end{pmatrix}.$$

Suppose that I verifies $\det(\widetilde{R}_{I^c}) \neq 0$ (such an I always exists: take a maximal free family of rows of $\widetilde{R}$ and choose I such that $\widetilde{R}_{I^c}$ is a restriction to these rows). Then, by (7), $b_I \neq 0$ and $\det(\widetilde{S}_I) \neq 0$. Multiplying (6) and (7), we set $a_I b_I = 1$. Thus, $a_I \in A^*$, and so the element a defined in Lemma 2.16belongs to A^*. Finally, using the lemma and (6), we obtain Equation (4) which ends the proof of the proposition. □

It could be more interesting to see Theorem 2.15 as follows. In particular, if R and S have independent columns, and if $\widetilde{f}$ and $\widetilde{g}$ are the standard bases, then relative determinants become usual ones (except the two denominators).

Corollary 2.17. *Let q be a positive integer and $S \in \mathcal{M}_{n,q}(A)$ be a matrix of rank s such that $R^\intercal S = 0$. Let $\widetilde{f}$ be a basis of $\overline{\mathrm{Im}}(R^\intercal)$ and $\widetilde{g}$ be a basis of $\overline{\mathrm{Im}}(S^\intercal)$. Let f be a basis of $\mathrm{Im}(R^\intercal)$ and g be a basis of $\mathrm{Im}(S^\intercal)$. Then, there exists an $a \in A^*$ such that, for all nonnegative integer k, both sides of the following equation are polynomials over A and*

$$\frac{1}{\det_{\widetilde{g}}(G)^k}\,\mathrm{Kir}_k(S, \widetilde{g}; \underline{x}) = a^k \frac{1}{\det_{\widetilde{f}}(F)^k}\,\mathrm{Sym}_k(R, \widetilde{f}; \underline{x}).$$

Moreover one can choose nonzero invariant factors of $A^n/\,\mathrm{Im}(S^\intercal)$ (resp. of $A^n/\,\mathrm{Im}(R^\intercal)$) such that the product of these factors is equal to $\det_{\widetilde{g}}(G)^k$ (respectively $\det_{\widetilde{f}}(F)^k$).

Proof of the equivalence between Theorem 2.15 and Corollary 2.17. The end of the corollary comes from the stacked bases theorem as in Lemma 2.3.

The equivalence between Corollary 2.17 and Theorem 2.15 comes from Lemma 2.12 about change of basis which shows that the stated equations are the same. □

Remark 2.18. There exists a direct proof of the corollary by proving that

$$\mathrm{Im}(S^\intercal)/\,\mathrm{Im}(S_I^\intercal) \simeq \mathrm{Im}(R^\intercal)/\,\mathrm{Im}(R_{I^c}^\intercal),$$

for all subsets $I \subset [1\cdot\cdot n]$ of size s, then by proving that $a \in A^*$ does not change when one changes one element of I.

Remark 2.19. In Theorem 2.15, any $a \in A^*$ can appear by choosing a different S or a different g. More precisely, multiplying an element of g by $b \in A^*$ or a column of S by b^{-1} will change a into ab. Even inverting two rows of R, and the corresponding two rows of S, will change the sign of a (one can see the value of a in Lemma 2.16 to be convinced).

Definition 2.20. In Theorem 2.15 if elements of A^n corresponding to columns of S form a basis of $\overline{\mathrm{Im}}(S)$ (i.e., of $\ker(R^\intercal)$), if g is the standard basis of A^s, and if $a = 1$ (i.e., $\mathrm{Kir}_k(S, e; \underline{x}) = \mathrm{Sym}_k(R, f; \underline{x})$ for e the standard basis), then S will be called a *normal kernel matrix of R with basis f*.

Looking at Remark 2.19, it is easy to see that one can always find a normal kernel matrix.

Let us now extend determinantal formulæ to Symanzik polynomials.

Proposition 2.21. *If f is a basis of $\mathrm{Im}(R^\intercal)$, if e is the standard basis of A^s, and if S is a normal kernel matrix of R with basis f, then*

$$\mathrm{Sym}_2(R, f; \underline{x}) = \det\left(S^\intercal X S\right),$$

where $X \in \mathcal{M}_n(A[\underline{x}])$ is the diagonal matrix $\mathrm{diag}(x_1, x_2, \dots, x_n)$.

Proposition 2.22. *If f is a basis of $\mathrm{Im}(R^\intercal)$, if e is the standard basis of A^s, if S is a normal kernel matrix of R with basis f, and if k is an even positive integer, then*

$$\mathrm{Sym}_k(R, f; \underline{x}) = \det\left(X \cdot_1 S \cdot_2 \cdots \cdot_k S\right),$$

where $X := \mathrm{diag}^k(x_1, \dots, x_n)$ defined in Proposition 2.7.

Proof of both claims. It suffices to apply the Definition 2.20 about normal kernel matrix to obtain

$$\mathrm{Sym}_k(R, f; \underline{x}) = \mathrm{Kir}_k(S, e; \underline{x}),$$

then to apply determinantal formulæ (Propositions 2.6 and 2.7). □

Remark 2.23. We enlighten that, with the notations of Theorem 2.15, $\det_f(R_I^\intercal)$ is nonzero if and only if $\det_g(S_{I^c}^\intercal)$ is nonzero.

2.3. **Symanzik polynomials with parameters.** Now we want to generalize the second Symanzik polynomials defined in the introduction (2). In fact, one can naturally add more than one parameter. Examples 3.42 and 3.44 will justify the usefulness of doing so.

Definition 2.24. Let l be a nonnegative integer, f be a family of size r in A^p overgenerating $\mathrm{Im}(R^\intercal)$ and $u_1, \dots, u_l$ be elements of $\mathrm{Im}(R^\intercal)$. The *Symanzik polynomial of order k of R associated to f with parameters $u_1, \dots, u_l$* denoted by $\mathrm{Sym}_k(R, f, u_1, \dots, u_l; \underline{x})$ is defined as

$$\mathrm{Sym}_k(R, f, u_1, \dots, u_l; \underline{x}) := \begin{cases} 0 & \text{if the family } (u_1, \dots, u_l) \text{ is not free,} \\ \frac{1}{\det_{\widetilde{f}}(F)^k} \mathrm{Kir}_k(S \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l, e; \underline{x}) & \text{otherwise,} \end{cases}$$

where $\widetilde{f}$ is any basis of $\mathrm{Im}(R^\intercal)$, S is a normal kernel matrix of R with basis $\widetilde{f}$, e is the standard basis of A^{s+l} and $v_i \in A^n$ verifies $R^\intercal \mathbf{v}_i = \mathbf{u}_i$ for all $i \in [1 \cdot\cdot l]$.

Remark 2.25. Reader might notice that the previous definition is consistent with Definition 2.10, about Symanzik polynomials, in the case $l = 0$ because of the definition of a normal kernel matrix (Definition 2.20) and of Lemma 2.12 about change of basis.

It is not obvious that this definition generalizes second Symanzik polynomials as defined in the introduction. We will see it further, in Example 3.38.

There are some simple properties about those parameters resulting from properties of the determinant.

Claim 2.26. *The Symanzik polynomial of order k is well-defined (i.e., it does not depend on the choice of $\widetilde{f}$, of S and of the v_is). Moreover, it is an alternating k-homogeneous map in its parameters (with the same notations as the previous definition):*

- *alternance: for all $i \in [1 \cdot\cdot l-1]$,*

$$\mathrm{Sym}_k(R, f, u_1, \dots, u_i, u_{i+1}, \dots, u_l; \underline{x}) = \mathrm{Sym}_k(R, f, u_1, \dots, -u_{i+1}, u_i, \dots, u_l; \underline{x}),$$

- *k-homogeneity: if $a \in A$, then, for all $i \in [1 \cdot\cdot l-1]$,*

$$\mathrm{Sym}_k(R, f, u_1, \dots, au_i, \dots, u_l; \underline{x}) = a^k \,\mathrm{Sym}_k(R, f, u_1, \dots, u_i, \dots, u_l; \underline{x}).$$

In particular, if the order is even, the Symanzik polynomial is symetric in its parameters.

Proof. The claim is obvious if the family $(u_1, \dots, u_l)$ is not free. Suppose now that this family is free. With the notations of the definition, let $\widetilde{f}'$ be another basis of $\mathrm{Im}(R^\intercal)$, S' be a normal kernel matrix of R with basis $\widetilde{f}'$ and $v'_1, \dots, v'_l$ be other elements of A^n such that $R^\intercal \mathbf{v}'_i = \mathbf{u}_i$ for $i \in [1 \cdot\cdot l]$. Then there exists an invertible matrix $P \in \mathcal{M}_s(A)$ such that $S' = SP$. Taking a subset $I \subset [1 \cdot\cdot n]$ of size s such that $R_{I^c} \neq 0$, because S and S' are normal kernel matrices, one can write, using Lemma 2.2 about change of basis,

$$\begin{aligned} \det(S_I^\intercal) &= \det_{\widetilde{f}}(R_{I^c}^\intercal) \\ &= \det_{\widetilde{f}}(\widetilde{F}') \det_{\widetilde{f}'}(R_{I^c}^\intercal) \\ &= \det_{\widetilde{f}}(\widetilde{F}') \det(S_I'^\intercal). \end{aligned}$$

Yet,

$$\det(S'_I) = \det(S_I P) = \det(S_I) \det(P).$$

Thus, once more using Lemma 2.2,

$$\det(P) = \frac{\det(S'^{\intercal}_I)}{\det(S^{\intercal}_I)} = \frac{1}{\det_{\widetilde{f}}(\widetilde{F}')} = \frac{\det_{\widetilde{f}'}(F)}{\det_{\widetilde{f}}(F)}. \tag{8}$$

Moreover, for any $i \in [1 \,..\, l]$, $R^{\intercal}(\mathbf{v}'_i - \mathbf{v}_i) = 0$. Thus, $v'_i - v_i$ is in $\ker(R^{\intercal})$, i.e., in $\mathrm{Im}(S)$. We deduce that there exists $w_1, \dots, w_l \in A^s$ such that $S\mathbf{w}_i = \mathbf{v}'_i - \mathbf{v}_i$ for all $i \in [1 \,..\, l]$. Then one can write

$$S' \star \mathbf{v}'_1 \star \cdots \star \mathbf{v}'_l = (S \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l) \begin{pmatrix} P & w_1 \cdots w_l \\ 0 & \mathrm{Id}_l \end{pmatrix}.$$

But the matrix between brackets has determinant $\det(P)$. Then, for all $I \subset [1 \,..\, n]$ of size $s + l$,

$$\det\left((S' \star \mathbf{v}'_1 \star \cdots \star \mathbf{v}'_l)_I\right) = \det\left((S \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l)_I\right) \det(P)$$

and, using (8),

$$\mathrm{Kir}_k(S' \star \mathbf{v}'_1 \star \cdots \star \mathbf{v}'_l, e; \underline{x}) = \mathrm{Kir}_k(S \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l, e; \underline{x}) \frac{\det_{\widetilde{f}'}(F)^k}{\det_{\widetilde{f}}(F)^k},$$

which concludes the well-definiteness.

The alternance and the k-homogeneity are very easy to see: transposing two u_is will transpose the corresponding columns in all $\widetilde{S}_I$, for all $I \subset [1 \,..\, n]$ of size $s+l$ where $\widetilde{S}$ denoted $S \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l$, thus it will change the sign of $\det(\widetilde{S}_I)$. Multiplying a u_i by $a \in A$ will multiply the corresponding columns in all $\widetilde{S}_I$, thus it will multiply $\det(\widetilde{S}_I)^k$ by a^k. □

Thus, far, our definitions of polynomials always depended on a basis. That is annoying because we have to make a choice. Remark 2.19 shows that polynomials really depend on the chosen bases and there is no canonical choice of basis possible in general (that will be even more true in Section 3). But the following proposition claims that the ratio of two Symanzik polynomials with parameters of the same order and of the same matrix does not depend on those choices.

Proposition 2.27. *Let f be a family of size r in A^p overgenerating $\mathrm{Im}(R^{\intercal})$, l be a nonnegative integer and $u_1, \dots, u_l$ be l elements of $\mathrm{Im}(R^{\intercal})$. Then the ratio*

$$\frac{\mathrm{Sym}_k(R, f, u_1, \dots, u_l; \underline{x})}{\mathrm{Sym}_k(R, f; \underline{x})}$$

only depends on $\overline{\mathrm{Im}}(R)$ and on u_is. Moreover it is equal to

$$\frac{\mathrm{Kir}_k(S \star \mathbf{v}_1 \star \ldots \star \mathbf{v}_l, h; \underline{x})}{\mathrm{Kir}_k(S, g; \underline{x})}$$

where $S \in \mathcal{M}_{n,q}(A)$ is a matrix of rank s, for q an arbitrary positive integer, such that $R^{\intercal}S = 0$, $v_1, \dots, v_l \in A^n$ are any elements verifying $R^{\intercal}\mathbf{v}_i = \mathbf{u}_i$, for all $i \in [1 \,..\, l]$, g is any family of size s in A^q overgenerating $\mathrm{Im}(S^{\intercal})$ and h is a family of size $s + l$ in A^{q+l} which is the standard completion of g:

$$H = \begin{pmatrix} G & 0 \\ 0 & \mathrm{Id}_l \end{pmatrix}.$$

Proof. There is no difficulty in the proof, one only has to make some changes of bases and use Lemma 2.12 in order to be able to use the definition of Symanzik polynomials with parameters (Definition 2.24). Let us take some variables as in the statement of the proposition. Let $\widetilde{g}$ be a basis of $\mathrm{Im}(S^\intercal)$, $\widetilde{h}$ be such that

$$\widetilde{H} = \begin{pmatrix} \widetilde{G} & 0 \\ 0 & \mathrm{Id}_l \end{pmatrix},$$

$\widetilde{S} \in \mathcal{M}_{n,s}(A)$ be such that $S^\intercal = \widetilde{G}\widetilde{S}^\intercal$ and e^s and e^{s+l} be the standard bases of A^s and on A^{s+l} respectively. First, notice that if $K \in \mathcal{M}_q(A)$ is the only matrix such that $\widetilde{G} = GK$, then

$$\widetilde{H} = H \begin{pmatrix} K & 0 \\ 0 & \mathrm{Id}_l \end{pmatrix},$$

thus,

$$\det_h(\widetilde{H})^k = \det_g(\widetilde{G})^k. \tag{9}$$

Next,

$$\det_{\widetilde{g}}(S^\intercal) = \det(\widetilde{S}^\intercal)$$

induces that

$$\mathrm{Kir}_k(S, \widetilde{g}; \underline{x}) = \mathrm{Kir}_k(\widetilde{S}, e^s; \underline{x}). \tag{10}$$

In the same way, as

$$(S \star \mathbf{v}_1 \star \ldots \star \mathbf{v}_l)^\intercal = \widetilde{H}(\widetilde{S} \star \mathbf{v}_1 \star \ldots \star \mathbf{v}_l)^\intercal,$$

we have

$$\mathrm{Kir}_k(S \star \mathbf{v}_1 \star \ldots \star \mathbf{v}_l, \widetilde{h}; \underline{x}) = \mathrm{Kir}_k(\widetilde{S} \star \mathbf{v}_1 \star \ldots \star \mathbf{v}_l, e^{s+l}; \underline{x}). \tag{11}$$

Yet, e^s is a basis of $\mathrm{Im}(\widetilde{S}^\intercal)$; indeed, it suffices to show that elements of e^s are in $\mathrm{Im}(\widetilde{S}^\intercal)$ but, if e_1^s and $\widetilde{g}_1^s$ denote the first elements of e^s and $\widetilde{g}^s$ respectively, as $\widetilde{g}$ is in $\mathrm{Im}(S^\intercal)$, there exists $\mathbf{w}_1 \in \mathcal{M}_{q,1}(A)$ such that $S^\intercal \mathbf{w}_1 = \widetilde{\mathbf{g}}_1^s$, and thus $\widetilde{G}\widetilde{S}^\intercal \mathbf{w}_1 = \widetilde{\mathbf{g}}_1^s$. We conclude, using the fact that columns of $\widetilde{G}$ are free, that $\widetilde{S}^\intercal \mathbf{w}_1 = \mathbf{e}_1^s$. The fact that e^s is a basis of $\mathrm{Im}(\widetilde{S}^\intercal)$ implies that

$$\mathrm{Im}(\widetilde{S}^\intercal) = \overline{\mathrm{Im}}(\widetilde{S}^\intercal) = A^s.$$

If $\widetilde{f}$ is a basis of $\mathrm{Im}(R^\intercal)$, using that e^s is a basis of $\mathrm{Im}(\widetilde{S}^\intercal)$, we can use the duality (Theorem 2.15):

$$\mathrm{Kir}_k(\widetilde{S}, e^s; \underline{x}) = a^k \,\mathrm{Sym}_k(R, \widetilde{f}; \underline{x})$$

for some $a \in A^*$. Let $\widetilde{S}_{a^{-1}}$ be the matrix $\widetilde{S}$ where we multiply the first column by a^{-1}. Thus, we have

$$\mathrm{Kir}_k(\widetilde{S}_{a^{-1}}, e^s; \underline{x}) = \mathrm{Sym}_k(R, \widetilde{f}; \underline{x}), \tag{12}$$

and

$$\widetilde{S}_{a^{-1}} \text{ is a normal kernel matrix of } R \text{ with basis } \widetilde{f}. \tag{13}$$

We now have all the elements to make final computations:

$$\begin{aligned} \mathrm{Kir}_k(S, g; \underline{x}) &= \det_g(\widetilde{G})^k \,\mathrm{Kir}_k(S, \widetilde{g}; \underline{x}) && \text{(Lemma 2.12)} \\ &= \det_g(\widetilde{G})^k \,\mathrm{Kir}_k(\widetilde{S}, e^s; \underline{x}) && (10) \end{aligned}$$

$$
\begin{aligned}
&= \det_g(\widetilde{G})^k a^k \operatorname{Kir}_k(\widetilde{S}_{a^{-1}}, e^s; \underline{x}) \\
&= \det_g(\widetilde{G})^k \operatorname{Sym}_k(R, \widetilde{f}; \underline{x}) && (12) \\
&= \frac{\det_g(\widetilde{G})^k}{\det_f(\widetilde{F})^k} \operatorname{Sym}_k(R, f; \underline{x}), && \text{(Lemma 2.12)}
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{Kir}_k(S \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l, h; \underline{x}) &= \det_h(\widetilde{H})^k \operatorname{Kir}_k(S \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l, \widetilde{h}; \underline{x}) && \text{(Lemma 2.12)} \\
&= \det_g(\widetilde{G})^k \operatorname{Kir}_k(S \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l, \widetilde{h}; \underline{x}) && (9) \\
&= \det_g(\widetilde{G})^k \operatorname{Kir}_k(\widetilde{S} \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l, e^{s+l}; \underline{x}) && (11) \\
&= \det_g(\widetilde{G})^k a^k \operatorname{Kir}_k(\widetilde{S}_{a^{-1}} \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l, e^{s+l}; \underline{x}) \\
&= \frac{\det_g(\widetilde{G})^k}{\det_f(\widetilde{F})^k} \operatorname{Sym}_k(R, f, u_1, \dots, u_l; \underline{x}). && \text{((13), Definition 2.24)}
\end{aligned}
$$

Now we can conclude that

$$
\frac{\operatorname{Kir}_k(S \star \mathbf{v}_1 \star \cdots \star \mathbf{v}_l, h; \underline{x})}{\operatorname{Kir}_k(S, g; \underline{x})} = \frac{\operatorname{Sym}_k(R, f, u_1, \dots, u_l; \underline{x})}{\operatorname{Sym}_k(R, f; \underline{x})} \tag{14}
$$

and, as one can choose the same S (and the same u_is, v_is and g) for two different matrices R and R' verifying $\overline{\operatorname{Im}}(R) = \overline{\operatorname{Im}}(R')$, and since the left hand-side member of (14) does not depend on R, the ratio only depends on $\overline{\operatorname{Im}}(R)$. ☐

Definition 2.28. Let l be a nonnegative integer and $u_1, \dots, u_l$ be elements of $\operatorname{Im}(R^\intercal)$. The *(normalized) Symanzik rational fraction of order k of R with parameters* $u_1, \dots, u_l$ denoted by $\widetilde{\operatorname{Sym}}_k(R, u_1, \dots, u_l; \underline{x})$ is defined as

$$
\widetilde{\operatorname{Sym}}_k(R, u_1, \dots, u_l; \underline{x}) := \frac{\operatorname{Sym}_k(R, f, u_1, \dots, u_l; \underline{x})}{\operatorname{Sym}_k(R, f; \underline{x})}
$$

where f is any family of size r of A^p overgenerating $\operatorname{Im}(R^\intercal)$.

We end this section with this question:

Question 2.29. In the same way, is there an interesting way of adding parameters to Kirchhoff polynomials?

2.4. **The matroids case.** We begin with recalling basic definitions and properties about matroids without any proof. We refer readers wanting more information to [8], [13].

A matroid can have many equivalent definitions. We will mainly use the following one.

Definition 2.30. A *matroid* $\mathfrak{M}$ is a pair which consists of a *ground set* E, which is any finite set, and a set of *independent sets* $\mathcal{I}$, which is a subset of $\mathcal{P}(E)$. We write $\mathfrak{M} = (E, \mathcal{I})$. A matroid has to verify three axioms:

(1) $\emptyset \in \mathcal{I}$,
(2) (hereditary property) $\mathcal{I}$ is stable by inclusion ($J \subset I \in \mathcal{I} \Rightarrow J \in \mathcal{I}$),
(3) (augmentation property) if $I, J \in \mathcal{I}$ and if $|J| < |I|$, then there exists $i \in I \setminus J$ such that $J + i \in \mathcal{I}$.

In this paper, the ground set will always be a set of integers.

Matroids encode all the combinatorial information about independency of subfamilies of a family of vectors in a linear space. More precisely, if l is a nonnegative integer and $(u_1, \ldots, u_l)$ is a family of vectors in some linear space, the *matroid representing the family* u is $\mathfrak{M}_u = (E_u = [1 \,..\, n], \mathcal{I}_u)$ where, for all nonnegative integer m, $I = \{i_1, \ldots, i_m\}$, with $i_j \neq i_{j'}$ if $j \neq j'$, is in $\mathcal{I}_u$ if $u_{i_1}, \ldots, u_{i_m}$ are independent. Similarly, if S is a matrix with l rows, the *matroid* $\mathfrak{M}_S(E_S, \mathcal{I}_S)$ *representing the matrix* S is equal to the matroid $\mathfrak{M}_u$ where $u = (u_1, \ldots, u_l)$ is such that $S^\intercal = \mathbf{u}_1 \star \ldots \star \mathbf{u}_l$. However, some matroids are not representable (i.e., do not represent any family of vectors) in any linear space.

Let $\mathfrak{M} = (E, \mathcal{I})$ be a matroid. If $I \subset E$, we call the *rank of* I

$$\mathrm{rk}(I) := \max_{J \in \mathcal{P}(I) \cap \mathcal{I}} |J|.$$

We call the *closure of* I the set

$$\mathrm{cl}(I) := \{i \in E \big| \,\mathrm{rk}(I + i) = \mathrm{rk}(I)\} \subset E.$$

Notice that if $I \in \mathcal{I}$, then $\mathrm{rk}(I) = |I|$. In a linear vector space, the closure operator corresponds to take the generated vector space. The closure operator has some properties enumerated below.

Claim 2.31. *If $I, J \in \mathcal{I}$, then*

1. *if $I \in \mathcal{I}$, then* $\mathrm{cl}(I) := I \cup \{i \in E \setminus I | (I + i) \notin \mathcal{I}\}$,
2. $I \subset \mathrm{cl}(I)$,
3. $I \subset J \Rightarrow \mathrm{cl}(I) \subset \mathrm{cl}(J)$,
4. $\mathrm{cl}(I) = \mathrm{cl}(\mathrm{cl}(I))$,
5. $\mathrm{rk}(\mathrm{cl}(I)) = \mathrm{rk}(I)$.

The *rank of a matroid* is $\mathrm{rk}(\mathfrak{M}) := \mathrm{rk}(E)$. A *basis* is an independent set maximal for the inclusion. The set of all bases is denoted by $\mathcal{B}(\mathfrak{M})$. If l is a nonnegative integer, the set of all independents of rank l is denoted by $\mathcal{I}_l$. Some properties of bases are enumerated below.

Claim 2.32. *Let $\mathfrak{M} = (E, \mathcal{I})$ be a matroid. Its bases have the following properties:*

1. *I is a basis if and only if $I \in \mathcal{I}$ and* $\mathrm{cl}(I) = E$,
2. *I is a basis if and only if $I \in \mathcal{I}$ and* $\mathrm{rk}(I) = \mathrm{rk}(\mathfrak{M})$,
3. *$\mathfrak{M}$ is entirely characterized by its bases and E,*
4. *(exchange property) if I_1, I_2 are two different bases, then there exist $i \in I_1 \setminus I_2$ and $j \in I_2 \setminus I_1$ such that $I_1 - i + j$ is a basis.*

A matroid $\mathfrak{M}' = (E', \mathcal{I}')$ is a *submatroid of* $\mathfrak{M}$ if $E' \subset E$ and $\mathcal{I}' \subset \mathcal{I}$. Moreover, if $E' = E$, then $\mathfrak{M}'$ is a *spanning submatroid of* $\mathfrak{M}$. Finally, the *dual of the matroid* $\mathfrak{M}$ is the only matroid $\widehat{\mathfrak{M}} = (E, \widehat{\mathcal{I}})$ where

$$\mathcal{B}(\widehat{\mathfrak{M}}) = \{I^c | I \in \mathcal{B}(\mathfrak{M})\}.$$

An example of duality for matroids in this article is given by Remark 2.23 which directly leads to following claim.

Claim 2.33. *If, for some positive integer q, $S \in \mathcal{M}_{n,q}(A)$ is a matrix of rank s verifying $R^\intercal S = 0$, then $\mathfrak{M}_S = \widehat{\mathfrak{M}}_R$.*

Proof. Remark 2.23 states that, for any basis g of $\mathrm{Im}(S^\intercal)$, with the notation of the claim, if $I \subset [1 \,..\, n]$ has size r, then $\det_f(R_I^\intercal) \neq 0$ if and only if $\det_g(S_{I^c}^\intercal) \neq 0$. But $\det_f(R_I^\intercal) \neq 0$ if and only if $\mathrm{rk}(R_I) = \mathrm{rk}(R) = r$, i.e., if $I \in \mathcal{B}(\mathfrak{M}_R)$. Similarly, $\det_g(S_{I^c}^\intercal) \neq 0$ if and only if $I^c \in \mathcal{B}(\mathfrak{M}_S)$. Thus, $I \in \mathcal{B}(\mathfrak{M}_R)$ if and only if $I^c \in \mathcal{B}(\mathfrak{M}_S)$. That matches the definition: $\mathfrak{M}_S = \widehat{\mathfrak{M}}_R$. □

Before continuing, we want to talk about an important type of matroid: graphic matroids.

Example 2.34. Matroids are powerful tools because they generalize at the same time independency in linear spaces, as we have already seen, and some properties of graphs, as we will see now.

Let $\mathcal{G}$ be a graph with the vertex set V and edge set E. One can associate to $\mathcal{G}$ a matroid $\mathfrak{M}_\mathcal{G}$ with ground set E and independent sets $\mathcal{I}$ such that, for all subset $I \subset E$, I is independent if and only if the spanning subgraph of $\mathcal{G}$ with the edge set I does not contain cycle.

Some basic properties of $\mathfrak{M}_\mathcal{G}$ are:

- bases of $\mathfrak{M}_\mathcal{G}$ correspond to maximal forests of $\mathcal{G}$ (to trees if $\mathcal{G}$ is connected),
- $\mathfrak{M}_\mathcal{G}$ represents the incidence matrix of $\mathcal{G}$ (see Example 2.9 for the definition of the incidence matrix),
- circuits, i.e., minimal dependent sets, of $\mathfrak{M}_\mathcal{G}$ correspond to cycles of $\mathcal{G}$.

We have seen that matroids encode bases, which are maximal independent sets, but Examples 2.8 and 2.11, about the case $k = 0$, show that Kirchhoff and Symanzik polynomials encode these too. That is why it should be possible to define these polynomials for a matroid. Indeed, here is such a definition for the order 0.

Definition 2.35. Let m be a positive integer and $\mathfrak{M} = (E = [1 \,..\, m], \mathcal{I})$ be a matroid. The *Kirchhoff polynomial (of order* 0*) of the matroid* $\mathfrak{M}$ *with variables* $x_1, \dots, x_n$ is defined by

$$\mathrm{Kir}_0(\mathfrak{M}; \underline{x}) = \sum_{I \in \mathcal{B}(\mathfrak{M})} x^I.$$

And the *Symanzik polynomial (of order* 0*) of the matroid* $\mathfrak{M}$ *with variables* $x_1, \dots, x_n$ is defined by

$$\mathrm{Sym}_0(\mathfrak{M}; \underline{x}) = \sum_{I \in \mathcal{B}(\widehat{\mathfrak{M}})} x^I.$$

These definitions are natural because of the following claim.

Claim 2.36. *Let* $\mathfrak{M}_R$ *be the matroid which* R *is a representation of. Then*

$$\mathrm{Kir}_0(R; \underline{x}) = \mathrm{Kir}_0(\mathfrak{M}_R; \underline{x}),$$
$$\mathrm{Sym}_0(R; \underline{x}) = \mathrm{Sym}_0(\mathfrak{M}_R; \underline{x}).$$

Proof. Clearly, $\mathrm{rk}(\mathfrak{M}_R) = \mathrm{rk}(R) = r$, and $I \in \mathcal{B}(\mathfrak{M}_R)$ if and only if $|I| = r = \mathrm{rk}(R_I)$. Using Example 2.8 we have

$$\mathrm{Kir}_0(R; \underline{x}) = \sum_{\substack{I \subset [1 \,..\, n] \\ |I| = \mathrm{rk}(I) = r}} x^I.$$

Looking at the definition of $\mathrm{Kir}_0(\mathfrak{M}_R; \underline{x})$, we exactly have

$$\mathrm{Kir}_0(R; \underline{x}) = \mathrm{Kir}_0(\mathfrak{M}_R; \underline{x}).$$

Using Example 2.11, Symanzik polynomials go the same way because $I \in \mathcal{B}(\widehat{\mathfrak{M}}_R)$ if and only if $I^c \in \mathcal{B}(\mathfrak{M}_R)$;, thus

$$\mathrm{Sym}_0(\mathfrak{M}_R; \underline{x}) = \sum_{I \in \mathcal{B}(\widehat{\mathfrak{M}})} x^I.$$

□

Remark 2.37. Let q be an arbitrary positive integer and $S \in \mathcal{M}_{n,q}(A)$ be a matrix of rank s such that $R^\intercal S = 0$. Then, Theorem 2.15 gives

$$\mathrm{Sym}_0(R; \underline{x}) = \mathrm{Kir}_0(S; \underline{x}).$$

By Claim 2.36 and Definition 2.35, this is equivalent to

$$\widehat{\mathfrak{M}}_R = \mathfrak{M}_S.$$

This was already stated in Claim 2.33. Thus, matroid duality naturally appears when we applied duality theorem to matroids.

Now, one can wonder what corresponds to Symanzik polynomials with parameters for matroids. Adding one parameter decrease the degree of the polynomial by one. Thus, it is natural to think that adding a parameter will lead to take a submatroid of rank decreased by one. We will now answer the following question.

What is the set of matroids $\mathfrak{M}$ of ground set E_R such that there exists a nonzero $u \in \mathrm{Im}(R^\intercal)$ verifying

$$\mathrm{Sym}_0(R, u; \underline{x}) = \mathrm{Sym}_0(\mathfrak{M}; \underline{x})?$$

Proposition 2.38. *A matroid $\mathfrak{M}$ of ground set E_R verifies that there exists a nonzero $u \in \mathrm{Im}(R^\intercal)$ such that*

$$\mathrm{Sym}_0(R, u; \underline{x}) = \mathrm{Sym}_0(\mathfrak{M}; \underline{x})$$

if and only if:

- *$\mathfrak{M}$ is a spanning submatroid of $\mathfrak{M}_R$ of rank $r-1$ and*
- $\left(\bigcap_{I \in (\mathcal{I}_R)_{r-1} \setminus \mathcal{B}(\mathfrak{M})} \ker(R^\intercal) + \mathcal{H}_I \right) \setminus \left(\bigcup_{I \in \mathcal{B}(\mathfrak{M})} \ker(R^\intercal) + \mathcal{H}_I \right) \neq \emptyset$
  where, for $I \subset [1 \cdot\cdot n]$, $\mathcal{H}_I$ is the A-submodule of A^n generated by the family $(e_i)_{i \in I}$ with $(e_1, \ldots, e_n)$ the standard basis.

Proof. We will need the following claim in the proof:

$$\text{if } I \subset [1 \cdot\cdot n], \text{ columns of } S_I \text{ are free iff } I^c \in \mathcal{I}_R. \tag{15}$$

Indeed this is true because a subset $I \subset [1 \cdot\cdot n]$ verifies that columns of S_I are free iff there exists $J \subset I$ such that $|J| = \mathrm{rk}(S_J) = \mathrm{rk}(S_I) = s$, iff there exists $J \in \mathcal{B}(\mathfrak{M}_S)$ such that $J \subset I$, iff $I^c \in \mathcal{I}_R$ ($\mathfrak{M}_S = \widehat{\mathfrak{M}}_R$ by Claim 2.33).

Similarly, one can prove that

$$\text{if } I \subset [1 \cdot\cdot n], \text{ columns of } (S \star v)_I \text{ are free iff } I^c \in \widehat{\mathcal{I}}_{S \star v}. \tag{16}$$

Let $\mathfrak{M} = (E_R, \mathcal{I}')$ be a matroid and u be a nonzero element in $\mathrm{Im}(R^\intercal)$ such that

$$\mathrm{Sym}_0(R, u; \underline{x}) = \mathrm{Sym}_0(\mathfrak{M}; \underline{x}). \tag{17}$$

Let $S \in \mathcal{M}_{n,s}(A)$ be a matrix of rank s such that $R^\intercal S = 0$ and $v \in A^n$ such that $R\mathbf{v} = \mathbf{u}$. Equation (17) is equivalent to

$$\mathrm{Kir}_0(S \star \mathbf{v}; \underline{x}) = \mathrm{Kir}_0(\widehat{\mathfrak{M}}; \underline{x}),$$

i.e., to

$$\widehat{\mathfrak{M}}_{S\star\mathbf{v}} = \mathfrak{M}. \tag{18}$$

In order to prove the proposition, we have to characterize $\widehat{\mathfrak{M}}_{S\star\mathbf{v}}$ for each possible $\mathbf{v}$.

As $u \neq 0$, $v \notin \overline{\mathrm{Im}}(S)$ and so $\mathrm{rk}(S \star \mathbf{v}) = s + 1$. Then

$$\mathrm{rk}(\widehat{\mathfrak{M}}_{S\star\mathbf{v}}) = r - 1. \tag{19}$$

Now we have to characterize bases of $\widehat{\mathfrak{M}}_{S\star\mathbf{v}}$. By Equations (16) and (19), $I \subset [1 \cdot\cdot n]$ is in $\mathcal{B}(\widehat{\mathfrak{M}}_{S\star\mathbf{v}})$ iff $|I| = r - 1$ and columns of $(S \star v)_{I^c}$ are free, i.e., iff

- $|I| = r - 1$, and
- columns of S_{I^c} are free, and
- $\mathbf{v}_{I^c}$ is independent of columns of S_{I^c}, i.e., $\mathbf{v}_{I^c} \notin \overline{\mathrm{Im}}(S_{I^c})$.

By (15), the second point is equivalent to $I \in \mathcal{I}_R$ (thus, the first two points give us the first point of the proposition). Now we focus on the third point. Let π_{I^c} be the projection along $\mathcal{H}_I$ onto $\mathcal{H}_{I^c}$ (it is well-defined). The second point is equivalent to $\pi_{I^c}(v) \notin \pi_{I^c}(\overline{\mathrm{Im}}(S))$, i.e., to $v \notin \overline{\mathrm{Im}}(S) + \mathcal{H}_I$. Moreover, $\overline{\mathrm{Im}}(S) = \ker(R^\intercal)$. To summarize,

$$I \in \mathcal{B}(\widehat{\mathfrak{M}}_{S\star\mathbf{v}}) \text{ iff } \begin{cases} I \in (\mathcal{I}_R)_{r-1} & \text{and} \\ v \notin \ker(R^\intercal) + \mathcal{H}_I. \end{cases} \tag{20}$$

Let $\mathfrak{M} = (E, \mathcal{I}')$ be a matroid. Then

$$\begin{aligned}
&\exists u \in \mathrm{Im}(R^\intercal), u \neq 0 \text{ and } \mathrm{Sym}_0(R, u; \underline{x}) = \mathrm{Sym}_0(\mathfrak{M}; \underline{x}) \\
&\overset{(18)}{\iff} \exists v \in A^p \setminus \ker(R^\intercal), \widehat{\mathfrak{M}}_{S\star\mathbf{v}} = \mathfrak{M}, \\
&\overset{2.32}{\iff} \exists v \in A^p \setminus \ker(R^\intercal), \mathcal{B}(\widehat{\mathfrak{M}}_{S\star\mathbf{v}}) = \mathcal{B}(\mathfrak{M}), \\
&\overset{(20)}{\iff} \exists v \in A^p \setminus \ker(R^\intercal), \forall I \subset [1 \cdot\cdot n], I \in \mathcal{B}(\mathfrak{M}) \text{ iff } \begin{cases} I \in (\mathcal{I}_R)_{r-1}, \\ v \notin \ker(R^\intercal) + \mathcal{H}_I, \end{cases} \\
&\iff \exists v \in A^p \setminus \ker(R^\intercal), \begin{cases} \forall I \in \mathcal{B}(\mathfrak{M}), I \in (\mathcal{I}_R)_{r-1} \text{ and } v \notin \ker(R^\intercal) + \mathcal{H}_I, \\ \forall I \notin \mathcal{B}(\mathfrak{M}), I \notin (\mathcal{I}_R)_{r-1} \text{ or } v \in \ker(R^\intercal) + \mathcal{H}_I, \end{cases} \\
&\iff \exists v \in A^p \setminus \ker(R^\intercal), \begin{cases} \mathcal{B}(\mathfrak{M}) \subset (\mathcal{I}_R)_{r-1}, \\ \forall I \in \mathcal{B}(\mathfrak{M}), v \notin \ker(R^\intercal) + \mathcal{H}_I, \\ \forall I \in (\mathcal{I}_R)_{r-1} \setminus \mathcal{B}(\mathfrak{M}), v \in \ker(R^\intercal) + \mathcal{H}_I, \end{cases} \\
&\iff \exists v \in A^p \setminus \ker(R^\intercal), \begin{cases} \mathcal{B}(\mathfrak{M}) \subset (\mathcal{I}_R)_{r-1}, \\ v \notin \bigcup_{I \in \mathcal{B}(\mathfrak{M})} \ker(R^\intercal) + \mathcal{H}_I, \\ v \in \bigcap_{I \in (\mathcal{I}_R)_{r-1} \setminus \mathcal{B}(\mathfrak{M})} \ker(R^\intercal) + \mathcal{H}_I, \end{cases} \\
&\iff \begin{cases} \mathfrak{M} \text{ is a spanning submatroid of } \mathfrak{M}_R \text{ of rank } r-1, \\ \left(\bigcap_{I \in (\mathcal{I}_R)_{r-1} \setminus \mathcal{B}(\mathfrak{M})} \ker(R^\intercal) + \mathcal{H}_I \right) \setminus \left(\bigcup_{I \in \mathcal{B}(\mathfrak{M})} \ker(R^\intercal) + \mathcal{H}_I \right) \neq \emptyset. \end{cases}
\end{aligned}$$

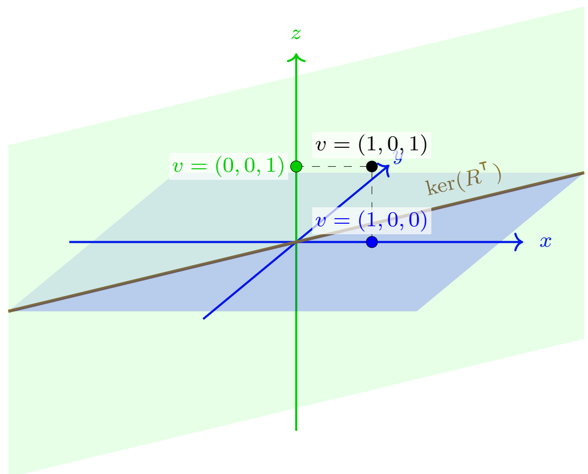


FIGURE 1. Illustration of the Example 2.39.

That concludes the proof. □

In general, not all spanning submatroids of rank $r-1$ of $\mathfrak{M}_R$ can verify the equation in the previous proposition, i.e., not all are of the form $\mathfrak{M}_{\widehat{S\star\mathbf{v}}}$.

Example 2.39. Set $A=\mathbb{R}, n=3, p=2, R=\left(\begin{smallmatrix}1&1\\1&1\\1&0\end{smallmatrix}\right)$. Then $r=2, s=1$, $\ker(R^{\intercal})$ is generated by $(1,-1,0)\in\mathbb{R}^3$ and $(\mathcal{I}_R)_{r-1}=\{\{1\},\{2\},\{3\}\}$. Now, $v\in\ker(R^{\intercal})+\mathcal{H}_{\{1\}}$ is equivalent to $v\in\ker(R^{\intercal})+\mathcal{H}_{\{2\}}$ (see Figure 1, both $\mathcal{H}_{\{1\}}$ and $\mathcal{H}_{\{2\}}$ are one of the blue lines and the sums are the only blue hyperplane). Thus, only submatroids containing both bases $\{1\}$ and $\{2\}$ or none of them can be obtain. In fact, the three such matroids with nonzero rank can be obtained:

v	$\mathcal{B}(\mathfrak{M}_{\widehat{S\star\mathbf{v}}})$
$(1,0,0)$	$\{\{3\}\}$
$(0,0,1)$	$\{\{1\},\{2\}\}$
$(1,0,1)$	$\{\{1\},\{2\},\{3\}\}$

However, the maximal submatroid of rank $r-1$, whose bases are $(\mathcal{I}_R)_{r-1}$, is always of the form $\widehat{\mathfrak{M}}_{S+\mathbf{v}}$.

Corollary 2.40. *If $|A|$ is infinite, then there exists $u\in A^p$ such that*

$$\mathrm{Sym}_0(R,u;\underline{x})=\mathrm{Sym}_0(\mathfrak{M}_R^{r-1};\underline{x}),$$

where $\mathfrak{M}_R^{r-1}=(E_R,\mathcal{I}_R^{r-1})$ is the matroid of bases $(\mathcal{I}_R)_{r-1}$.

Proof. By Proposition 2.38, we only have to prove that

$$\bigcup_{I\in(\mathcal{I}_R)_{r-1}}\ker(R^{\intercal})+\mathcal{H}_I\neq A^n.$$

As

$$\mathrm{rk}(\ker(R^{\intercal})+\mathcal{H}_I)\leqslant\mathrm{rk}(\ker(R^{\intercal}))+\mathrm{rk}(\mathcal{H}_I)=s+r-1<n,$$

this is a corollary of the following more general result.

Lemma 2.41. *If A is infinite, then, for any positive integer l, a finite union of affine A-submodules of positive corank in A^l is never equal to the entire space A^l where we call an affine A-submodule any subset of the form*

$$\mathcal{H} + u := \{h + u | h \in \mathcal{H}\}$$

where $\mathcal{H}$ is any A-submodule and u is any element of A and the corank of $\mathcal{H}+u$ is $l - \mathrm{rk}(\mathcal{H}+u)$ where $\mathrm{rk}(\mathcal{H} + u) := \mathrm{rk}(\mathcal{H})$.

There are some clashes of notations: in the following proof, $\mathcal{H}+u$ will never denote $\mathcal{H} \cup \{u\}$, and $\mathrm{rk}(\mathcal{H} + u)$ will denote the affine rank.

Proof. This lemma can easily be proven by induction on l.

If $l = 1$, A^l is infinite but an affine A-submodule of positive corank is just a point, thus the lemma is true.

If $l \geqslant 1$, suppose that the lemma is true at order l. Let $U_1, \dots, U_m$ by m affine A-submodules of positive coranks in A^{l+1}. Let $\mathcal{H}$ be any linear A-submodule of A^{l+1} of corank 1 and let u be an element which is not in $\mathcal{H}$. Necessarily, if there exist $i \in [1 \cdot\cdot m]$ and $j_0 \in \mathbb{Z}$ such that $\mathrm{rk}((\mathcal{H} + j_0 u) \cap U_i) = l$ (where we take the affine rank), then $(\mathcal{H} + ju) \cap U_i = \emptyset$ for any $j \in (\mathbb{Z} \setminus \{j_0\})$. Thus, there exists $j_0 \in \mathbb{Z}$ such that $\mathrm{rk}((\mathcal{H} + j_0 u) \cap U_i) < l$ for all $i \in [1 \cdot\cdot m]$. By induction,

$$\bigcup_{i=1}^{m} (\mathcal{H} + j_0 u) \cap U_i \neq \mathcal{H} + j_0 u.$$

Thus,

$$\mathcal{H} + j_0 \not\subset \bigcup_{i=1}^{m} U_i,$$

so the lemma is true for the order $l + 1$.

We can conclude the proof of the lemma by induction. □

By the lemma,

$$\bigcup_{I \in (\mathcal{I}_R)_{r-1}} \ker(R^{\intercal}) + \mathcal{H}_I \neq A^n,$$

thus, using Proposition 2.38, the corollary is true. □

To prove the corollary, we strongly use the fact that A is infinite. Indeed, the corollary is no more true is A is not infinite, i.e., when A is a finite field.

Example 2.42. Set $A = \mathbb{Z}/2\mathbb{Z}$, $n = 3$, $R = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $r = 2$, $s = 1$, $S = \left(\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right)$ and $(\mathcal{I}_R)_{r-1} = \{\{1\}, \{2\}, \{3\}\}$. Whatever $v \in A^3$ one chooses, $S \star \mathbf{v}$ will have two identical rows, thus $\mathcal{B}(\widehat{\mathfrak{M}}_{S\star\mathbf{v}})$ can't be equal to $(\mathcal{I}_R)_{r-1}$. One can see it in another way, using Proposition 2.38:

$$\bigcup_{i=1}^{3} \ker(R^{\intercal}) + \mathcal{H}_{\{i\}} = A^3.$$

Remark 2.43. A rough counting shows that if $|A| \geqslant \binom{n}{r-1}$, then the corollary is still true.

Proof. If $\binom{n}{r-1} = 1$, then $r - 1 = 0$ and the corollary is obvious. Else, if $|A| \geqslant \binom{n}{r-1}$, then, as a hyperplane (i.e., a maximal proper submodule) of A^n has cardinality $|A|^{n-1}$ and two hyperplanes are never disjoint,

$$|\bigcup_{I \in (\mathcal{I}_R)_{r-1}} \ker(R^\intercal) + \mathcal{H}_I| < |(\mathcal{I}_R)_{r-1}| \cdot |A|^{n-1},$$

and

$$\begin{aligned} |(\mathcal{I}_R)_{r-1}| \cdot |A|^{n-1} &\leqslant \binom{n}{r-1} |A|^{n-1}, \\ &\leqslant |A^n|. \end{aligned}$$

Thus,

$$\bigcup_{I \in (\mathcal{I}_R)_{r-1}} \ker(R^\intercal) + \mathcal{H}_I \neq A^n$$

and the corollary is true in this case. □

We end this section with the question: "under which conditions is Corollary 2.40 true for a finite A?" It can be reformulated in the following terms.

Question 2.44. What are the finite fields A, the positive integers n and r and the subspaces V of A^n of rank $n - r$ such that

$$\bigcup_{\substack{I \subset [1 \cdot\cdot n] \\ |I| = r-1}} V + \mathcal{H}_I \neq A^n\,?$$

Here $\mathcal{H}_I$ denotes the subspace generated by the elements of the standard basis whose indices are in I.

3. Symanzik polynomials on simplicial complexes

Every results in this section can be extended to CW-complexes.

Before defining Symanzik polynomials, we will generalize the notion of forests on graph theory to the case of simplicial complexes. Generalized forests will reveal interesting properties of those polynomials.

3.1. **Simplicial complexes and forests.** Let V be a finite set of *vertices*. An *abstract simplicial complex on* V is a nonempty set Δ of subsets of V called *faces* such that Δ is stable by inclusion: if δ is a face and if $\gamma \subset \delta$, then γ is also a face. A simplicial complex Γ is a subcomplex of Δ if $\Gamma \subset \Delta$. If δ is a face, its *dimension* is $\dim(\delta) := |\delta| - 1$. Notice that Δ has always a unique face of dimension -1: the empty set. The dimension of Δ is the maximal dimension of its faces. We call it d. The d-dimensional faces are called *facets*. If l is an integer, $l \geqslant -1$, then Δ_l is the set of faces of Δ of dimension l. The *l-skeleton* $\Delta_{(l)}$ *of* Δ is the subcomplex of all faces of dimension at most l of Δ:

$$\Delta_{(l)} := \bigcup_{i=-1}^{l} \Delta_i.$$

In this article, we will suppose that a complex Δ is always endowed with an enumeration on each set of faces Δ_l, $l \in [-1 \cdot\cdot d]$, by numbers from 1 to $|\Delta_l|$.

Let ∂_Δ be the d-th boundary operator of the augmented simplicial chain complex associated to Δ relative to A for the standard orientation of the faces associated to the chosen enumeration of vertices: endowing V with an order $<$ corresponding to the enumeration on Δ_0, one has

$$\partial_\Delta : \begin{array}{rcl} A\langle\Delta_d\rangle & \longrightarrow & A\langle\Delta_{d-1}\rangle, \\ \{i_0,\dots,i_d\} \text{ with } i_0 < \dots < i_d & \longmapsto & \sum_{j=0}^d (-1)^j \{i_0,\dots,i_{j-1},i_{j+1},\dots,i_d\}. \end{array} \tag{21}$$

If δ_i denotes the i-th facet for the chosen enumeration, then $(\delta_1,\dots,\delta_{|\Delta_d|})$ is a basis of $A\langle\Delta_d\rangle$. We obtain in the same way a basis of $A\langle\Delta_{d-1}\rangle$. Now we can represent ∂_Δ by a matrix in $\mathcal{M}_{|\Delta_{d-1}|,|\Delta_d|}(A)$ which we will call the *d-th incidence matrix of* Δ. The kernel of ∂_Δ is denoted by $\mathrm{Z}_d(\Delta)$ and its elements are called *d-cycles*. The image of ∂_Δ is denoted by $\mathrm{B}_{d-1}(\Delta)$ and its elements are called *$(d-1)$-boundaries*.

Example 3.1. Figure 2 is an example of a 2-dimensional simplicial complex Δ called the bipyramide. The enumeration of the vertices is indicated, as well as the standard orientation of facets and edges for this enumeration: arrows of the figure are such that the border of an edge is its head minus its tail and the border of a facet is equal to the sum of its edges with a sign -1 if the edge is not oriented with respect to the indicated direction on the facet. Edges are enumerated in the list on the left of the matrix, from top to bottom, and facets are enumerated in the list above the matrix, from left to right. Then, this matrix is the d-th incidence matrix of Δ. Moreover, one can see on the bipyramide:

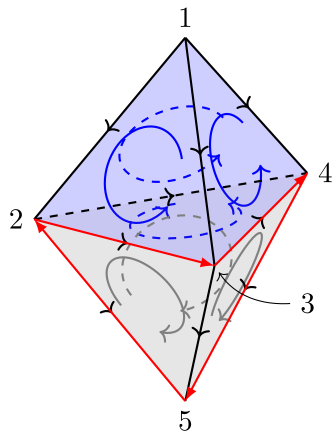
- a 1-boundary in red, with the orientation indicated by arrows on the end of edges (i.e., if the orientation indicated for a face is the opposite of the standard orientation, then the face is counted negatively):
$$\{2,3\} - \{2,5\} + \{4,5\} + \{3,4\} = \partial_\Delta\big(\{2,3,5\} + \{3,4,5\}\big),$$
- a 2-cycle in blue, with the non-indicated outer orientation (i.e., for each facet of the cycle, the outer orientation is the counterclockwise direction if one looks at the facet from the exterior of the cycle):
$$\partial_\Delta\big(\{1,2,3\} + \{1,3,4\} - \{1,2,4\} - \{2,3,4\}\big) = 0.$$

For the rest of this section, we fix V a finite set, Δ an abstract simplicial complex on V, $d := \dim(\Delta)$, $n := |\Delta_d|$, $p := |\Delta_{d-1}|$. We fix $R \in \mathcal{M}_{n,p}(A)$ being the transpose of the d-th incidence matrix of Δ. Let $\mathfrak{M} = (E,\mathcal{I}) := \mathfrak{M}_R$ be the matroid representing R. As in the previous section, we set r the rank of R and $s := n - r$.

Now we define (simplicial) κ-forests of Δ following the idea of article [4]. This definition is one possibility to generalize the notion of forests in graphs to higher dimension. Indeed, in dimension one, our definition will coincide with the usual one if one sees graphs as 1-dimensional simplicial complexes.

Definition 3.2. Let $\Gamma \subset \Delta$ be a simplicial subcomplex of Δ such that $\Gamma_{(d-1)} = \Delta_{(d-1)}$. Let κ be a nonnegative integer. Then Γ is a *κ-forest of the simplicial complex* Δ if it verifies the following three properties.

(1) *acyclicity*: Γ has no nonzero d-cycle,
(2) $\mathrm{rk}(\partial_\Delta) - \mathrm{rk}(\partial_\Gamma) = \kappa$,
(3) $|\Gamma_d| = |\Delta_d| - \mathrm{rk}\left(\mathrm{Z}_d(\Delta)\right) - \kappa$.



	$\{1,2,3\}\downarrow$	$\{1,2,4\}\downarrow$	$\{1,3,4\}\downarrow$	$\{2,3,4\}\downarrow$	$\{2,3,5\}\downarrow$	$\{2,4,5\}\downarrow$	$\{3,4,5\}\downarrow$
$\{1,2\}\rightarrow$	1	1	0	0	0	0	0
$\{1,3\}\rightarrow$	−1	0	1	0	0	0	0
$\{1,4\}\rightarrow$	0	−1	−1	0	0	0	0
$\{2,3\}\rightarrow$	1	0	0	1	1	0	0
$\{2,4\}\rightarrow$	0	1	0	−1	0	1	0
$\{2,5\}\rightarrow$	0	0	0	0	−1	−1	0
$\{3,4\}\rightarrow$	0	0	1	1	0	0	1
$\{3,5\}\rightarrow$	0	0	0	0	1	0	−1
$\{4,5\}\rightarrow$	0	0	0	0	0	1	1

FIGURE 2. A bipyramide and its d-th incidence matrix.

Example 3.3. Figure 3 is an example of a 0-forest, called Γ, on the bipyramide, called Δ (using the same enumeration on faces of Δ as in Example 3.1). Γ is obtained from Δ by removing the two crossed facets. To check that it is a 0-forest, look at the d-th incidence matrix D_Γ of ∂_Γ which is on the right (columns of the d-th incidence matrix of Δ which are not anymore in D_Γ are still indicated in bright gray). Clearly $\Gamma_{(1)} = \Delta_{(1)}$. Moreover, the three conditions are verified:

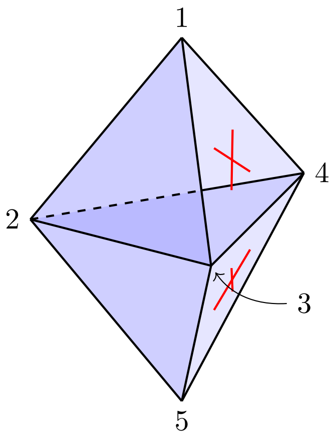
(1) the kernel of D_Γ is trivial. Thus, Γ is acyclic.
(2) $\mathrm{Im}(D_\Gamma) = \mathrm{Im}(D_\Delta)$. Thus, $\mathrm{rk}(\partial_\Delta) - \mathrm{rk}(\partial_\Gamma) = 0$.
(3) $|\Gamma_2| = 5$, $|\Delta_2| = 7$ and $\mathrm{rk}\,\mathrm{Z}_2(\Delta) = \mathrm{rk}\ker(\partial_\Delta) = 2$. Thus,

$$|\Gamma_2| = |\Delta_2| - \mathrm{rk}\,\mathrm{Z}_2(\Delta) - 0.$$

Finally, Γ is a 0-forest of Δ.

We will see easier ways to check if a subcomplex is or is not a κ-forest in this section. But before, let us motivate the name of forest.

Example 3.4. If Δ is a 1-dimensional simplicial complex, then it can be seen as a simple graph $\mathcal{G}$ with set of vertices V and set of edges Δ_1. Put the same orientation of edges and same numerations of edges and of vertices on Δ and on $\mathcal{G}$ (therefore edges in $\mathcal{G}$ are oriented from the vertex of lower number to the other one). Then, the 1st incidence matrix of Δ is exactly the incidence matrix of $\mathcal{G}$ (which has been defined in Example 2.9). To an oriented cycle (or circuit) of $\mathcal{G}$ one can associate the oriented sum of the edges in this cycle. This gives an element of $\mathrm{Z}_1(\Delta)$. Moreover, if g is the genus of $\mathcal{G}$, which is equal to the number of edges minus the number of vertices plus the number of connected components, then one can find a g-tuple of oriented cycles of $\mathcal{G}$ such that the associated elements in $\mathrm{Z}_1(\Delta)$ form a basis (see Figure 5 further: (c_1, c_2, c_3) is a possible choice). In particular, $g = \mathrm{rk}(\mathrm{Z}_1(\Delta))$. If v, v' are two vertices, $\{v\} - \{v'\}$ is a 0-boundary of Δ if and only if $\{v\}$ and $\{v'\}$ are in the same connected component of $\mathcal{G}$ (indeed, it is the boundary of any oriented path going from v' to v). More generally, a 0-boundary of Δ is any linear combination of such a difference of two vertices.



	$\{1,2,3\}\downarrow$	$\{1,2,4\}\downarrow$	$\{1,3,4\}\downarrow$	$\{2,3,4\}\downarrow$	$\{2,3,5\}\downarrow$	$\{2,4,5\}\downarrow$	$\{3,4,5\}\downarrow$
$\{1,2\}\rightarrow$	1	1	0	0	0	0	0
$\{1,3\}\rightarrow$	−1	0	1	0	0	0	0
$\{1,4\}\rightarrow$	0	−1	−1	0	0	0	0
$\{2,3\}\rightarrow$	1	0	0	1	1	0	0
$\{2,4\}\rightarrow$	0	1	0	−1	0	1	0
$\{2,5\}\rightarrow$	0	0	0	0	−1	−1	0
$\{3,4\}\rightarrow$	0	0	1	1	0	0	1
$\{3,5\}\rightarrow$	0	0	0	0	1	0	−1
$\{4,5\}\rightarrow$	0	0	0	0	0	1	1

FIGURE 3. A 0-forest of the bipyramide.

A 0-forest Γ of Δ can be associated to a subgraph $\mathcal{H}$ of $\mathcal{G}$ and verifies:

(1) Γ is acyclic, i.e., $\mathcal{H}$ is acyclic,
(2) $\mathrm{rk}(\Gamma) = \mathrm{rk}(\Delta)$, i.e., two vertices which are in a same connected component of $\mathcal{G}$ are in the same connected component of $\mathcal{H}$,
(3) $|\Gamma_1| = |\Delta_1| - \mathrm{rk}(\mathrm{Z}_1)$. We have seen that $\mathrm{rk}(\mathrm{Z}_1)$ is equal to the number of edges minus the number vertices plus the number of connected components. Thus, last formula said that the number of edges of Γ is equal to its number of vertices minus its number of connected components.

In the case $\mathcal{G}$ is connected, we recognize the conditions for $\mathcal{H}$ to be a subtree of $\mathcal{G}$. In the general case, these three conditions characterize maximal forests on $\mathcal{G}$, i.e., a union of a spanning subtree for each connected component of $\mathcal{G}$.

As we will see later, removing κ more edges will give a κ-forest. If $\mathcal{G}$ is connected, a κ-forest is a subforest of $\mathcal{G}$ with $\kappa+1$ connected components (thus, $(\kappa+1)$-forest should be a better name but the definition of κ-forest would be less natural).

It is well-known that, in graphs, only two out of the three conditions enumerated in previous example are needed to be a κ-forest. This is still true in simplicial complexes.

Proposition 3.5. *A subcomplex Γ of Δ such that $\Gamma_{(d-1)} = \Delta_{(d-1)}$ is a κ-forest for a non-negative integer κ if and only if it verifies two out of the three conditions of the Definition 3.2.*

Proof. By the rank-nullity theorem, acyclicity (i.e., triviality of $\ker(\partial_\Gamma)$) is equivalent to $\mathrm{rk}(\partial_\Gamma) = |\Gamma_d|$. The same theorem implies that $\mathrm{rk}(\mathrm{Z}_d(\Delta)) = |\Delta_d| - \mathrm{rk}(\partial_\Delta)$. Thus, the three conditions can be rewritten as:

(1) $\mathrm{rk}(\partial_\Gamma) = |\Gamma_d|$,
(2) $\mathrm{rk}(\partial_\Delta) - \mathrm{rk}(\partial_\Gamma) = \kappa$,
(3) $|\Gamma_d| = |\Delta_d| - (|\Delta_d| - \mathrm{rk}(\partial_\Delta)) - \kappa$.

Now, the proposition is clear. □

If Γ is a subcomplex of Δ, $\mathrm{Fac}(\Gamma)$ will be the subset of $[1 \cdot\cdot n]$ consisting of the numbers of the facets of Δ present in Γ. Reciprocally, if $I \subset [1 \cdot\cdot n]$, $\mathrm{Sub}_\Delta(I)$ will be the simplical subcomplex Γ of Δ verifying $\Gamma_{(d-1)} = \Delta_{(d-1)}$ and $\mathrm{Fac}(\Gamma) = I$. For any nonnegative integer κ, we set

$$\mathcal{F}_\kappa(\Delta) := \left\{ \mathrm{Fac}(\Gamma) \middle| \Gamma \text{ is a } \kappa\text{-forest of } \Delta \right\}.$$

With these notations, it can be useful to see κ-forests in the following way. We recall that, if $l \in [1 \cdot\cdot r]$, $(\mathcal{I}_R)_l$ is the set of independents of rank l in $\mathfrak{M}_R$.

Proposition 3.6. *Let κ be a nonnegative integer. Then*

$$\mathcal{F}_\kappa(\Delta) = (\mathcal{I}_R)_{r-\kappa}.$$

Proof. Let Γ be a subcomplex of Δ with the same $(d-1)$-skeleton. The first condition for Γ to be a κ-forest, for some nonnegative κ, is that $\ker(\partial_\Gamma)$ is trivial, i.e., that rows of R whose index is in $\mathrm{Fac}(\Gamma)$ form a free family, that is

$$\mathrm{Fac}(\Gamma) \in \mathcal{I}_R. \tag{22}$$

The second condition is $\mathrm{rk}(\partial_\Gamma) = \mathrm{rk}(\partial_\Delta) - \kappa$, i.e., rows of R whose index is in $\mathrm{Fac}(\Gamma)$ form a family of rank $r - \kappa$. In other words, seeing $\mathrm{Fac}(\Gamma)$ as a subset of E_R,

$$\mathrm{rk}(\mathrm{Fac}(\Gamma)) = r - \kappa. \tag{23}$$

Finally, as the third condition is unnecessary by Proposition 3.5, Equations (22) and (23) imply the claim. ☐

Before concluding this subsection, we want to deal with a practical way of finding κ-forests. First we begin with $\kappa = 0$.

If one remove a facet δ of Δ, obtaining some subcomplex Γ, two cases can happen.

- Either $\partial_\Delta(\delta)$ is in $\mathrm{B}_{d-1}(\Gamma)$, then $\mathrm{B}_{d-1}(\Gamma) = \mathrm{B}_{d-1}(\Delta)$. Using the rank-nullity theorem, as $|\Gamma_d| = |\Delta_d| - 1$, we deduce
$$\mathrm{rk}(\mathrm{Z}_d(\Gamma)) = \mathrm{rk}(\mathrm{Z}_d(\Delta)) - 1.$$
- Or, $\partial_\Delta(\delta) \notin \mathrm{B}_{d-1}(\Gamma)$. Thus, we have
$$\mathrm{rk}(\mathrm{B}_{d-1}(\Gamma)) < \mathrm{rk}(\mathrm{B}_{d-1}(\Delta)).$$

Now suppose that the second case happens. The above strict inegality will clearly remained true after removing other facets of Γ. Thus, one cannot obtain anymore 0-forest of Δ taking a subcomplex of Γ because of the second condition in the definition of forests (the boundary of a 0-forest has to be of rank r).

But suppose that one makes $\mathrm{rk}(\mathrm{Z}_d(\Delta))$ times the choice of facets of Δ such that in each step the first case happens (i.e., that each time we remove a facet which still is in a cycle). Thus, we finally obtain a subcomplex Γ' of Δ such that

$$\mathrm{rk}(\partial_{\Gamma'}) = \mathrm{rk}(\partial_\Delta)$$

and

$$|\Gamma'_d| = |\Delta'_d| - \mathrm{rk}(\mathrm{Z}_d(\delta)).$$

Therefore, in this way we obtain a 0-forest Γ' of Δ. Now, by Proposition 3.6, removing κ more facets will lead to a κ-forest.

To summarize, one can, and necessarily will, obtain a 0-forest removing facets which are in a cycle until it is no more possible. In fact, what we truly do is choosing a basis of $\widehat{\mathfrak{M}}_R$ consisting of the removed facets. Then one can remove any κ more facets to obtain κ-forests.

Now we have enough results on forests to understand the combinatorics behind Kirchhoff and Symanzik polynomials on simplicial complexes.

3.2. **Kirchhoff and Symanzik polynomials for simplicial complexes.**

Definition 3.7. The *Kirchhoff polynomial class of order k of Δ* is an element of $A[\underline{x}]/A^{*k}$ denoted by $\mathrm{Kir}_k(\Delta;\underline{x})$ which is defined as

$$\mathrm{Kir}_k(\Delta;\underline{x}) := \mathrm{Kir}_k(R,f;\underline{x}) \pmod{A^{*k}},$$

where f is any basis of $\mathrm{Im}(R^\intercal)$.

Remark 3.8. In this remark, we will see what happens if one chooses another enumeration or another orientation on the faces of Δ. Another choice of the enumeration of facets of Δ and of their orientation induces a permutation of rows of R or a change in their sign. Thus, that can change the order of variables and their sign.

Moreover, one has to choose the enumeration on $(d-1)$-dimensional faces of Δ and their orientation. But changing it will only change the sign of $\mathrm{Kir}_k(R,f;\underline{x})$, which will disappear modulo A^{*k}.

Finally, changing the basis f will only multiply $\mathrm{Kir}_k(R,f;\underline{x})$ by an element of A^{*k} (see Lemma 2.12).

Theorem 3.9 (Kirchhoff's theorem for simplicial complexes)**.** *If $A=\mathbb{Z}$ and k is a nonnegative even integer, then $A^{*k}=\{1\}$ and*

$$\mathrm{Kir}_k(\Delta;\underline{x}) = \sum_{\Gamma \text{ 0-forest of } \Delta} |H_{d-1}(\Gamma)/H_{d-1}(\Delta)|^k x^{\mathrm{Fac}(\Gamma)},$$

where $H_{d-1}(\Delta)$, resp. $H_{d-1}(\Gamma)$, are the $(d-1)$-th reduced homology group of Δ, resp. of Γ: in this case

$$H_{d-1}(\Gamma)/H_{d-1}(\Delta) \simeq \mathrm{B}_{d-1}(\Delta)/\mathrm{B}_{d-1}(\Gamma).$$

Proof. Let f be a basis of $\mathrm{Im}(R^\intercal)$. We recall the Definition 2.4 about Kirchhoff polynomials:

$$\mathrm{Kir}_k(R,f;\underline{x}) := \sum_{\substack{I\subset[1\,..\,n]\\ |I|=r}} \det\nolimits_f(R_I^\intercal)^k \underline{x}^I.$$

As the subsets $I\subset[1\,..\,n]$ of size r such that $\det_f(R_I^\intercal)\neq 0$ are exactly those which are in $\mathcal{B}(\mathfrak{M}_R)$ and as Proposition 3.6 states that $\mathcal{F}_0(\Delta)=\mathcal{B}(\mathfrak{M}_R)$, we get the following equation.

$$\sum_{I\in\mathcal{F}_0(\Delta)} \det\nolimits_f(R_I^\intercal)^k \underline{x}^I = \sum_{\substack{I\subset[1\,..\,n]\\ |I|=r}} \det\nolimits_f(R_I^\intercal)^k \underline{x}^I.$$

It remains to show that, if $I\in\mathcal{F}_0(\Delta)$, then

$$|\det\nolimits_f(R_I^\intercal)| = |\,\mathrm{B}_{d-1}(\Delta)/\,\mathrm{B}_{d-1}(\mathrm{Sub}_\Delta(I))|. \tag{24}$$

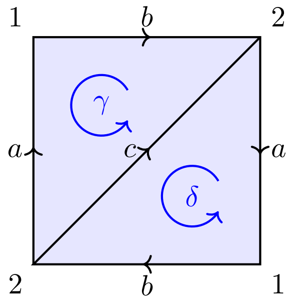
As columns of $R_I^\intercal$ are free and f overgenerates $\mathrm{Im}(R_I^\intercal)$, one can use Lemma 2.3 to obtain

$$\det\nolimits_f(R_I^\intercal) = |\,\mathrm{Im}(F)/\,\mathrm{Im}(R_I^\intercal)|.$$

But $\mathrm{Im}(F)=\mathrm{Im}(R^\intercal)$ and $R_I^\intercal$ is the d-th incidence matrix of $\mathrm{Sub}_\Delta(I)$ with respect to the same basis of A^p as $R^\intercal$, the d-th incidence matrix of Δ. Thus,

$$\mathrm{Im}(R^\intercal)/\,\mathrm{Im}(R_I^\intercal) \simeq \mathrm{B}_{d-1}(\Delta)/\,\mathrm{B}_{d-1}(\mathrm{Sub}_\Delta(I)),$$

which concludes the proof. ☐



$$R = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{matrix} \leftarrow \gamma \\ \leftarrow \delta \end{matrix} \quad \text{(columns } a, b, c\text{)}$$

Figure 4. Decomposition of $\mathbb{R}\mathrm{P}^2$.

Remark 3.10. The statement of the previous theorem could appear difficult to use for practical computations. The same theorem can be found in the article [4] under a form closer to the usual theorem: for any nonnegative even integer k (actually in the article $k = 2$),

$$\mathrm{Kir}_k(\widetilde{R}, e; \underline{x}) = |\operatorname{Tor}(H_{d-1}(\Delta))|^k \sum_{\Gamma \text{ 0-forest of } \Delta} |H_{d-1}(\Gamma)/H_{d-1}(\Delta)|^k x^{\mathrm{Fac}(\Gamma)},$$

where $\widetilde{R} = (R^\intercal)_J^\intercal$ for any $J \subset [1 \cdot\cdot n]$ of size r such that $\mathrm{rk}(R_J^\intercal) = r$, e is the standard basis of $\mathbb{Z}^r$ and $\operatorname{Tor}(H_{d-1}(\Delta))$ denotes the torsion part of $H_{d-1}(\Delta)$.

Example 3.11. The previous theorem cannot be used, a priori, in order to count the number of 0-forests. Indeed, the coefficients are not all equal to 1, unlike the case of a graph ($d = 1$) we have already seen (Example 2.9). Of course, if we get the entire polynomial, then it suffices to count the number of coefficients. Moreover, if $k = 0$, then all coefficients are equal to 1 but it is often easier to only compute $\mathrm{Kir}_k(\Delta; 1, \ldots, 1)$ using determinantal formulæ (see Propositions 2.6 and 2.7) and such formulæ do not exist for $k = 0$. Knowing $\mathrm{Kir}_k(\Delta; 1, \ldots, 1)$ for all even positive k would suffice to retrieve the number of 0-forests, but this method seems hard to apply.

Nevertheless, not all simplicial complexes have coefficient different from 1. Let us make the example of the real projective plane $\mathbb{R}\mathrm{P}^2$ which is one such case.

In order to make computations easier, we will study the real projective plane as a Δ-complex (or a CW-complex) and not as a simplicial one. But it is easy to find a simplicial decomposition of the real projective plane and the results are the same. Let us take the decomposition of Figure 4.

We have to take some basis of $\mathrm{Im}(R^\intercal)$, for example $f = \left(\left(\begin{smallmatrix}1\\1\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\0\\2\end{smallmatrix}\right)\right)$. Then $\widetilde{R} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ verifies $R^\intercal = F\widetilde{R}^\intercal$. We have $\det(\widetilde{R}) = -1$. Now we can calculate the Symanzik polynomial in $\mathbb{Z}$ for an even order k

$$\mathrm{Kir}_k(\mathbb{R}\mathrm{P}^2; x_1, x_2) = x_1 x_2.$$

Thus, $\mathbb{R}\mathrm{P}^2$ is a 0-forest of itself.

However, one can notice that $|\overline{\mathrm{Im}}(F)/\mathrm{Im}(F)| = 2$. Then the theorem of article [4] (Remark 3.10) would give the polynomial $2^k x_1 x_2$. The difference comes from that we choose to take a basis of $\mathrm{Im}(R^\intercal)$ instead of $\overline{\mathrm{Im}}(R^\intercal)$ in order to make the formula of generalized Kirchhoff's Theorem 3.9 simpler.

If the reader is interested in an example where all the coefficients are not equal to 1, it could look at Example 3.20.

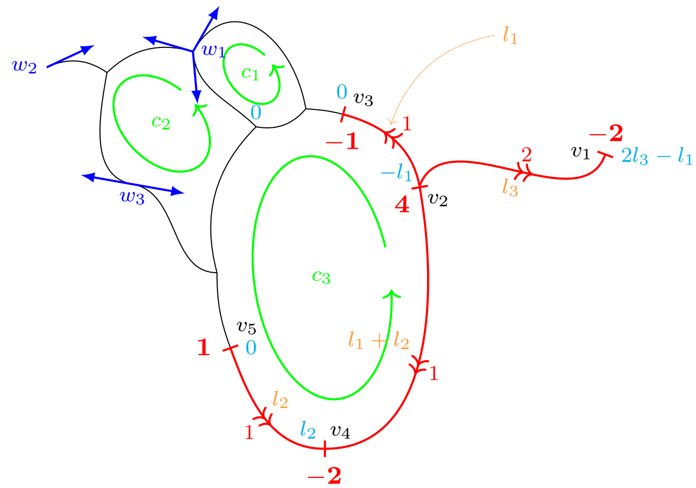


Figure 5. Divisors and rational functions.

To conclude this example, even if the Kirchhoff polynomials do not count directly the number of 0-forests, in the article [4] the authors explain that $|\operatorname{Tor}(H_{d-1}(\Delta))|^2 \operatorname{Kir}_2(\Delta; 1, \ldots, 1)$ counts the number of 0-forests of Δ taking into account a *fitted orientation*.

Now we focus only on Symanzik polynomials.

Definition 3.12. The *Symanzik polynomial class of order k of Δ* is an element of $A[\underline{x}]/A^{*k}$ denoted by $\operatorname{Sym}_k(\Delta; \underline{x})$ which is defined as

$$\operatorname{Sym}_k(\Delta; \underline{x}) := \operatorname{Sym}_k(R, f; \underline{x}) \pmod{A^{*k}},$$

where f is any basis of $\operatorname{Im}(R^\intercal)$.

A direct corollary of Theorem 3.9 follows.

Corollary 3.13. *If $A = \mathbb{Z}$ and k is a nonnegative even integer, then $A^{*k} = \{1\}$ and*

$$\operatorname{Sym}_k(\Delta; \underline{x}) = \sum_{\Gamma \text{ 0-forest of } \Delta} |H_{d-1}(\Gamma)/H_{d-1}(\Delta)|^k x^{\operatorname{Fac}(\Gamma)^c},$$

where $H_{d-1}(\Delta)$, resp. $H_{d-1}(\Gamma)$, is the $(d-1)$-th reduced homology group of Δ, resp. of Γ: in this case

$$H_{d-1}(\Gamma)/H_{d-1}(\Delta) \simeq \mathrm{B}_{d-1}(\Delta)/\mathrm{B}_{d-1}(\Gamma).$$

Next example is the first important example: Symanzik polynomials compute interesting data about metric graphs.

Example 3.14. The Kirchhoff's weighted theorem has a dual which could compute the volume of an important torus associated to a metric graphs: the tropical Jacobian of that graph. We will now summarize how this torus is defined and the Symanzik polynomials of even order compute its volume. All the details can be found in [12], [11], or [3].

A metric graph can be seen as a disjoint union of a finite number of closed bounded real intervals where some endpoints has been glued. There is a natural metric on it. Let n be a positive integer and $\mathfrak{G}$ be a connected metric graph composed of intervals $\mathrm{I}_1, \ldots, \mathrm{I}_n$. One can associate to $\mathfrak{G}$ a graph $\mathcal{G}$, whose edges correspond to the intervals. If $\mathcal{G}$ is simple, then one can associate to $\mathfrak{G}$ a 1-dimensional simplicial complex Γ whose facets are enumerated following the enumeration of the intervals (if $\mathcal{G}$ was not simple, Γ would be a Δ-complex). Let g be the genus of $\mathcal{G}$ (see Example 3.4). Then g is the rank of $\mathrm{Z}_1(\Gamma)$. Let $c_1, \dots, c_g$ be a bases of $\mathrm{Z}_1(\Gamma)$ (see an example on Figure 5 in green).

We set $A = \mathbb{Z}$. A divisor of $\mathfrak{G}$ is a finite abstract linear combination of points of $\mathfrak{G}$ over $\mathbb{Z}$. For example, on Figure 5 in bold red there is the following linear combination

$$D_0 = -2(v_1) + 4(v_2) - (v_3) - 2(v_4) + (v_5).$$

The *degree* $\deg(D)$ of a divisor D is the sum of its coefficient. For example,

$$\deg(D_0) = -2 + 4 - 1 - 2 + 1 = 0.$$

The set of divisors of degree l, for some integer l, is denoted by $\mathrm{Div}^l(\mathfrak{G})$.

At each point p of $\mathfrak{G}$ we associate the *directions* from this point denoted $\vec{p}_1, \dots, \vec{p}_l$ where l is the degree $\deg(p)$ of p; for example, in Figure 5 in blue, w_1 has three directions whereas w_2 only has one. Most of the points (i.e., all but a finite number) have degree two. Note that we use deg for both divisors and vertices with different meanings.

Let f be a continuous piecewise linear real-valued function on $\mathfrak{G}$, i.e., a real-valued function such that, for each $i \in [1 \cdot\cdot n]$, the induced function on the interval I_i is piecewise linear. The function f is called *rational* if all the slopes are in $\mathbb{Z}$. Such a function can be seen on the Figure 5 where the slopes (in red, next to double arrows; double arrows follow the increasing direction), the value of the function on some points (in cyan), and the length of the slopes (in orange: l_1, l_2, l_3) are indicated. One can naturally define the *derivative of f on one point p along one direction $\vec{p}_i$* denoted by $\mathrm{d}_{p_i} f(p)$, for $i \in [1 \cdot\cdot \deg(p)]$. We define the following function, which is zero for almost every p,

$$\mathrm{ord}_p(f) = \sum_{i=1}^{\deg(p)} \mathrm{d}_{p_i} f(p).$$

For example, if f is the function of the figure,

$$\mathrm{ord}_{v_2}(f) = 1 + 2 + 1 = -4.$$

A divisor is *principal* if it is of the form

$$\sum_{p \in \mathfrak{G}} \mathrm{ord}_p(f) \cdot (p),$$

for some rational function f on the metric graph. The set of principal divisors is denoted by $\mathrm{Prin}(\mathfrak{G}) \subset \mathrm{Div}^0(\mathfrak{G})$. For example, the red divisor of Figure 5 is the principal divisor associated to the rational function of the figure. Two divisors are called*equivalent* if their difference is in $\mathrm{Prin}(\mathfrak{G})$. The *Picard group* is by definition

$$\mathrm{Pic}(\mathfrak{G}) := \mathrm{Div}(\mathfrak{G}) / \mathrm{Prin}(\mathfrak{G}),$$

and, if l is any integer, we define the set

$$\mathrm{Pic}^l(\mathfrak{G}) := \mathrm{Div}^l(\mathfrak{G}) / \mathrm{Prin}(\mathfrak{G}).$$

If ℓ is a path on $\mathfrak{G}$, i.e., a locally isometric function from a closed bounded real interval to $\mathfrak{G}$, (red, blue and purple paths are some examples on Figure 6), its *boundary* $\partial(\ell)$ is the divisor $(\ell^+) - (\ell^-)$ where ℓ^+ denoted the endpoint of ℓ and ℓ^- its beginning point. Figure 6 presents three paths with the same boundary: $(v_2) - (v_2)$. We call *multipath* the $\mathbb{Z}$-module of finite abstract linear combination of paths over $\mathbb{Z}$ quotiented by concatenation; $\mathrm{Path}(\mathfrak{G})$

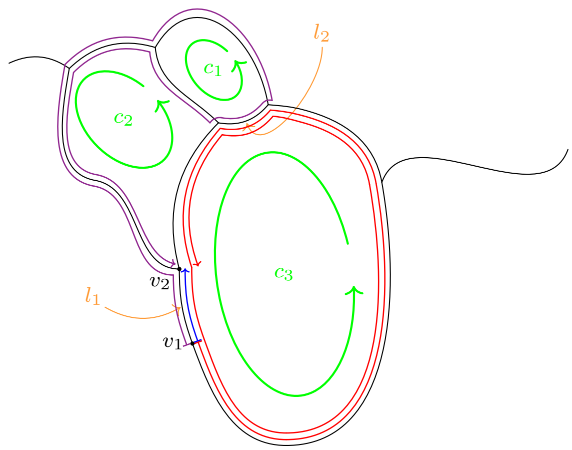


FIGURE 6. Paths and bounds on metric graphs.

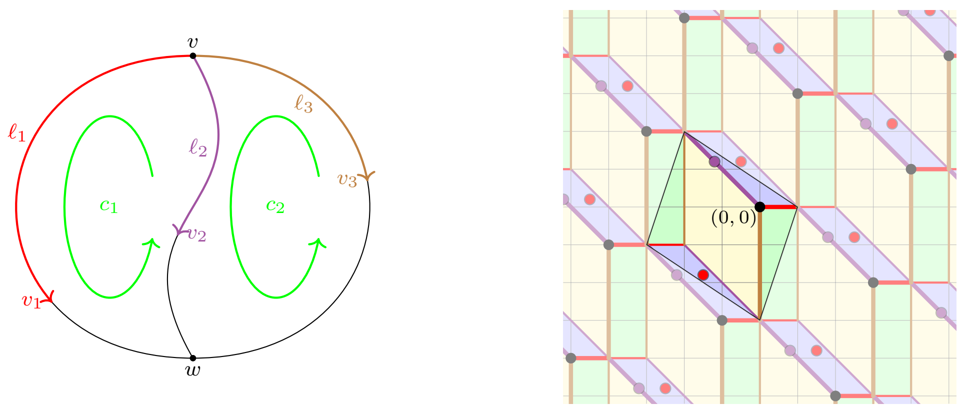


FIGURE 7. Bijection between the set of break divisors and the tropical Jacobian.

will denote the $\mathbb{Z}$-module of multipaths. The map ∂ can naturally be extended to $\mathrm{Path}(\mathfrak{G})$. We have the following claim. Any divisor of degree 0 has a nonempty preimage by ∂ and the kernel of ∂ is generated by $\mathfrak{c}_1, \dots, \mathfrak{c}_g$ where $\mathfrak{c}_i$, $i \in [1 \cdot\cdot g]$, is any path making one turn along c_i following its orientation. For example, in Figure 6, the red path has the same boundary as the blue one and can be obtaines from the blue one making two more turns along c_3. Similarly, in term of multipaths, the violet path can be obtained from the blue one adding one counter-turn along c_1.

If ℓ is a path, one can define the *intersection product* of ℓ with a cycle c_i, $i \in [1 \cdot\cdot g]$, denoted by $\langle \ell, c_i \rangle$, to be the oriented length, in $\mathbb{R}$, traveled by ℓ through c_i. For example, in Figure 6, the intersection product of the blue path with c_3 is equal to $-l_1$ and the intersection product of the violet one with c_3 is equal to $l_2 - l_1$. The intersection product can be extended linearly to $\mathrm{Path}(\mathfrak{G}) \times \mathrm{Z}(\Gamma)$. We define the map

$$\begin{array}{rrcl} \phi : & \mathrm{Path}(\mathfrak{G}) & \longrightarrow & \mathbb{R}^g, \\ & \mathcal{L} & \longmapsto & (\langle \mathcal{L}, c_1 \rangle, \dots, \langle \mathcal{L}, c_g \rangle). \end{array}$$

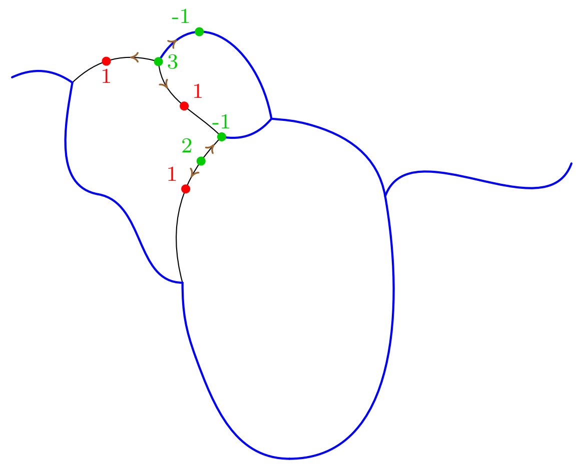


FIGURE 8. A break divisor and the corresponding tree.

For example the violet dot in the right of Figure 7 represents the image $\phi(\ell_2)$, ℓ_2 being the violet path represented on the left of length 1.2. By the universal property of quotient we can naturally define

$$\widetilde{\phi} : \mathrm{Path}(\mathfrak{G})/\ker(\partial) \longrightarrow \mathrm{Jac}(\mathfrak{G}),$$

where

$$\mathrm{Jac}(\mathfrak{G}) := \mathbb{R}^g/\phi(\ker(\partial))$$

is (isometric to) the tropical Jacobian of $\mathfrak{G}$. As $\ker(\partial) \simeq \mathrm{Z}(\Gamma)$ and as $\phi(\ker(\partial))$ is a $\mathbb{Z}$-submodule of rank g in $\mathbb{R}^g$ (black dots in the right of Figure 7 is this $\mathbb{Z}$-module), $\mathrm{Jac}(\mathfrak{G})$ is a torus of dimension g.

Now fix a divisor D_0 of degree g (in Figure 7, $D_0 = 2{\cdot}(v)$). For another divisor $D \in \mathrm{Div}^g(\mathfrak{G})$, the preimage $\partial^{-1}(D - D_0)$ is an element of $\mathrm{Path}(\mathfrak{G})/\ker(\partial)$. Thus, one can define the Abel-Jacobi map

$$\begin{array}{rccl} \mu : & \mathrm{Div}^g(\mathfrak{G}) & \to & \mathrm{Jac}(\mathfrak{G}), \\ & D & \mapsto & \widetilde{\phi}(\partial^{-1}(D - D_0)). \end{array}$$

For example, in Figure 7, the red dots are equal to the image of $\partial^{-1}((v_1) + (v_2))$ by ϕ. Actually, the kernel of μ is exactly the set of divisors which are equivalent to D_0. Thus, μ can be factorized by

$$\psi : \mathrm{Pic}^g(\mathfrak{G}) \xrightarrow{\sim} \mathrm{Jac}(\mathfrak{G}).$$

Now we want to calculate the volume of $\mathrm{Jac}(\mathfrak{G})$. A *generic break divisor* is any divisor in $\mathrm{Div}^g(\mathfrak{G})$ of the form $(p_1) + \cdots + (p_g)$ for different points $p_i \in \mathfrak{G}$, $i \in [1 \cdot\cdot g]$, which are all in the interior of an edge such that $\mathfrak{G} \setminus \{p_1, \dots, p_g\}$ is connected (or equivalently has no cycle). On Figure 8, the red divisor is a generic break divisor. It is easy to see that a divisor is a generic break divisor if and only if there is one point of the divisor in the interior of each edge which is in the complement of some spanning subtree of $\mathcal{G}$ (see Figure 8: there are a subtree of $\mathcal{G}$ in blue and a generic break divisor on its complementary in red). Thus, the set of generic breaks divisors are naturally in bijection with the disjoint union

$$\Upsilon := \sqcup_{T \in \mathcal{T}(\mathcal{G})} \bigtimes_{e \notin E(T)} \mathring{\mathrm{I}}_e,$$

where $\mathcal{T}(\mathcal{G})$ is the set of spanning subtrees of $\mathcal{G}$ and if T is a subtree, then $E(T)$ denoted the set of edges of T (if e is the i-th edge of $\mathcal{G}$, $\mathring{\mathrm{I}}_e := \mathring{\mathrm{I}}_i$ is the interior of the interval associated to e). Let $\overline{\Upsilon}$ be the topological adherence of Υ. Elements of $\overline{\Upsilon}$ are called *break divisor*. The volume of $\overline{\Upsilon}$ for the flat metric induced by the intersection pairing on $\mathrm{Jac}(\mathfrak{G})$ is

$$\sum_{J\in\mathcal{F}_0(\Gamma)}\prod_{i\in J^c}|\mathrm{I}_i|,$$

where $|\mathrm{I}_e|$ is the length of the interval I_e and e_i, $i\in[1\cdot\cdot n]$, denotes the i-th edge of $\mathcal{G}$. But we have seen in Example 2.14 that, if k is a positive even integer, as Γ corresponds to a graph, $A^{*k}=\{1\}$, and so

$$\mathrm{Sym}_k(\Gamma;\underline{x})=\sum_{J\in\mathcal{F}_0(\Gamma)}\underline{x}^{J^c}.$$

Thus,

$$\mathrm{Vol}(\overline{\Upsilon})=\mathrm{Sym}_k(\Gamma;|\mathrm{I}_1|,\dots,|\mathrm{I}_n|).$$

The following result, from [12] and [3], will be assumed. Each equivalence class of $\mathrm{Pic}^g(\mathfrak{G})$ contains exactly one break divisor. Thus, there exist a bijection Φ from $\overline{\Upsilon}$ to $\mathrm{Jac}(\mathfrak{G})$. In fact, the images of $\bigtimes_{e\notin E(T)}\mathrm{I}_e$, for $T\in\mathcal{T}$, form a quasi-partition of $\mathrm{Jac}(\mathfrak{G})$ into parallelotopes. Let us make an example.

On Figure 7, $\mathfrak{G}$ is represented on the left. The three edges are called, from left to right, e_1, e_2 and e_3 which are respectively of length $l_1=1$, $l_2=2$ and $l_3=3$, the cycles c_1 and c_2 are indicated. $\mathcal{G}$ has three trees T_1, T_2 and T_3, each one being reduced to an only edge, respectively e_1, e_2 and e_3. A break divisor associated to T_1 is, for example, $(v_1)+(v_2)$. Taking $D_0:=2\cdot(v)$, on the right of the figure many informations are represented.

- Black dots are the image of $\ker(\partial)$ by ϕ. To compute it, let $\mathfrak{c}_1$, resp. $\mathfrak{c}_2$, be the path going from (v) to itself making one turn of c_1, resp. c_2. We have

$$\begin{aligned}\phi(\ell_1)&=(\langle\ell_1,c_1\rangle,\langle\ell_1,c_2\rangle)\\&=(l_1+l_2,-l_2)\\&=(3,-2)\text{ and}\\\phi(\ell_2)&=(\langle\ell_2,c_1\rangle,\langle\ell_2,c_2\rangle)\\&=(-l_2,l_2+l_3)\\&=(-3,5).\end{aligned}$$

  Thus, $\phi(\ker(\partial))$ is the lattice generated by $(3,-2)$ and $(-3,5)$.
- The parallelogram is a fundamental domain of $\mathrm{Jac}(\mathfrak{G})$.
- The thick purple lines are the image by ϕ (for all choose of multipaths at the same time) of $((v)+(v_2))$ as (v_2) travels along (e_2). Similarly, the red ones correspond to $\phi((v)+(v_1))$ and the brown one to $\phi((v)+(v_3))$.
- The thin purple lines are the image by ϕ of $((w)+(v_2))$ as (v_2) travels along (e_2). Red and brown thin lines are defined similarly.
- Yellow domain corresponds to the parallelotope associated to the tree T_1 in $\mathrm{Jac}(\mathfrak{G})$, the green one to T_2 and the blue one to T_3. An interesting thing is that, for example, the volume of the blue domain is equal to the product of the length of the edges in T_1: $6=l_2\times l_3$.

Actually, the bijection Φ from $\overline{\Upsilon}$ to $\mathrm{Jac}(\mathfrak{G})$ is an isometry. As Φ is an isometry, we finally have

$$\mathrm{Vol}(\mathrm{Jac}(\mathfrak{G})) = \mathrm{Vol}(\overline{\Upsilon}) = \mathrm{Sym}_k(\Gamma; |I_1|, \dots, |I_q|).$$

In the previous example, the value of the Symanzik polynomial obtained at the end will not change if one adds or deletes some vertices inside an edge of the metric graph. We will see in the next subsection that this result is more general.

3.3. **Stability of the Symanzik polynomial by subdivision.** We will define here abstract oriented non degenerate subdivisions of an abstract simplicial complex, which we will simply call a *subdivision*. Let us give a geometrical intuition: a subdivision is obtained cutting some faces in smaller faces of the same dimension. We will need boundary maps of every dimension. Let Δ be the simplicial complex already fixed above. For all $l \in [0 \,..\, d]$, one can define the *l-th (reduced) boundary map on* Δ denoted by $\partial_{\Delta,l}$ as

$$\begin{array}{rccl} \partial_{\Delta,l}: & A\langle \Delta_l \rangle & \to & A\langle \Delta_{l-1} \rangle, \\ & \zeta & \mapsto & \partial_{\Delta_{(l)}}(\zeta), \end{array}$$

where $\partial_{\Delta_{(l)}}$ is defined as in the last subsection (Equation (21)) for the induced enumeration of vertices on the l-dimensional skeleton of Δ.

Until now, we only used the standard orientation relative to an arbitrary enumeration of the vertices. One can define an *orientation relative to this standard orientation* as a function from the set of faces to $\{-1, 1\}$.

Let Γ be a simplicial complex and let $\mu : \Gamma \to \Delta$ be a map from the set of faces of Γ to the set of faces of Δ. Let $\varepsilon_l : \Gamma_l \to \{-1, 1\}$, $l \in [-1 \,..\, d]$, be an orientation of faces of Γ relative to the standard orientation. For all $l \in [-1 \,..\, d]$, we define the linear map

$$\begin{array}{rccl} \varphi_l : & A\langle \Gamma_l \rangle & \to & A\langle \Delta_l \rangle, \\ & \gamma & \mapsto & \left\{ \begin{array}{ll} \varepsilon(\gamma)\mu(\gamma) & \text{if } \dim(\gamma) = \dim(\mu(\gamma)), \\ 0 & \text{otherwise.} \end{array} \right. \end{array}$$

Its adjoint, denoted by φ_l^*, verifies that, for all $\delta \in \Delta_l$,

$$\varphi_l^*(\delta) = \sum_{\substack{\gamma \in \mu^{-1}(\delta) \\ \dim(\gamma) = l}} \varepsilon(\gamma)\gamma.$$

Definition 3.15. With the above notations, (Γ, μ) *is a subdivision of* Δ *for the orientation* ε if it verifies the following four conditions.

(1) μ is increasing for the inclusion, i.e., if $\gamma_1 \subset \gamma_2$, then $\mu(\gamma_1) \subset \mu(\gamma_2)$.
(2) $\dim(\mu(\gamma)) \geqslant \dim(\gamma)$ for all $\gamma \in \Gamma$.
(3) For all $l \in [0 \,..\, d]$ and for all $\delta \in \Delta_l$,
$$\partial_{\Gamma,l}(\varphi_l^*(\delta)) = \varphi_{l-1}^*(\partial_{\Delta,l}(\delta)).$$
(4) For all face $\delta \in \Delta_l$ of dimension $l \in [-1 \,..\, d]$, there is no nontrivial cycle of $\mathrm{Z}_l(\partial_{\Gamma,l})$ in the A-submodule of $A\langle \Gamma_l \rangle$ generated by the set
$$\{\gamma \in \mu^{-1}(\delta) \,|\, \dim(\gamma) = \dim(\delta)\}.$$

For an Example of such subdivision, see Example 3.19.

If (Γ, μ) is a subdivision of Δ for an orientation ε, one can roughly summarize the four conditions as follows: each face of Γ has a non degenerate image included in a face of Δ, with the induced orientation if the dimension of the image is the same, these images make a kind of partition for each face of Δ, and a cycle of Γ cannot be mapped into a face of Δ of the same dimension. For example, because of the fourth condition, a subdivision of a triangle cannot contain a sphere. Actually, one would expect something stronger: a subdivision of a part of a surface which does not contain any sphere should not contain any sphere. This idea corresponds to the following lemma.

If $l \in [-1 \,..\, d]$, let

$$\widetilde{\Gamma}_l := \{\gamma \in \Gamma_l \,|\, \dim(\mu(\gamma)) = l\},$$

and let $\widetilde{\partial}_l$ be the restriction of $\partial_{\Gamma,l}$ to $A\langle\widetilde{\Gamma}_l\rangle$.

Lemma 3.16. *With the above notations, if* (Γ, μ) *is a subdivision of* Δ *for some orientation, then*

$$\ker(\widetilde{\partial}_d) \subset \mathrm{Im}(\varphi_d^*).$$

Before proving this lemma, we need to state another lemma.

Lemma 3.17. *With the notations of Definition 3.15, one has the following results.*

- μ *is surjective.*
- *For all* $l \in [-1 \,..\, n]$, φ_l *is surjective.*
- *For all* $l \in [-1 \,..\, n]$, φ_l^* *is injective.*

Proof of Lemma 3.17. Let us proof the third point by induction on l. If $l = -1$, $\varphi_{-1}^*(\emptyset) = \pm\emptyset \neq 0$, thus φ_{-1}^* is injective. Let $l \in [0 \,..\, d]$ be an integer, and suppose that φ_{l-1}^* is injective. Let $\delta \in \Delta_l$. Clearly, $\partial_{\Delta,l}(\delta) \neq 0$. Thus, by the induction hypothesis, $\varphi_{l-1}^*(\partial_{\Delta,l}(\delta)) \neq 0$. Therefore, the third condition of Definition 3.15 implies that $\varphi_l^*(\delta) \neq 0$. Now let $\zeta \in A\langle\Delta_l\rangle$ be an arbitrary nonzero element. Let δ be an element of Δ_l such that δ has a nonzero coefficient in ζ. As $\varphi_l^*(\delta) \neq 0$, there exists $\gamma \in \Gamma_l$ such that γ has a nonzero coefficient in $\varphi_l^*(\delta)$, i.e., such that $\mu(\gamma) = \delta$. As $\mu(\gamma) = \delta$, γ could not have a nonzero coefficient in $\varphi_l^*(\delta')$ if $\delta' \in \Delta_l - \delta$. Thus, γ has a nonzero coefficient in $\varphi_l^*(\zeta)$. Finally $\varphi_l^*(\zeta) \neq 0$ and so $\ker(\varphi_l^*)$ is trivial, which is the induction hypothesis for order l.

We conclude by induction that the third point is true. The second point comes automatically from the third one: the adjoint of an injective map is surjective. If $\delta \in \Delta$ for some $l \in [-1 \,..\, d]$, $\varphi_l^*(\delta) \neq 0$ implies that $\mu^{-1}(\delta) \neq \emptyset$. Thus the third point implies the first one, which finishes the proof. ☐

Proof of Lemma 3.16. First, notice that, for all $l \in [1 \,..\, d]$,

$$\partial_{\Gamma,l-1} \circ \partial_{\Gamma,l} = \partial_{\Delta,l-1} \circ \partial_{\Delta,l} = 0. \tag{25}$$

This can easily be proven using the definition (cf Equation (21)). Let us show by induction on l that

$$\ker(\widetilde{\partial}_l) \subset \mathrm{Im}(\varphi_l^*). \tag{26}$$

Suppose that $l = 0$. Let us prove that μ is a bijection between $\widetilde{\Gamma}_0$ and Δ_0. By Lemma 3.17, μ is surjective. Suppose that $\mu(\gamma_1) = \mu(\gamma_2)$ where $\gamma_1, \gamma_2 \in \widetilde{\Gamma}_0$. Then, $\gamma_1 - \gamma_2$ is in the kernel of $\widetilde{\partial}_0$. Thus, because of the fourth condition of Definition 3.15, $\gamma_1 - \gamma_2 = 0$. That implies the injectivity. As μ is a bijection, $\mathrm{Im}(\varphi_0^*) = A\langle\widetilde{\Gamma}_0\rangle$. Thus, the induction property (26) is true for $l = 0$.

Let $l \in [1 \,..\, d]$ be an integer, and suppose that the induction property is true for $l - 1$. Let $\zeta = \sum_{i \in I} \alpha_i \gamma_i$ be an element of $\ker(\widetilde{\partial}_l)$ where γ_i is the i-th face of Γ_l, I is a subset of $[1 \,..\, |\Gamma_l|]$

and α_is are elements of A. Let δ be a face of Δ_l and K be the set of integers i such that $\gamma_i \in \mu^{-1}(\delta) \cap \widetilde{\Gamma}_l$. Let $J = I \cap K$. Suppose that J is nonempty. Let $\zeta_\delta = \sum_{i\in J} \alpha_i\gamma_i$. One has

$$\zeta = \sum_{\delta' \in \Delta_l} \zeta'_\delta.$$

Thus,

$$\sum_{\delta' \in \Delta_l} \widetilde{\partial}_l(\zeta_{\delta'}) = 0. \tag{27}$$

We will show that the boundary of ζ_δ is in $\mathrm{Im}(\varphi^*_{l-1})$.

First, let us prove by contradiction that $\widetilde{\partial}_l(\zeta_\delta)$ is in $A\langle\widetilde{\Gamma}_{l-1}\rangle$. Suppose that there is a face $\gamma' \in \Gamma_{l-1} \setminus \widetilde{\Gamma}_{l-1}$ such that the coefficient of γ' in $\widetilde{\partial}_l(\zeta_\delta)$ is nonzero. Then there exists $i' \in J$ such that the coefficient of γ' in $\widetilde{\partial}_l(\gamma_{i'})$ is nonzero. Thus, γ' is a face of $\gamma_{i'}$, i.e., $\gamma' \subset \gamma_{i'}$. Moreover, by condition (1), $\mu(\gamma') \subset \mu(\gamma_{i'}) = \delta$. But we supposed that $\gamma' \notin \widetilde{\Gamma}_{l-1}$, i.e., that $\dim(\mu(\gamma')) > l-1$. The only possibility is that $\mu(\gamma') = \delta$. Now looking at Equation (27), we obtain that γ' must have a nonzero coefficient in $\widetilde{\partial}_l(\zeta_{\delta'})$ for some $\delta' \in \Delta_l - \delta$. But the same argument implies $\mu(\gamma') = \delta'$. Thus, $\delta = \delta'$ which is absurd. Finally,

$$\widetilde{\partial}_l(\zeta_\delta) \in A\langle\widetilde{\Gamma}_{l-1}\rangle.$$

Last equation implies that, if γ' has a nonzero coefficient in $\widetilde{\partial}_l(\zeta_\delta)$, then $\gamma' \in \widetilde{\Gamma}_{l-1}$, thus $\dim(\mu(\gamma')) = l-1$. But we also have that $\mu(\gamma') \subset \delta$. Thus,

$$\mu(\gamma') \text{ is a subset of } \delta \text{ of dimension } l-1. \tag{28}$$

We have seen (Equation (25)) that $\widetilde{\partial}_{l-1} \circ \widetilde{\partial}_l(\zeta_\delta) = 0$. Thus, $\widetilde{\partial}_l(\zeta_\delta) \in \ker(\widetilde{\partial}_{l-1})$. The induction hypothesis (26) implies that $\widetilde{\partial}_l(\zeta_\delta) \in \mathrm{Im}(\varphi^*_{l-1})$. Let ξ_δ be such that

$$\varphi^*_{l-1}(\xi_\delta) = \widetilde{\partial}_l(\zeta_\delta). \tag{29}$$

Let $\delta' \in \Delta_{l-1}$ having a nonzero coefficient in ξ_δ. Let $\gamma' \in \Gamma_{l-1}$ having a nonzero coefficient in $\varphi^*_{l-1}(\delta')$, i.e., such that $\mu(\gamma') = \delta'$. Then γ' has a nonzero coefficient in $\varphi^*_{l-1}(\xi_\delta)$ (because it could not have a nonzero coefficient in $\varphi^*_{l-1}(\delta'')$ if $\delta'' \neq \delta'$, else $\mu(\gamma') = \delta''$). Thus, by (28), $\delta' \subset \delta$. Let $\Delta_{\delta,l-1}$ be the subcomplex of Δ composed of all proper faces of δ:

$$\Delta_{\delta,l-1} := \{\delta'' \subsetneq \delta\}.$$

It is easy to see that $\mathrm{Z}_{l-1}(\Delta_{\delta,l-1})$ is generated by $\partial_{\Delta,l}(\delta)$. Notice that Equation (28) implies $\xi_\delta \in A\langle(\Delta_{\delta,l-1})_{l-1}\rangle$. Let us show that ξ_δ is in $\mathrm{Z}_{l-1}(\Delta_{\delta,l-1})$. By Equation (25),

$$\widetilde{\partial}_{l-1} \circ \varphi^*_{l-1}(\xi_\delta) = \widetilde{\partial}_{l-1} \circ \widetilde{\partial}_l(\zeta_\delta) = 0.$$

If we use the third condition of Definition 3.15, we obtain

$$\widetilde{\partial}_{l-1} \circ \varphi^*_{l-1}(\xi_\delta) = \varphi^*_{l-2} \circ \widetilde{\partial}_{l-1}(\xi_\delta) = 0.$$

We have already seen in Lemma 3.17 that the kernel of φ^*_{l-2} is trivial. Thus,

$$\widetilde{\partial}_{l-1}(\xi_\delta) = 0.$$

Then $\xi_\delta \in \mathrm{Z}_{l-1}(\Delta_{\delta,l-1})$ and so there exists an $a \in A$ such that

$$\xi_\delta = a\partial_{\Delta,l}(\delta). \tag{30}$$

Finally,

$$\begin{aligned}\widetilde{\partial}_l(\zeta_\delta) &= \varphi^*_{l-1}(\xi_\delta) && \text{(Equation (29))}\\ &= \varphi^*_{l-1}(a\partial_{\Delta,l}(\delta)) && \text{(Equation (30))}\\ &= \partial_{\Gamma,l}\circ\varphi^*_l(a\delta). && \text{(third condition)}\end{aligned}$$

As $\varphi^*_l(a\delta)$ is an element of $\widetilde{\Gamma}_l$, one can replace $\partial_{\Gamma,l}$ by $\widetilde{\partial}_l$ and obtain

$$0 = \widetilde{\partial}_l(\zeta_\delta - \varphi^*_l(a\delta)).$$

But by definition of ζ_δ and of φ^*_l, $\zeta_\delta - \varphi^*_l(a\delta)$ is in the A-submodule generated by the elements in

$$\{\gamma \in \mu^{-1}(\delta) \,\big|\, \dim(\gamma) = \dim(\delta)\}.$$

We can conclude, using the fourth condition of Definition 3.15, that $\zeta_\delta = \varphi^*_l(a\delta)$. Thus, $\zeta_\delta \in \mathrm{Im}(\varphi^*_l)$ and

$$\zeta = \sum_{\delta'\in\Delta_l} \zeta_{\delta'} \in \mathrm{Im}(\varphi^*_l).$$

That implies the induction hypothesis (Equation (26)).

We conclude the lemma by induction: setting $l = d$, the induction hypothesis is

$$\ker(\widetilde{\partial}_d) \subset \mathrm{Im}(\varphi^*_d).$$

□

In our case, subdivision is interesting because of the following proposition.

Proposition 3.18. *Suppose* (Γ, μ) *is a subdivision of* Δ *for some orientation* ε *of the faces of* Γ*, and denote by* m *the number of facets of* Γ*. Then* $\dim(\Gamma) = \dim(\Delta)$ *and, modulo* A^{*k}*, for any even positive integer* k*, we have*

$$\mathrm{Sym}_k(\Gamma; \underline{x}) = \mathrm{Sym}_k(\Delta; \underline{y})$$

for any even positive integer k*, where, for* $i \in [1 \mathinner{..} n]$*,*

$$y_i = \sum_{\substack{j\in[1\mathinner{..}m]\\ \mu(\gamma_j)=\delta_i}} x_j$$

with δ_i *being the* i*-th facet of* Δ *and* γ_j *the* j*-th facet of* Γ*.*

This proposition is really important because it means that one can define Symanzik polynomials of a topological triangulable space endowed with a measure. If the measure of any simple path is 0, and if the triangulation of the space is unique up to subdivision, then the Symanzik polynomial does not depend on the chosen triangulation. One can see that on Example 3.19 below.

Proof of the proposition. First, Lemma 3.17 implies that any facet of Δ has a nonempty preimage. Moreover, the second condition of Definition 3.15 implies that this preimage only contains d-dimensional faces of Γ. This condition also implies that Γ has no faces of dimension greater than d. Thus, $\dim(\Gamma) = \dim(\Delta)$.

Next, let T be the transpose of the d-th incidence matrix of Γ. We want to rewrite the third condition for facets in term of matrices. Let $P = (\mathbf{p}_{i,j}) \in \mathcal{M}_{n,m}(A)$ be the matrix associated to φ_d (for the standard bases corresponding to the enumerations of facets and of $(d-1)$-faces of Δ). It verifies

$$\mathbf{p}_{i,j} = \begin{cases} \varepsilon(\gamma_j) & \text{if } \mu(\gamma_j) = \delta_i, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

Let $Q \in \mathcal{M}_{p,q}(A)$, where p and q are the number of $(d-1)$-dimensional faces of Δ and Γ respectively, be the matrix associated to φ_{d-1}. Thus, the third equation of Definition 3.15 for $l = d$ is

$$\partial_{\Gamma,d} \circ \varphi_d^* = \varphi_{d-1}^* \circ \partial_{\Delta,d},$$

which is clearly equivalent to

$$T^\intercal P^\intercal = Q^\intercal R^\intercal. \tag{32}$$

Now let $\widetilde{f}$ be a basis of $\ker(R^\intercal)$ and let $S := \widetilde{F}$. Let $\widetilde{g}$ be a family of A^m such that $\widetilde{G} = P^\intercal S$. We want to show that $\widetilde{g}$ forms a basis of $\ker(T^\intercal)$. By Equation (32),

$$T^\intercal P^\intercal S = Q^\intercal R^\intercal S = 0.$$

Thus, $\widetilde{g}$ is included in $\ker(T^\intercal)$. In order to show that $\widetilde{g}$ is free, it suffices to prove that columns of S are free, which is clearly true, and that columns of $P^\intercal$ are free, i.e., that φ_d^* is injective. But this is stated by Lemma 3.17.

It remains to show that $\widetilde{g}$ generates $\ker(R^\intercal)$. Lemma 3.16 implies that $\ker(T^\intercal) \subset \operatorname{Im}(P^\intercal)$. Thus, if $u \in \ker(T^\intercal)$, then there exists $v \in A^n$ such that $\mathbf{u} = P^\intercal \mathbf{v}$. One has

$$\begin{aligned} T^\intercal \mathbf{u} &= 0, \\ T^\intercal P^\intercal \mathbf{v} &= 0, \\ Q^\intercal R^\intercal \mathbf{v} &= 0. \end{aligned}$$

As Lemma 3.17 implies that $\ker(Q^\intercal)$ is trivial, we deduce from the last equation that

$$R^\intercal \mathbf{v} = 0.$$

Thus, $\mathbf{v} \in \ker(R^\intercal)$, i.e., $\mathbf{v} \in \operatorname{Im}(S)$, and so $\mathbf{u} \in \operatorname{Im}(\widetilde{G})$. Finally, $\ker(T^\intercal) \subset \operatorname{Im}(\widetilde{G})$ and so $\widetilde{g}$ is a basis of $\ker(T^\intercal)$.

Now one can use the determinantal formula for Symanzik polynomials for $k = 2$ (Proposition 2.21). If g is a basis of $\operatorname{Im}(T^\intercal)$, multiplying an element of $\widetilde{g}$ by a unit of A if necessary, one can assume that $\widetilde{G}$ is a normal kernel matrix of T with basis g (see Definition 2.20 about normal kernel matrices). Setting $Y = PXP^\intercal$, where $X = \operatorname{diag}(x_1, \dots, x_m)$, one has, modulo A^{*2},

$$\begin{aligned} \operatorname{Sym}_2(\Gamma; \underline{x}) &= \operatorname{Sym}_2(T, g; \underline{x}) && \text{(Definition 3.12)} \\ &= \det(\widetilde{G}^\intercal X \widetilde{G}) && \text{(Proposition 2.21)} \\ &= \det(S^\intercal P X P^\intercal S) \\ &= \det(S^\intercal Y S). \end{aligned}$$

Let us compute Y. Looking at the definition of P (Equation (31)), columns of P have at most one nonzero entry. Thus, rows of P are orthogonal for the standard inner product. Therefore, Y is a diagonal matrix. The i-th diagonal entry of Y is equal to

$$\sum_{j=0}^{m} \mathbf{p}_{i,j}^2 x_j,$$

i.e., to

$$\sum_{\substack{j\in[1\,..\,m]\\ \mu(\gamma_j)=\delta_i}} x_j,$$

where γ_j is the j-th facet of Γ and δ_i is the i-th facets of Δ. But last equation is exactly the definition of y_i in the statement of the proposition. Thus, $Y = \operatorname{diag}(y_1, \dots, y_n)$. Then, using Proposition 2.21 as above, modulo A^{*2},

$$\begin{aligned}\operatorname{Sym}_2(\Gamma; \underline{x}) &= \det(S^\intercal Y S)\\ &= \operatorname{Sym}_2(\Delta; \underline{y}).\end{aligned}$$

Finally, the proposition is true for $k = 2$ and so it is true for all even order (see Remark 2.5). □

Here is the second important example.

Example 3.19. For this example we set $A = \mathbb{Z}$. Let $\mathcal{S}$ be a compact orientable surface. Let Δ be a 2-dimensional simplicial complex and V be its set of vertices. Set a map $\nu : V \to U$ in some real vector space U. Set

$$\begin{array}{rrcl}\widetilde{\nu}: & \Delta & \to & \mathcal{P}(U),\\ & \{i_0,\dots,i_l\} & \mapsto & \operatorname{conv}(\nu(i_0),\dots,\nu(i_l)),\end{array}$$

where $\operatorname{conv}(v_0, \dots, v_l)$ is the convex hull of $\{v_0, \dots, v_l\}$ in U for any $l \in [0\,..\,d]$ and any points $v_0, \dots, v_l \in U$. Suppose that $\widetilde{\nu}$ verifies that, for all ordered pair of faces (γ, δ) of Δ,

$$\dim(\widetilde{\nu}(\gamma)) = \dim(\gamma),$$

where $\dim(\widetilde{\nu}(\gamma))$ denotes the affine dimension, and that

$$\widetilde{\nu}(\gamma \cap \delta) = \widetilde{\nu}(\gamma) \cap \widetilde{\nu}(\delta).$$

In this case, the union

$$|\Delta|_\nu := \bigcup_{\delta\in\Delta} \widetilde{\nu}(\delta)$$

is called a *geometric realization of* Δ (actually, we have already seen geometric realizations of the abstract bipyramide in Figures 2 and 3). Now, if there exists a homeomorphism $\Phi : |\Delta|_\nu \xrightarrow{\sim} \mathcal{S}$, then (Δ, ν, Φ) will be called an *abstract triangulation of* $\mathcal{S}$ (in Figure 9 the blues lines form a triangulation of the disk). If such a triangulation of $\mathcal{S}$ exists, $\mathcal{S}$ is said *triangulable*. It is known that all compact surfaces are triangulable.

In this paper we will said that a triangulation $(\Gamma, \upsilon, \Phi')$ is a *subtriangulation of* (Δ, ν, Φ) if $|\Gamma|_\upsilon = |\Delta|_\nu$, if $\Phi' = \Phi$, and if, for any $\gamma \in \Gamma$, there exists $\delta \in \Delta$ such that $\widetilde{\upsilon}(\gamma) \subset \nu(\delta)$. Therefore, Γ is a subdivision of Δ for the map μ such that $\mu(\gamma)$ is the minimal δ for inclusion such that $\upsilon(\gamma) \subset \nu(\delta)$. For example, in Figure 9 there are two triangulations of the disk, one in blue and one in red, and adding the thin arcs to red and blue ones we obtain a common subtriangulation (i.e., a subtriangulation of the blue triangulation by some simplicial complex and another subtriangulation of the red triangulation by the same simplicial complex). If we said that two triangulations of the same compact surface are equivalent if they have a common subtriangulation, and if we extend this relation in order to have an equivalence relation, then all triangulations of a compact surface are equivalent.

Let (Δ, ν, Φ) be a triangulation of the compact orientable surface $\mathcal{S}$, let (Γ, μ) be a subdivision of Δ for some orientation, and let (Γ, υ, Φ) be a subtriangulation of Δ with respect to μ. Suppose now there is some finite measure π on $\mathcal{S}$ such that the measure of simple path on

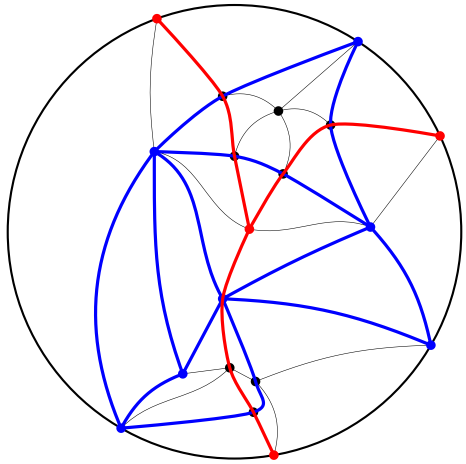


FIGURE 9. Two triangulations of the disk and a common triangulation.

$\mathcal{S}$ (i.e., of any part homeomorphic to a bounded real interval) is zero. We extend π to facets of Δ by

$$\pi(\delta) := \pi(\Phi(\widetilde{\nu}(\Delta))),$$

for any facet δ of Δ. Extend π to facets of Γ in the same way. Notice that the extension verifies the following property:

$$\pi(\delta) = \sum_{\substack{\gamma\in\Gamma_2\\ \mu(\gamma)=\delta}} \pi(\gamma).$$

Then, if k is even, Proposition 3.18 affirms that

$$\mathrm{Sym}_k(\Delta; \pi(\delta_1), \dots, \pi(\delta_n)) = \mathrm{Sym}_k(\Gamma; \pi(\gamma_1), \dots, \pi(\gamma_m)), \tag{33}$$

where m is the number of facets of Γ, δ_i, $i \in [1 \,..\, n]$, is the i-th facet of Δ, and γ_i, $i \in [1 \,..\, m]$, is the i-th facet of Γ. Thus, the value (33) does not depend on the choice of the triangulation.

Let us compute this value. One has the following intuitive property. $\mathrm{Z}_d(\Delta)$ has rank 1 and is generated by the sum of facets of Δ, with a good sign for each facet corresponding to an orientation of the facets with respect to an orientation of $\mathcal{S}$. Thus, one can obtain a 0-forest of Δ removing any facet of Δ. But we know by Corollary 3.13 that

$$\mathrm{Sym}_k(\Delta; \underline{x}) = \sum_{\Gamma \text{ 0-forest of } \Delta} |H_{d-1}(\Delta)/H_{d-1}(\Gamma)|^k x^{\mathrm{Fac}(\Gamma)^c}$$

Here, $|H_{d-1}(\Delta)/H_{d-1}(\Gamma)|$ is always equal to 1. Thus,

$$\mathrm{Sym}_k(\Delta; \underline{x}) = x_1 + \cdots + x_n,$$

and finally

$$\begin{aligned}\mathrm{Sym}_k(\Delta; \pi(\delta_1), \dots, \pi(\delta_n)) &= \pi(\delta_1) + \cdots + \pi(\delta_n)\\ &= \pi(\mathcal{S}).\end{aligned}$$

Thus, the Symanzik polynomial with variables assigned with the measure of corresponding facets is equal to the total measure of $\mathcal{S}$.

We end this subsection with an interesting example about the Symanzik polynomials which can be obtained from a simplicial complex.

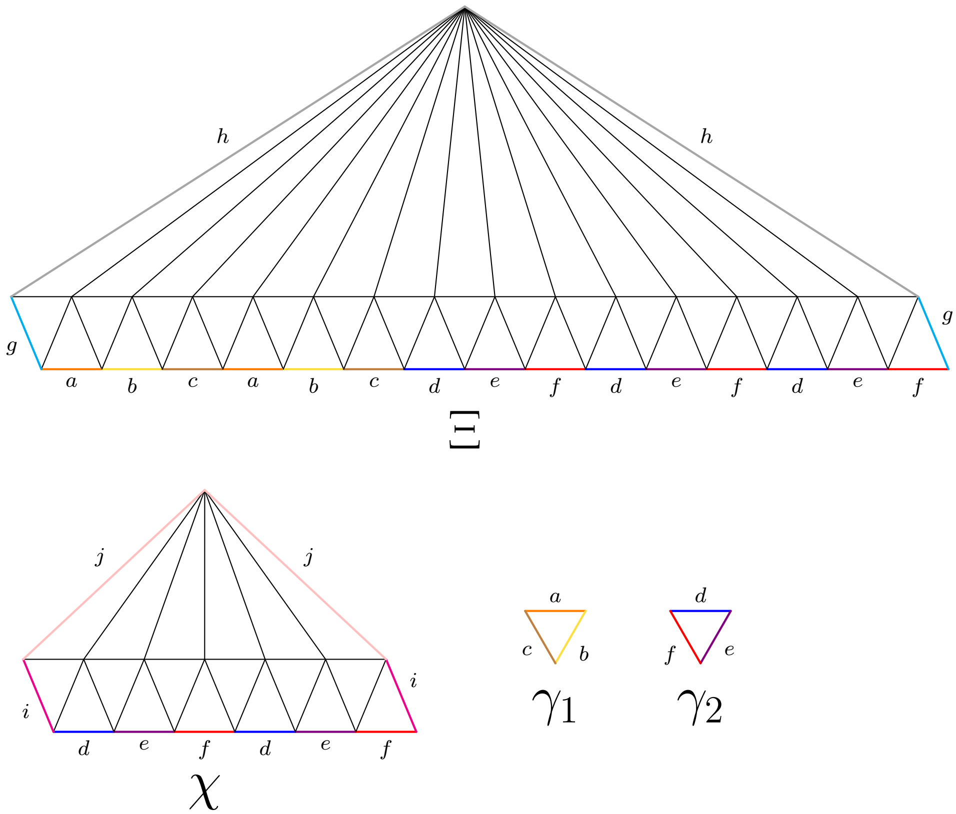


FIGURE 10. A simplicial complex linked to $\begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$.

Example 3.20. In this example $A = \mathbb{Z}$. Let Δ be the simplicial complex on Figure 10 where all edges with the same label are glued together. We allow to put a nonstandard orientation on facets of Δ. Suppose that all facets have a counterclockwise orientation on the figure. The d-cycles of Δ are generated by the following two cycles:

$$2\gamma_1 + 3\gamma_2 + \sum_{\gamma \in \Xi} \gamma \text{ and} \tag{34}$$

$$2\gamma_2 + \sum_{\gamma \in \chi} \gamma, \tag{35}$$

where sums are over all facets of the parts of the complex denoted by Ξ and χ, respectively. If we enumerate faces of Δ beginning with γ_1, then γ_2, then facets of Ξ and finally facets of χ, the following matrix $S \in \mathcal{M}_{n,2}(\mathbb{Z})$ is a normal kernel matrix of R for a well chosen basis of $\mathrm{Im}(R^\intercal)$

$$S := \begin{pmatrix} 2 & 3 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 & 1 & \cdots & 1 \end{pmatrix}^\intercal.$$

Proposition 2.21, about determinantal formula for Symanzik p olynomials for $k = 2$, states that

$$\mathrm{Sym}_2(\Delta; \underline{x}) = \det(S^\intercal X S),$$

with $X = \operatorname{diag}(x_1, \dots, x_n)$. Now set $x_3 = \cdots = x_n = 0$. Thus, we obtain

$$\begin{aligned}\operatorname{Sym}_2(\Delta; x_1, x_2, 0, \dots, 0) &= \det\left(\begin{pmatrix} \begin{matrix} 2 & 3 \\ 0 & 2 \end{matrix} & * \end{pmatrix} \begin{pmatrix} \begin{matrix} x_1 & 0 \\ 0 & x_2 \end{matrix} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \begin{matrix} 2 & 0 \\ 3 & 2 \end{matrix} \\ * \end{pmatrix} \right) \\ &= \det\left(\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix} \right) \\ &= \operatorname{Kir}_2(\begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}, e; \underline{x}),\end{aligned}$$

where e is the standard basis of $\mathbb{Z}^2$.

This example obviously generalizes in order to obtain any matrix instead of $\left(\begin{smallmatrix} 2 & 0 \\ 3 & 2 \end{smallmatrix}\right)$. Thus, just putting some variables to zero allows us to create a simplicial complex whose Symanzik polynomial is equal to the Symanzik polynomial (or, equivalently, Kirchhoff polynomial) of any integer matrix.

In the next subsection, we will add parameters. It turns out that orientations play an essential role in this case.

3.4. **Symanzik rational fractions, orientations and oriented matroid.** If $l \in [-1 \cdot\cdot d]$, we will denote be λ_l the natural isomorphism from $A\langle \Delta_l \rangle$ to $A^{|\Delta_l|}$, or, more simply, λ if there is no ambiguity on l. The reader would not be surprised by the two following definitions.

Definition 3.21. Let l be a nonnegative integer and let $u_1, \dots, u_l$ be elements of $\mathrm{B}_{d-1}(\Delta)$. The *Symanzik polynomial class of order k of Δ with parameters* $u_1, \dots, u_l$ is an element of $A[\underline{x}]/A^{*k}$ denoted by $\operatorname{Sym}_k(\Delta, u_1, \dots, u_l; \underline{x})$ defined as

$$\operatorname{Sym}_k(\Delta, u_1, \dots, u_l; \underline{x}) := \operatorname{Sym}_k(R, f, \lambda(u_1), \dots, \lambda(u_l); \underline{x}) \pmod{A^{*k}}$$

where f is any basis of $\operatorname{Im}(R^\intercal)$.

Definition 3.22. Let l be a nonnegative integer and let $u_1, \dots, u_l$ be elements of $\mathrm{B}_{d-1}(\Delta)$. The *(normalized) Symanzik rational fraction of order k of Δ with parameters* $u_1, \dots, u_l$ denoted by $\widetilde{\operatorname{Sym}}_k(\Delta, u_1, \dots, u_l; \underline{x})$ is defined as

$$\widetilde{\operatorname{Sym}}_k(\Delta, u_1, \dots, u_l; \underline{x}) := \widetilde{\operatorname{Sym}}_k(R, \lambda(u_1), \dots, \lambda(u_l); \underline{x}).$$

Notice that $\widetilde{\operatorname{Sym}}_k(\Delta, u_1, \dots, u_l; \underline{x})$ is an element of A and not of $A[\underline{x}]/A^{*k}$.

From now on, we take A a PID which is a subring of $\mathbb{R}$.

We want to deal with orientations because they will naturally appear in Symanzik polynomials with parameters. To do so, we will extend $\mathfrak{M}$ to an oriented matroid. We do not want to define precisely oriented matroids, but we will use specific ones: the ones representing family of vectors in A^n or matrices over A. Readers interested in more details could check [7], [8].

Before stating the definition, we need few more notations. If l is a nonnegative integer and $I = \{i_1, \dots, i_l\} \subset \mathbb{Z}$, we write $-I := \{-i_1, \dots, -i_l\}$. If we use the notation $\vec{I}$ for some subset $\{i_1, \dots, i_l\}$ of $\mathbb{Z}$, then it means that $-\vec{I} \cap \vec{I} = \emptyset$ and that I will denote the set $\{|i_1|, \dots, |i_l|\}$.

If we use the notation $\underline{I}$ for a tuple $(i_1, \dots, i_l)$, then it means that the i_js, $j \in [1 \cdot\cdot n]$, are all different and that I will denote the set of elements of $\underline{I}$: $I := \{i_1, \dots, i_l\}$.

If n, p are positive integers and if $u = (u_1, \dots, u_n)$ is a family of elements of A^p, then the *oriented matroid representing* u is denoted by $\vec{\mathfrak{M}}_u$ and is the matroid $\vec{\mathfrak{M}}_u = (\vec{E}_u, \vec{\mathcal{I}}_u)$ where $\vec{E}_u = -E_u \sqcup +E_u$, $E_u := [1 \cdot\cdot n]$, and $\vec{\mathcal{I}}_u \subset \mathcal{P}(\vec{E}_u)$ contains all subsets $I \in \vec{\mathcal{I}}_u$ such that there is no nontrivial nonnegative linear combination of elements in $(u_i)_{i \in I}$ equal to zero, where $u_{-i} := -u_i$, for $i \in [1 \cdot\cdot n]$.

If n, p are positive integers, and if $R \in \mathcal{M}_{n,p}(A)$ is a matrix, then the *oriented matroid representing* R is denoted by $\vec{\mathfrak{M}}_R = (\vec{E}_R, \vec{\mathcal{I}}_R)$ and is equal to $\vec{\mathfrak{M}}_u$ where $u = (u_1, \dots, u_n)$ and $R^\intercal = \mathbf{u}_1 \star \cdots \star \mathbf{u}_n$.

To an oriented matroid $\vec{\mathfrak{M}} = (\vec{E} = [-n \cdot\cdot -1] \sqcup [1 \cdot\cdot n], \vec{\mathcal{I}})$ one can naturally associate the nonoriented matroid $\mathfrak{M} := (E = [1 \cdot\cdot n], \mathcal{I})$ where $\mathcal{I}$ is the maximal subset of $\mathcal{P}(E)$ verifying

$$\forall\, \vec{I} \notin \vec{\mathcal{I}}, \qquad I \notin \mathcal{I}.$$

In this paper, we call *basis of the oriented matroid* $\vec{\mathfrak{M}}$ any tuple $\underline{I}$ of elements of $\vec{E}$ such that $I \in \mathcal{B}(\mathfrak{M})$. The set of all bases of $\vec{\mathfrak{M}}$ is denoted by $\mathcal{B}(\vec{\mathfrak{M}})$. The *rank of* $\vec{\mathfrak{M}}$ is $\mathrm{rk}(\vec{\mathfrak{M}}) := \mathrm{rk}(\mathfrak{M})$.

Now we can set $\vec{\mathfrak{M}} = (\vec{E}, \vec{\mathcal{I}}) := \vec{\mathfrak{M}}_R$ the oriented matroid representing R.

An *orientation of the bases of a matroid* is a map from the bases to the set $\{-1, +1\}$.

Definition 3.23. An orientation of the bases of a matroid is said *canonical* if it verifies the three following conditions for any basis $(i_1, \dots, i_r)$.

(1) It is an alternating map: for all permutation $\tau \in \mathfrak{S}_r$,
$$\varepsilon(i_{\tau(1)}, \dots, i_{\tau(r)}) = \sigma(\tau)\varepsilon(i_1, \dots, i_r).$$
(2) It is homogeneous: for every $j \in [1 \cdot\cdot r]$,
$$\varepsilon(i_1, \dots, i_{j-1}, -i_j, i_{j+1}, \dots, i_r) = -\varepsilon(i_1, \dots, i_r).$$
(3) If $(i_1, \dots, i_{r-1}, i'_r)$ is a second basis, then, for any $\zeta_1, \dots, \zeta_r, \zeta'_r \in \{-1, +1\}$ such that $\{\zeta_1 i_1, \dots, \zeta_r i_r, \zeta'_r i'_r\} \subset \mathcal{P}(\vec{E})$ is dependent,
$$\varepsilon(i_1, \dots, i_{r-1}, i'_r) = -\zeta_r \zeta'_r \varepsilon(i_1, \dots, i_r).$$

This orientation are called *chirotopes* in the theory of oriented matroid.

Remark 3.24. In the previous definition, Conditions (1) and (3) implies Condition (2). Indeed, in Condition (3) one can take $i'_r = -i_r$ and $\zeta'_r = \zeta_r$ obtaining

$$\varepsilon(i_1, \dots, i_{r-1}, -i_r) = -\varepsilon(i_1, \dots, i_r),$$

which is Condition (2) for $j = r$.

Proposition 3.25. *An oriented matroid has exactly two canonical orientations of its bases, and they are opposite.*

Justification of the proposition. The unicity, up to a sign, comes from the exchange property between bases (Claim 2.32, (4)): if one chooses the orientation of one basis, all the other bases can be obtained with the three operations available in the proposition. Thus, their orientation will be uniquely defined.

The existence in the case of oriented matroid representing a matrix over $A \subset \mathbb{R}$ will be clear after next example. For the general case, we refer to [8]. ☐

Example 3.26. Let f be a free family of size r overgenerating $\mathrm{Im}(R^\intercal)$. In this example we will show that the map

$$\begin{array}{rccl} \varepsilon: & \mathcal{B}(\vec{\mathfrak{M}}) & \to & \{-1,1\}, \\ & \underline{\vec{I}} & \mapsto & \mathrm{sgn}(\det_f(R^\intercal_{\underline{\vec{I}}})), \end{array}$$

is one of the two canonical orientations (of the bases) of $\vec{\mathfrak{M}}$ where

$$R^\intercal_{\underline{\vec{I}}} := \mathbf{r}_{i_1} \star \ldots \star \mathbf{r}_{i_r},$$

with $R^\intercal = \mathbf{r}_1 \star \ldots \star \mathbf{r}_n$, $\mathbf{r}_{-i} := -\mathbf{r}_i$ if $i \in [1\,..\,n]$, and $\underline{\vec{I}} = (i_1, \ldots, i_r)$.

First, ε is well defined because if $\underline{\vec{I}} \in \mathcal{B}(\vec{\mathfrak{M}})$, then $I \in \mathcal{B}(\mathfrak{M})$, thus columns of R_I are free and $\det(R_I)$ is nonzero.

Then the first two points of Definition 3.23 are simply properties of the determinant.

Finally it remains to prove the last point. Let $(i_1, \ldots, i_{r-1}, i'_r)$ be a second basis and let $\zeta_1, \ldots, \zeta_r, \zeta'_r \in \{-1, +1\}$ be such that $\{\zeta_1 i_1, \ldots, \zeta_r i_r, \zeta'_r i'_r\}$ is dependent. Then there exist $a_1, \ldots, a_r, a'_r \in \mathbb{R}$ not all equal to zero such that

$$\zeta_i = \begin{cases} 1 & \text{if } a_i = 0, \\ \mathrm{sgn}(a_i) & \text{otherwise}, \end{cases}$$
$$\zeta'_r = \mathrm{sgn}(a'_r),$$

and

$$a_1\mathbf{r}_{i_1} + \cdots + a_r\mathbf{r}_{i_r} + a'_r\mathbf{r}_{i'_r} = \mathbf{0},$$

where $\mathbf{0} \in \mathcal{M}_{n,1}(\mathbb{R})$ is the column matrix with only zero entries. Notice that $a'_r \neq 0$ because $(r_{i_1}, \ldots, r_{i_r})$ is a basis. Then one has, setting $\underline{\vec{J}} := (i_1, \ldots, i_{r-1})$,

$$\begin{aligned} \varepsilon(i_1, \ldots, i_{r-1}, i'_r) &= \mathrm{sgn} \circ \det(R_{\underline{\vec{J}}} \star \mathbf{r}_{i'_r}) \\ &= \mathrm{sgn} \circ \det\left(R_{\underline{\vec{J}}} \star \big(-\frac{1}{a'_r}(a_1\mathbf{r}_{i_1} + \cdots + a_r\mathbf{r}_{i_r})\big)\right) \\ &= \mathrm{sgn}\left(-\frac{1}{a'_r}\det\left(R_{\underline{\vec{J}}} \star (a_r\mathbf{r}_{i_r})\right)\right) \\ &= -\,\mathrm{sgn}\left(\frac{a_r}{a'_r}\right)\mathrm{sgn} \circ \det(R_{\underline{\vec{I}}}) \\ &= -\zeta_r\zeta'_r\varepsilon(i_1, \ldots, i_r). \end{aligned}$$

Thus, ε verifies the three conditions to be a canonical orientation of $\vec{\mathfrak{M}}$. By Proposition 3.25, the two orientations on $\vec{\mathfrak{M}}$ are ε and $-\varepsilon$.

Definition 3.27. Let r be a nonnegative integer, $\vec{\mathfrak{M}} = (\vec{E}, \vec{\mathcal{I}})$ be an oriented matroid of rank r, $\mathfrak{M} = (E, \mathcal{I})$ be the associated nonoriented matroid and $l \leqslant r$ be a nonnegative integer. Let $I \in \mathcal{I}$ be an independent set of rank l. Let $\underline{\vec{I}} = (i_1, \ldots, i_l)$ be a tuple of elements in I with an arbitrary order and with arbitrary signs.

We define $\underline{\overrightarrow{\mathrm{Compl}}}(I)$ as the set of *tuples completing* $\underline{\vec{I}}$ *into a basis*, i.e., of $(r-l)$-tuples $\underline{\vec{J}} := (j_{l+1}, \ldots, j_r)$ such that $(i_1, \ldots, i_l, j_{l+1}, \ldots, j_r) \in \mathcal{B}(\vec{\mathfrak{M}})$.

A *canonical orientation of* $\vec{\mathfrak{M}}$ *relative to* I, denoted by ε_I, is a function from $\underline{\overrightarrow{\mathrm{Compl}}}(I)$ to $\{-1, +1\}$ verifying: there exists a canonical orientation ε of $\vec{\mathfrak{M}}$ such that, for all elements $\underline{\vec{J}} \in \underline{\overrightarrow{\mathrm{Compl}}}(I)$,

$$\varepsilon_I(\underline{\vec{J}}) = \varepsilon(i_1, \ldots, i_l, j_{l+1}, \ldots, j_r).$$

Claim 3.28. *With the notations of the definition, the definitions of* $\underline{\overrightarrow{\mathrm{Compl}}}(I)$ *and of canonical orientations relative to* I *does not depend on the choice of* $\underline{\vec{I}}$.

Proof. It is clear that $\underline{\vec{J}} \in \underline{\overrightarrow{\mathrm{Compl}}}(I)$ if and only if I and J are disjoint and $I \sqcup J \in \mathcal{B}(I)$. Thus, the definition of $\underline{\overrightarrow{\mathrm{Compl}}}(I)$ does not depend on the choice of $\underline{\vec{I}}$.

Let ε be one of the two canonical orientations on $\vec{\mathfrak{M}}$. Let us define

$$\begin{array}{rrcl} \varepsilon_{\underline{\vec{I}}}: & \underline{\overrightarrow{\mathrm{Compl}}}(I) & \to & \{-1,+1\}, \\ & \underline{\vec{J}} & \mapsto & \varepsilon(i_1,\dots,i_l,j_{l+1},\dots,j_r). \end{array}$$

It is clear that $\varepsilon_{\underline{\vec{I}}}$ and $-\varepsilon_{\underline{\vec{I}}}$ are the two opposite canonical orientations relative to I.

Let $\underline{\vec{I'}} = (i'_1,\dots,i'_l)$ be a second tuple of elements in I with some signs. It remains to show that $\varepsilon_{\underline{\vec{I'}}} = \pm\varepsilon_{\underline{\vec{I}}}$. There exists $\tau \in \mathfrak{S}_l$ and $\zeta_1,\dots,\zeta_l \in [1\,..\,l]$ such that

$$i'_j = \zeta_j i_{\tau(j)}$$

for all $j \in [1\,..\,l]$. Let $\underline{\vec{J}} \in \underline{\overrightarrow{\mathrm{Compl}}}(I)$. One has

$$\begin{aligned} \varepsilon_{\underline{\vec{I'}}}(\underline{\vec{J}}) &= \varepsilon(i'_1,\dots,i'_l,j_{l+1},\dots,j_r) \\ &= \varepsilon(\zeta_1 i_{\tau(1)},\dots,\zeta_l i_{\tau(l)},j_{l+1},\dots,j_r). \end{aligned}$$

Thus, using Conditions (1) and (2) of Definition 3.23,

$$\begin{aligned} \varepsilon_{\underline{\vec{I'}}}(\underline{\vec{J}}) &= \zeta_1\cdots\zeta_l\,\sigma(\tau)\varepsilon(i_1,\dots,i_l,j_{l+1},j_r) \\ &= \zeta_1\cdots\zeta_l\,\sigma(\tau)\varepsilon_{\underline{\vec{I}}}(\underline{\vec{J}}). \end{aligned}$$

As the factor $\zeta_1\cdots\zeta_l\,\sigma(\tau)$ does not depend on $\underline{\vec{J}}$,

$$\varepsilon_{\underline{\vec{I}}} = \zeta_1\cdots\zeta_l\,\sigma(\tau)\varepsilon_{\underline{\vec{I'}}}.$$

Finally, the definition of canonical orientations relative to I does not depend on the choice of $\underline{\vec{I}}$. □

Now the following proposition is straightforward.

Proposition 3.29. *If* $\vec{\mathfrak{M}}$ *is an oriented matroid, and if* $\mathfrak{M}$ *is its associated nonoriented matroid, then, for any independent set* I *of* $\mathfrak{M}$*, there are exactly two canonical orientations relative to* I *and, they are opposite.*

Canonical relative orientations often have simple expression, as in the following two examples.

Example 3.30. Let Δ be a simplicial complex of dimension d. Using the usual notations, set $l = r-1$. Let $I \in \mathcal{I}_l$, $\underline{\vec{I}} = \{i_1,\dots,i_l\}$ be an l-tuple of elements of I with some signs, and ε_I be a canonical orientation relative to I. We know, by Proposition 3.6, that $\mathcal{I}_l$ corresponds to 1-forests of Δ. Therefore, ε_I is defined on the set of facets of Δ completing $\mathrm{Sub}_\Delta(I)$ into a 0-forest of I. If one takes two such different facets δ and δ' of respective numbers j_r and j'_r, then $\mathrm{Z}(\mathrm{Sub}_\Delta(I + j_r + j'_r))$ has rank one and is generated by a cycle c. In this cycle, δ and δ' have nonzero coefficients a and a' of signs ζ and ζ'. Using the third point of Definition 3.23 about canonical orientations,

$$\varepsilon_I(j_r) = -\zeta\zeta'\varepsilon_I(j'_r).$$

That is to say that orientations relative to I are such that, if the orientation of δ follows the orientation of the cycle c, then the orientation of δ' follows the opposite orientation.

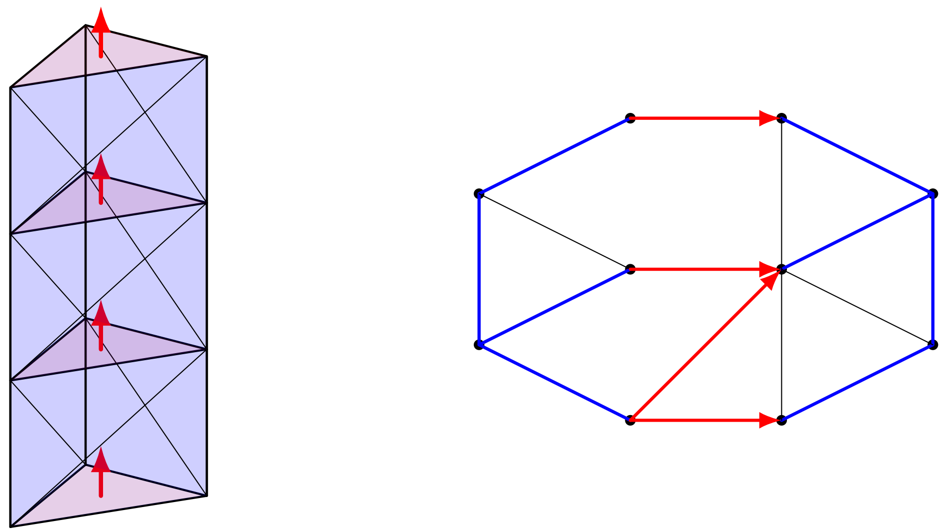


FIGURE 11. Canonical orientations relative to some 1-forests.

Thus, one can compute a canonical orientation relative to a 1-forest in the following way: take an arbitrary facet δ which completes $\mathrm{Sub}_\Delta(I)$ into a 0-forest; set an arbitrary orientation on δ; for each other facet δ' completing $\mathrm{Sub}_\Delta(I)$ into a 0-forest, compute the only orientation compatible with the chosen orientation of δ. There are two examples of such an orientation on Figure 11. In each example, the 1-forest is in blue, ε_I is defined on red facets, and the associated orientation is indicated.

In the case of a connected graph (see left hand-side of the figure), the orientation is easy to compute. A 1-forest is composed of two subtrees, and a canonical orientation relative to this 1-forest is defined on edges going from one tree to the other, and all these edges are oriented from always the same tree to always the other.

Example 3.31. Another way to understand relative orientation is to use *contraction* on matroids. Let I be an independent set in $\mathfrak{M}$. Let us begin with nonoriented contraction. The contraction of $\mathfrak{M}$ by I is the matroid denoted by $\mathfrak{M}/I = (E/I, \mathcal{I}/I)$, where $E/I = E \setminus I$ and $J \in \mathcal{I}/I$ if and only if $J \cup I \in \mathcal{I}$. Contractions correspond to projections in linear spaces. It happens that J is a basis of $\mathfrak{M}/I$ if and only if $I \cup J$ is a basis of $\mathfrak{M}$.

Now the oriented case. The contraction of $\vec{\mathfrak{M}}$ by I (I is still an element of $\mathcal{I}$) is denoted by $\vec{\mathfrak{M}}/I = (\vec{E}/I, \vec{\mathcal{I}}/I)$ where $\vec{E}/I := \vec{E} \setminus (I \cup (-I))$, and $\vec{\mathcal{I}}/I$ contains $\vec{J} \subset \vec{E}/I$ if and only if: for all $\vec{I}$, where we put some signs on elements of I, $\vec{J} \cup \vec{I}$ is independent in $\vec{\mathfrak{M}}$. It happens that $\mathcal{B}(\vec{\mathfrak{M}}/I) = \underline{\overrightarrow{\mathrm{Compl}}}(I)$ (cf the end of the last paragraph).

The interesting property is that the canonical orientations on $\vec{\mathfrak{M}}/I$ are exactly the canonical orientations relative to I in $\vec{\mathfrak{M}}$. It suffices to check that a canonical orientation relative to I verifies the three conditions of Definition 3.23, which can be easily done.

For example, Figure 12 shows what happens for a graphic matroid and a 2-forest. The first figure is the graph associated to $\mathfrak{M}$, and the 2-forest in blue corresponds to I. Then, the second graph represents $\mathfrak{M}/I$. Actually, the second graph can be reduced to the third one. Finally, it suffices to study canonical orientations on the third graph to get canonical orientations relative to I.

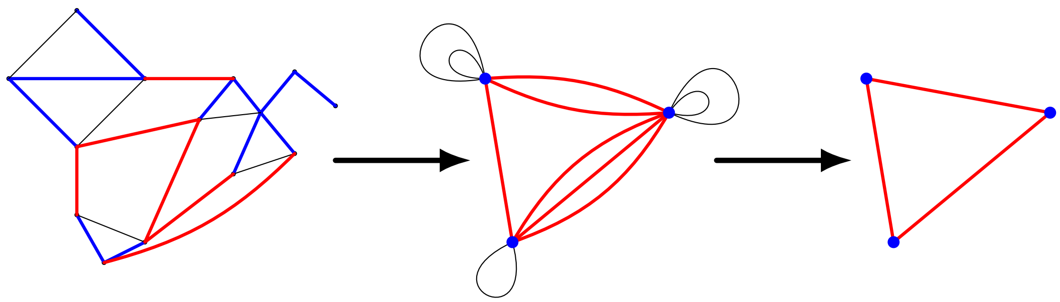


Figure 12. A contraction of a 2-forest.

Theorem 3.32 below naturally introduces a link between relative orientations and Symanzik polynomials. This theorem will be useful in order to understand why Symanzik polynomials with parameters generalize the second Symanzik polynomials of the introduction (see (2)).

We have to introduce two new sets. If $I \subset E$ is any subset of the ground set of $\mathfrak{M}$, then

$$\mathrm{Compl}(I) := \{J \subset I^c \big| I \cup J \in \mathcal{B}(\mathfrak{M})\} \text{ and}$$
$$\underline{\mathrm{Compl}}(I) := \{\underline{J} \big| J \in \mathrm{Compl}(I)\}.$$

Notice that the set $\underline{\mathrm{Compl}}(I)$ is the subset of completions in $\underline{\overrightarrow{\mathrm{Compl}}}(I)$ which only contain positive elements.

Theorem 3.32. *If l is a nonnegative integer, if f is a free family of size r overgenerating $\mathrm{Im}(R^\intercal)$, and if $u_1, \dots, u_l \in \mathrm{Im}(R^\intercal)$, then there is a canonical orientation ε_I relative to I, for each $I \in \mathcal{I}_{r-l}$, such that*

$$\mathrm{Sym}_k(R, f, u_1, \dots, u_l; \underline{x}) = \sum_{I \in \mathcal{I}_{r-l}} \Big(\sum_{\underline{J} \in \underline{\mathrm{Compl}}(I)} \varepsilon_I(\underline{J}) |\det{}_f(R^\intercal_{I \cup J})| \mathbf{v}_{1,j_1} \cdots \mathbf{v}_{l,j_l} \Big)^k x^{I^c},$$

where, for all $i \in [1\,..\,l]$, $v_i = (\mathbf{v}_{i,1}, \dots, \mathbf{v}_{i,n}) \in A^n$ is such that $R\mathbf{v}_i = \mathbf{u}_i$.

The proof of this theorem is a big computation. We will cut it into some lemmas.

With the notations of the theorem, let $\widetilde{f}$ be a basis of $\mathrm{Im}(R^\intercal)$, S be a normal kernel matrix of R with basis $\widetilde{f}$, and e be the standard basis of A^{s+l}.

Let $V = \mathbf{v}_1 \star \dots \star \mathbf{v}_l$ (be careful, $\mathbf{v}_{i,j}$ is the entry in the (i)-th column and in the j-th row of V). For all $i \in I$ and $j \in [1\,..\,s]$, $\mathbf{s}_{i,j}$ denotes the entry of S of index (i,j). If I is a set, we write $i_1, \dots, i_{|I|}$ its elements such that $i_1 < \dots < i_{|I|}$. Then, $\mathrm{ord}(I)$ denotes the tuple $(i_1, \dots, i_{|I|})$. The set of permutations on I is denoted by $\mathfrak{S}_I$.

If $J, J' \subset I$ are such that $J \sqcup J' = I$, then we denote by $\tau_{J',J} \in \mathfrak{S}_I$ the permutation such that

$$(\tau_{J',J}(i_1), \dots, \tau_{J',J}(i_{|I|})) = (j'_1, \dots, j'_{|J'|}, j_1, \dots, j_{|J|}), \tag{36}$$

and by $\zeta_{J',J} = \sigma(\tau_{J',J})$ its signature. With these notations, we have the following generalization of the Laplace decomposition.

Lemma 3.33. *If I is a subset of $[1\cdot\cdot n]$ of size $s+l$, then*
$$\det((S\star V)_I)=\sum_{\substack{J\subset I\\|J|=l}}\zeta_{I\setminus J,J}\det(S_{I\setminus J})\det(V_J).$$

Proof. If J is a subset of I and if $\tau_J\in\mathfrak{S}_J$, then τ_J^I denotes the permutation on I which is equal to τ_J on J and to identity on $I\setminus J$. Notice that $\sigma(\tau_J^I)=\sigma(\tau_J)$.

Let us study the map
$$\begin{array}{rccc}\Phi: & \{(J',J,\tau_{J'},\tau_J)\big|J'\sqcup J=I,|J|=l,\tau_{J'}\in\mathfrak{S}_{J'},\tau_J\in\mathfrak{S}_J\} & \to & \mathfrak{S}_I\\ & (J',J,\tau_{J'},\tau_J) & \mapsto & \tau_{J'}^I\circ\tau_J^I\circ\tau_{J',J}.\end{array}$$

If we choose an element $(J',J,\tau_{J'},\tau_J)$ in the set of definition of Φ, and if we denote by τ_I its image, then, using (36),
$$J'=\tau_I(\{i_1,\dots,i_{|J'|}\}),$$
$$J=\tau_I(\{i_{|J'|+1},\dots,i_{|I|}\})\text{ and}$$
$$\big(\tau_I(i_1),\dots,\tau_I(i_{s+l})\big)=\big(\tau_{J'}(j'_1),\dots,\tau_{J'}(j'_s),\tau_J(j_1),\dots,\tau_J(j_l)\big).$$
One can create an inverse function of Φ thanks to these three equations. Thus, Φ is a bijection.

Now we can write the decomposition of the determinant.
$$\begin{aligned}\det((S\star V)_I)&\\
&=\sum_{\tau_I\in\mathfrak{S}_I}\sigma(\tau_I)\mathbf{s}_{\tau_I(i_1),1}\cdots\mathbf{s}_{\tau_I(i_s),s}\mathbf{v}_{1,\tau_I(i_{l+1})}\cdots\mathbf{v}_{l,\tau_I(i_{l+s})}\\
&=\sum_{\substack{J',J\\J'\sqcup J=I,|J|=l}}\ \sum_{\tau_{J'}\in\mathfrak{S}_{J'}}\sum_{\tau_J\in\mathfrak{S}_J}\sigma(\tau_{J'}^I\circ\tau_J^I\circ\tau_{J',J})\mathbf{s}_{\tau_{J'}(j'_1),1}\cdots\mathbf{s}_{\tau_{J'}(j'_s),s}\mathbf{v}_{1,\tau_J(j_1)}\cdots\mathbf{v}_{l,\tau_J(j_l)}\\
&=\sum_{\substack{J',J\\J'\sqcup J=I,|J|=l}}\sigma(\tau_{J',J})\sum_{\tau_{J'}\in\mathfrak{S}_{J'}}\sigma(\tau_{J'})\mathbf{s}_{\tau_{J'}(j'_1),1}\cdots\mathbf{s}_{\tau_{J'}(j'_s),s}\sum_{\tau_J\in\mathfrak{S}_J}\sigma(\tau_J)\mathbf{v}_{1,\tau_J(j_1)}\cdots\mathbf{v}_{l,\tau_J(j_l)}\\
&=\sum_{\substack{J',J\\J'\sqcup J=I,|J|=l}}\zeta_{J',J}\det(S_{J'})\det(V_J),\\
&=\sum_{\substack{J\subset I\\|J|=l}}\zeta_{I\setminus J,J}\det(S_{I\setminus J})\det(V_J).\end{aligned}$$
which concludes this first lemma. □

The second lemma deals with signs.

Lemma 3.34. *If $I\subset[1\cdot\cdot n]$ is of size $r-l$, then there exists a canonical orientation ε_I relative to I such that, for all $J\in\mathrm{Compl}(I)$,*
$$\zeta_{I^c\setminus J,J}\,\sigma(I\cup J)\,\mathrm{sgn}\circ\det_{\widehat{f}}(R^{\intercal}_{I\cup J})=\varepsilon_I(j_1,\dots,j_l).$$

Proof. We compute the different signs.

First, $\zeta_{I^c\setminus J,J}$ is by definition the signature of $\tau_{I^c\setminus J,J}$. Set $J':=I^c\setminus J$. Let us count the number of inversions in $\tau_{J',J}$. This permutation is the only permutation which is increasing from $\{i_1,\dots,i_s\}$ to J' and increasing from $\{i_{s+1},\dots,i_{s+l}\}$ to J. The set of elements in

$\{i_1, \dots, i_s\}$ implied in an inversion with i_{s+1} is $\tau^{-1}_{J',J}([j_1 \mathinner{..} n] \cap J')$. A similar result is true for $i_{s+2}, \dots, i_{s+l}$. Thus,

$$\zeta_{I^c \setminus J, J} = (-1)^{\left|[j_1 \mathinner{..} n] \cap (I^c \setminus J)\right| + \dots + \left|[j_l \mathinner{..} n] \cap (I^c \setminus J)\right|}.$$

Second,

$$\sigma(I \cup J) = \sigma(I)(-1)^{j_1 + \dots + j_l}.$$

Then, as we have seen in Example 3.26, there exists a canonical orientation ε' on $\vec{\mathfrak{M}}_R$ such that

$$\operatorname{sgn} \circ \det_{\widetilde{f}}(R^{\intercal}_{I \cup J}) = \varepsilon'(\operatorname{ord}(I \cup J)).$$

Finally, set ε'_I the canonical orientation relative to I associated to ε', i.e., such that, for all $\underline{\vec{J}} \in \underline{\overrightarrow{\operatorname{Compl}}}(I)$,

$$\varepsilon'_I(j_1, \dots, j_l) = \varepsilon'(i_1, \dots, i_s, j_1, \dots, j_l).$$

Thus, using the third point of Definition 3.23,

$$\frac{\varepsilon'(\operatorname{ord}(I \cup J))}{\varepsilon'(i_1, \dots, i_s, j_1, \dots, j_l)}$$

is equal to the signature of the permutation τ such that

$$(\tau(i_1), \dots, \tau(i_s), \tau(j_1), \dots, \tau(j_l)) = \operatorname{ord}(I \cup J).$$

But this is exactly the permutation $\tau^{-1}_{I,J}$ (see Equation (36)). With the same argument as above, we obtain

$$\sigma(\tau^{-1}_{I,J}) = (-1)^{\left|[j_1 \mathinner{..} n] \cap I\right| + \dots + \left|[j_l \mathinner{..} n] \cap I\right|}.$$

We can end the proof:

$$\begin{aligned}
&\zeta_{I^c \setminus J, J}\, \sigma(I \cup J) \frac{\operatorname{sgn} \circ \det_{\widetilde{f}}(\mathbb{R}^{\intercal}_{I \cup J})}{\varepsilon'_I(j_1, \dots, j_l)} \\
&\quad = (-1)^{\left|[j_1 \mathinner{..} n] \cap (I^c \setminus J)\right| + \dots + \left|[j_l \mathinner{..} n] \cap (I^c \setminus J)\right|} \sigma(I)(-1)^{j_1 + \dots + j_l} \frac{\varepsilon'(\operatorname{ord}(I \cup J))}{\varepsilon'(i_1, \dots, i_s, j_1, \dots, j_l)} \\
&\quad = (-1)^{\left|[j_1 \mathinner{..} n] \cap (I^c \setminus J)\right| + \dots + \left|[j_l \mathinner{..} n] \cap (I^c \setminus J)\right|} \sigma(I)(-1)^{j_1 + \dots + j_l} (-1)^{\left|[j_1 \mathinner{..} n] \cap I\right| + \dots + \left|[j_l \mathinner{..} n] \cap I\right|} \\
&\quad = \sigma(I) \prod_{\alpha=1}^{l} (-1)^{\left|[j_\alpha \mathinner{..} n] \cap (I^c \setminus J)\right| + j_\alpha + \left|[j_\alpha \mathinner{..} n] \cap I\right|}.
\end{aligned}$$

But, if $\alpha \in [1 \mathinner{..} l]$,

$$\begin{aligned}
\left|[j_\alpha \mathinner{..} n] \cap (I^c \setminus J)\right| + \left|[j_\alpha \mathinner{..} n] \cap I\right| &= |[j_\alpha \mathinner{..} n] \cap (I^c \setminus J) \cup [j_\alpha \mathinner{..} n] \cap I| \\
&= |[j_\alpha \mathinner{..} n] \cap (I \cup (I^c \setminus J))| \\
&= |[j_\alpha \mathinner{..} n] \setminus J|.
\end{aligned}$$

Moreover $[j_\alpha \mathinner{..} n] \cap J = \{j_\alpha, j_{\alpha+1}, \dots, j_l\}$, whose cardinality is $l - \alpha + 1$. That is why

$$|[j_\alpha \mathinner{..} n] \setminus J| = n - j_\alpha + 1 - (l - \alpha + 1) = n - l - j_\alpha + \alpha.$$

We continue the computation. The j_αs disappear:

$$\zeta_{I^c \setminus J, J}\, \sigma(I \cup J) \frac{\operatorname{sgn} \circ \det_{\widetilde{f}}(R^{\intercal}_{I \cup J})}{\varepsilon'_I(j_1, \dots, j_l)} = \sigma(I)(-1)^{l(n-l) + 1 + \dots + l}.$$

The right hand-side member is independent of J. Thus, the orientation

$$\varepsilon_I := \sigma(I)(-1)^{l(n-l)+1+\cdots+l}\varepsilon'_I,$$

verifies the lemma: for all $J \in \mathrm{Compl}(I)$,

$$\zeta_{I^c\setminus J,J}\,\sigma(I\cup J)\,\mathrm{sgn}\circ\det_{\widetilde{f}}(R^{\intercal}_{I\cup J}) = \varepsilon_I(j_1,\dots,j_l).$$

□

The third lemma is a development of the determinant of V.

Lemma 3.35. *If $I \subset [1\cdot\cdot n]$ is of size r, and if ε_I is a canonical orientation relative to I, then*

$$\sum_{J\in\mathrm{Compl}(I)} \varepsilon_I(j_1,\dots,j_l)\det(V_J) = \sum_{\underline{J}\in\underline{\mathrm{Compl}}(I)} \varepsilon_I(\underline{J})\mathbf{v}_{1,j_1}\cdots\mathbf{v}_{l,j_l}.$$

Notice that in the left hand-side of the equation above we have $j_1 < \cdots < j_l$ but this is no more true in the right hand-side.

Proof. The proof is essentially the definitions of the determinant and of the canonical orientations. Let $I \subset [1\cdot\cdot n]$ be of size r. The following map is clearly a bijection.

$$\begin{array}{rcl}\mathrm{Compl}(I)\times\mathfrak{S}_l & \to & \underline{\mathrm{Compl}}(I),\\ (J,\tau) & \mapsto & (j_{\tau(1)},\dots,j_{\tau(l)}).\end{array}$$

Moreover, the first condition to be a canonical orientation (Definition 3.23), which clearly applies for relative canonical orientation, implies that if $J \in \mathrm{Compl}(I)$ and if $\tau \in \mathfrak{S}_l$, then

$$\varepsilon_I(j_{\tau(1)},\dots,j_{\tau(l)}) = \sigma(\tau)\varepsilon_I(j_1,\dots,j_l).$$

Thus, we have

$$\begin{aligned}\sum_{J\in\mathrm{Compl}(I)} \varepsilon_I(j_1,\dots,j_l)\det(V_J) &= \sum_{J\in\mathrm{Compl}(I)} \varepsilon_I(j_1,\dots,j_l)\sum_{\tau\in\mathfrak{S}_l}\sigma(\tau)\mathbf{v}_{1,j_{\tau(1)}}\cdots\mathbf{v}_{l,j_{\tau(l)}}\\ &= \sum_{J\in\mathrm{Compl}(I)}\sum_{\tau\in\mathfrak{S}_l} \varepsilon_I(j_{\tau(1)},\dots,j_{\tau(l)})\mathbf{v}_{1,j_{\tau(1)}}\cdots\mathbf{v}_{l,j_{\tau(l)}}\\ &= \sum_{\underline{J}\in\underline{\mathrm{Compl}}(I)} \varepsilon_I(\underline{J})\mathbf{v}_{1,j_1}\cdots\mathbf{v}_{l,j_l},\end{aligned}$$

which ends the proof. □

Proof of Theorem 3.32. Now we can do the main computation. We recall that $\widetilde{f}$ is a basis of $\mathrm{Im}(R^{\intercal})$, S is a normal kernel matrix of R with basis $\widetilde{f}$, and e is the standard basis of A^{s+l}.

$$\begin{aligned}&\mathrm{Sym}_k(R,\widetilde{f},u_1,\dots,u_l;\underline{x})\\ &\quad= \mathrm{Kir}_k(S\star\mathbf{v}_1\star\cdots\star\mathbf{v}_l,e;\underline{x}) && \text{(Definition 2.24)}\\ &\quad= \sum_{\substack{I\subset[1\cdot\cdot n]\\|I|=s+l}}\det((S\star V)_I)^k x^I && \text{(Definition 2.4)}\\ &\quad= \sum_{\substack{I\subset[1\cdot\cdot n]\\|I|=s+l}}\Big(\sum_{\substack{J\subset I\\|J|=l}}\zeta_{I\setminus J,J}\det(S_{I\setminus J})\det(V_J)\Big)^k x^I && \text{(Lemma 3.33)}\end{aligned}$$

$$= \sum_{\substack{I\subset[1\mathinner{\ldotp\ldotp}n]\\|I|=s+l}} \Big(\sum_{\substack{J\subset I\\|J|=l}} \zeta_{I\setminus J,J}\,\sigma((I\setminus J)^c)\det_{\widetilde{f}}(R^{\intercal}_{(I\setminus J)^c})\det(V_J)\Big)^k x^I \qquad \left(\begin{matrix}\text{Equation (4),}\\ \text{Definition 2.20}\end{matrix}\right)$$

$$= \sum_{\substack{I\subset[1\mathinner{\ldotp\ldotp}n]\\|I|=r-l}} \Big(\sum_{\substack{J\subset I^c\\|J|=l}} \zeta_{I^c\setminus J,J}\,\sigma(I\cup J)\det_{\widetilde{f}}(R^{\intercal}_{I\cup J})\det(V_J)\Big)^k x^{I^c}. \qquad (I\leftarrow I^c)$$

Notice that $\det_{\widetilde{f}}(R^{\intercal}_{I\cup J})\neq 0$ if and only if $I\in\mathcal{I}_{r-l}$ and $J\in\mathrm{Compl}(I)$. Thus, there exist relative orientations ε_I for each $I\in\mathcal{I}_{r-l}$ such that

$$\mathrm{Sym}_k(R,\widetilde{f},u_1,\dots,u_l;\underline{x})$$

$$(37)\qquad = \sum_{I\in\mathcal{I}_{r-l}} \Big(\sum_{J\in\mathrm{Compl}(I)} \zeta_{I^c\setminus J,J}\,\sigma(I\cup J)\,\mathrm{sgn}\circ\det_{\widetilde{f}}(R^{\intercal}_{I\cup J})|\det_{\widetilde{f}}(R^{\intercal}_{I\cup J})|\det(V_J)\Big)^k x^{I^c}$$

$$= \sum_{I\in\mathcal{I}_{r-l}} \Big(\sum_{J\in\mathrm{Compl}(I)} \varepsilon_I(j_1,\dots,j_l)|\det_{\widetilde{f}}(R^{\intercal}_{I\cup J})|\det(V_J)\Big)^k x^{I^c} \qquad \text{(Lemma 3.34)}$$

$$= \sum_{I\in\mathcal{I}_{r-l}} \Big(\sum_{\underline{J}\in\underline{\mathrm{Compl}}(I)} \varepsilon_I(\underline{J})|\det_{\widetilde{f}}(R^{\intercal}_{I\cup J})|\mathbf{v}_{1,j_1}\cdots\mathbf{v}_{l,j_l}\Big)^k x^{I^c}. \qquad \text{(Lemma 3.35)}$$

Actually, in the left equality, we use a result a little stronger than Lemma 3.35. Looking carefully at the proof of this lemma, the reader could be convinced that this result is true.

Multiplying each side of the last equation by $\det_f(\widetilde{F})^k$ and replacing ε_I by $\mathrm{sgn}(\det_f(\widetilde{F})^k)\varepsilon_I$, for all $I\subset[1\mathinner{\ldotp\ldotp}n]$ of size $r-l$, we obtain what we wanted to prove (using Lemmas 2.2 and 2.12):

$$\mathrm{Sym}_k(R,f,u_1,\dots,u_l;\underline{x}) = \sum_{I\in\mathcal{I}_{r-l}} \Big(\sum_{\underline{J}\in\underline{\mathrm{Compl}}(I)} \varepsilon_I(\underline{J})|\det_f(R^{\intercal}_{I\cup J})|\mathbf{v}_{1,j_1}\cdots\mathbf{v}_{l,j_l}\Big)^k x^{I^c}.$$

□

Remark 3.36. In Theorem 3.32, if the order k is even, then one can choose any relative canonical orientation for each I. Moreover, a factorized form of the theorem can be found in the proof:

$$\sum_{I\in\mathcal{I}_{r-l}} \Big(\sum_{J\in\mathrm{Compl}(I)} \varepsilon_I(j_1,\dots,j_l)|\det(R_{I\cup J})|\det(V_J)\Big)^k x^{I^c}.$$

Now we restate Theorem 3.32 in terms of simplicial complexes.

Corollary 3.37. *If l is a nonnegative integer and if $u_1,\dots,u_l\in\mathrm{B}_{d-1}(\Delta)$, then for each $I\in\mathcal{F}_l(\Delta)$ there is a canonical orientation ε_I relative to I such that*

$$\mathrm{Sym}_k(\Delta,u_1,\dots,u_l;\underline{x}) = \sum_{I\in\mathcal{F}_l(\Delta)} \Big(\sum_{\underline{J}\in\underline{\mathrm{Compl}}(I)} \varepsilon_I(\underline{J})|H_{d-1}(\mathrm{Sub}_\Delta(I\cup J))/H_{d-1}(\Delta)|\mathbf{v}_{1,j_1}\cdots\mathbf{v}_{l,j_l}\Big)^k x^{I^c}$$

modulo A^{*k}, where, for all $i\in[1\mathinner{\ldotp\ldotp}l]$, $v_i\in A\langle\Delta_d\rangle$ is such that $\partial_\Delta v_i=u_i$, and $\lambda(v_i)=(\mathbf{v}_{i,1},\dots,\mathbf{v}_{i,n})\in A^n$.

Proof. By definition (Definition 3.21), if f is a basis of $\mathrm{Im}(R^{\intercal})$

$$\mathrm{Sym}_k(\Delta,u_1,\dots,u_l;\underline{x}) = \mathrm{Sym}_k(R,\widetilde{f},\lambda(u_1),\dots,\lambda(u_l);\underline{x})$$

modulo A^{*k}. Then Theorem 3.32 gives

$$\mathrm{Sym}_k(\Delta, u_1, \dots, u_l; \underline{x}) = \sum_{I \in \mathcal{I}_{r-l}} \Big(\sum_{\underline{J} \in \underline{\mathrm{Compl}}(I)} \varepsilon_I(\underline{J}) |\det_f(R^{\intercal}_{I \cup J})| \mathbf{v}_{1,j_1} \cdots \mathbf{v}_{l,j_l} \Big)^k x^{I^c}.$$

But Proposition 3.6 affirms that $\mathcal{I}_{r-l} = \mathcal{F}_l(\Delta)$, and we have already seen in the proof of the generalized Kirchhoff's theorem (see Equation (24)) that, since f is a basis of $\mathrm{B}(\Delta)$,

$$|\det_f(R^{\intercal}_{I \cup J})| = |\,\mathrm{B}_{d-1}(\Delta)/\,\mathrm{B}_{d-1}(\mathrm{Sub}_\Delta(I \cup J))|,$$

and that

$$\mathrm{B}_{d-1}(\Delta)/\,\mathrm{B}_{d-1}(\mathrm{Sub}_\Delta(I \cup J)) \simeq H_{d-1}(\mathrm{Sub}_\Delta(I \cup J))/H_{d-1}(\Delta),$$

which concludes the proof. □

We can finally explain the link between Symanzik polynomials with parameters defined in this paper, and the second Symanzik polynomial used in quantum field theory (see the introduction, (2)).

Example 3.38. In this example, $A = \mathbb{R}$. Suppose that Δ is a connected 1-dimensional complex. Let $\mathcal{G}$ be the oriented graph corresponding to Δ, with the corresponding enumeration of the edges and of the vertices. Let V be the set of vertices of $\mathcal{G}$ and E be its set of edges. Suppose that Δ is such that $\mathcal{G}$ is connected. We rewrite differently the definition of the second Symanzik polynomial ((2) with D=1):

$$\phi_{\mathcal{G}}(\mathfrak{p}, \underline{x}) := \sum_{F \in \mathcal{SF}_2(\mathcal{G})} q(T_1, T_2) \underline{x}^{(\mathrm{num}(T_1) \cup \mathrm{num}(T_2))^c} \tag{38}$$

where $\mathfrak{p} \in \mathbb{R}\langle V \rangle$ is the momentum on each vertex of $\mathcal{G}$, $\mathcal{SF}_2(\mathcal{G})$ is the set of pairs of subtrees $\{T_1, T_2\}$ forming a spanning forest with 2 connected components of $\mathcal{G}$ (i.e., such that the pair consisting of the set of vertices of T_1 and of the set of vertices of T_2 forms a partition of V),

$$q(T_1, T_2) := -\mathfrak{p}_{T_1} \mathfrak{p}_{T_2}, \tag{39}$$

with $\mathfrak{p}_{T_i}$, $i \in \{1, 2\}$, being the sum of coefficients of $\mathfrak{p}$ corresponding to vertices in T_i, and $\mathrm{num}(T_i)$, $i \in \{1, 2\}$, is the set of numbers of edges in E which are edges of T_i. Suppose that coefficients of $\mathfrak{p} \in \mathbb{R}\langle V \rangle$ sum to zero. Then, $u := \lambda^{-1}(\mathfrak{p})$ is an element of $\mathrm{B}_0(\Delta)$ (see Example 3.4 for an explicitation of $\mathrm{B}_0(\Delta)$). We will show that

$$\phi_{\mathcal{G}}(\mathfrak{p}, \underline{x}) = \mathrm{Sym}_2(\Delta, u; \underline{x}).$$

By Corollary 3.37,

$$\mathrm{Sym}_2(\Delta, u; \underline{x}) = \sum_{I \in \mathcal{F}_1(\Delta)} \Big(\sum_{\underline{J} \in \underline{\mathrm{Compl}}(I)} \varepsilon_I(\underline{J}) |H_{d-1}(\mathrm{Sub}_\Delta(I \cup J))/H_{d-1}(\Delta)| \mathbf{v}_{j_1} \Big)^2 x^{I^c} \tag{40}$$

where $v \in \mathbb{R}\langle \Delta_1 \rangle$ verifies that $\partial_\Delta v = u$ and $\lambda(v) = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in \mathbb{R}^n$, and, if $I \in \mathcal{F}_1(\Delta)$, ε_I is any canonical orientation relative to I ($k = 2$ is even). We have seen in Example 3.4 that $\mathcal{F}_1(\Delta)$ corresponds to subforests of $\mathcal{G}$ with two connected components, that we have called 1-forest. More precisely, subsets I of $[1 \cdot\cdot n]$ verify

$$I \in \mathcal{F}_1(\Delta) \iff \exists \{T_1, T_2\} \in \mathcal{SF}_2(\mathcal{G}), I = \mathrm{num}(T_1) \cup \mathrm{num}(T_2).$$

Let $I \in \mathcal{F}_1(\Delta)$ and let $\{T_1, T_2\}$ be the unique pair in $\mathcal{SF}_2(\mathcal{G})$ such that $I = \mathrm{num}(T_1) \cup \mathrm{num}(T_2)$. Comparing polynomials (38) and (40), as $k = 2$ is even, it remains to prove that

$$\Big(\sum_{\underline{J} \in \underline{\mathrm{Compl}}(I)} \varepsilon_I(\underline{J}) \big| H_{d-1}(\mathrm{Sub}_\Delta(I \cup J))/H_{d-1}(\Delta) \big| \mathbf{v}_{j_1} \Big)^2 = q(T_1, T_2).$$

Looking at the definition of $q(T_1, T_2)$ (Equation (39)), since the sum of coefficients of $\mathfrak{p}$ is zero,

$$q(T_1, T_2) = \mathfrak{p}_{T_1}^2.$$

If $\underline{J} \in \underline{\mathrm{Compl}}(I)$, then $I \cup J$ corresponds to a 0-forests of Δ. Since Δ is 1-dimensional, one obtains that $\mathrm{B}_{d-1}(\Delta) = \mathrm{B}_{d-1}(\mathrm{Sub}_\Delta(I \cup J))$ (once more, the reader can look at Example 3.4 to see that the set of $(d-1)$-boundaries only depends on connected components). Thus,

$$|H_{d-1}(\mathrm{Sub}_\Delta(I \cup J))/H_{d-1}(\Delta)| = 1.$$

If $u' \in \mathbb{R}\langle \Delta_0 \rangle$, we denoted by u'_{T_1} the value of $\mathfrak{p}'_{T_1}$, where $\mathfrak{p}' = \lambda^{-1}(u')$. Then $(\partial v)_{T_1}^2 = \mathfrak{p}_{T_1}^2$. It remains to show that

$$(\partial_v)_{T_1} = \sum_{(j) \in \underline{\mathrm{Compl}}(I)} \varepsilon_I(j) \mathbf{v}_j. \tag{41}$$

One can choose ε_I such that, in the last equation, $\varepsilon_I(j)$ is equal to 1 if the edge numbered j goes from T_2 to T_1 and to -1 otherwise (see Example 3.30). Let $e \in \Delta_1$ be any edge of $\mathcal{G}$. Let a_e be the coefficient of e in v. If e goes from T_2 to T_2, then e does not contribute to $(\partial v)_{T_1}$. If e goes from T_1 to T_1, then contribution of e is equal to $a_e - a_e = 0$. If e goes from T_1 to T_2, then only the tail of e is in T_1, thus its contribution is $-a_e$. Finally, if e goes from T_2 to T_1, contribution of e is a_e. Thus, Equation (41) is true. Finally, the second Symanzik polynomial corresponds to our Symanzik polynomial of order 2:

$$\phi_{\mathcal{G}}(\mathfrak{p}, \underline{x}) = \mathrm{Sym}_2(\Delta, u; \underline{x}).$$

Now we will discuss what happens when boundaries chosen as parameters are simple enough. Let $u \in \mathrm{B}_{d-1}(\Delta)$. Let U be the subset of faces in Δ_{d-1} which have a nonzero coefficient in u. We say that u' *is included in* u if all the faces which have a nonzero coefficient in u' are in U.

Definition 3.39. A boundary $u \in \mathrm{B}_{d-1}(\Delta)$ is said *simple* if, for all $u' \in \mathrm{B}_{d-1}(\Delta)$ such that u' is included in u, there exists an $a \in A$ such that $u' = au$.

Example 3.40. For example, if $\mathcal{G}$ is a graph and if v_1, v_2 are two vertices of a same connected component of $\mathcal{G}$, then $\{v_2\} - \{v_1\}$ is a simple boundary of $\mathcal{G}$. Actually, all simple boundaries on $\mathcal{G}$ are of this form.

Example 3.41. A second example: let $\mathcal{S}$ be a compact orientable surface. A simple closed path on $\mathcal{S}$ which is a boundary is always simple. More precisely, let (Δ, ν, Φ) be an abstract triangulation of $\mathcal{S}$. Let u be a 1-boundary of Δ. Suppose that u as only coefficients in $\{-1, 0, 1\}$ and let U be the set of faces which have a nonzero coefficient in u. Let

$$\widetilde{\nu}(u) := \bigcup_{\delta \in U} \widetilde{\nu}(\delta).$$

Finally, suppose that $\Phi(\widetilde{\nu}(u))$ (thus, $\widetilde{\nu}(u)$) is a simple closed path (i.e., is homeomorphic to the circle). Then u is a simple boundary. Let us sketch the proof. If u' is a 1-boundary

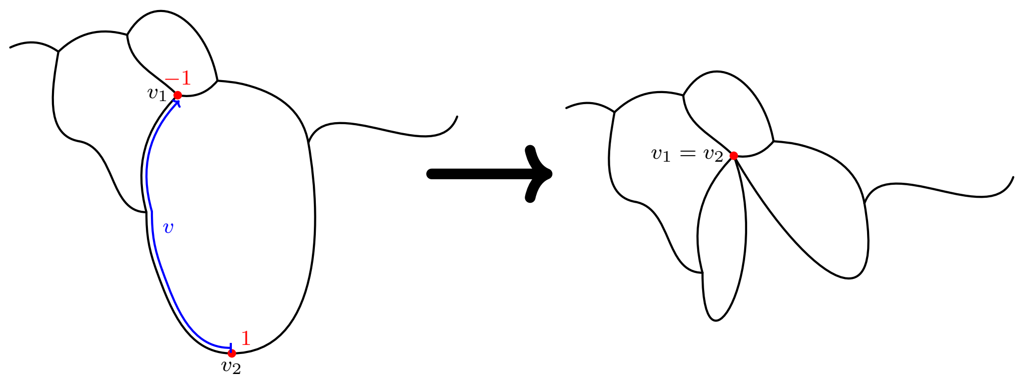


FIGURE 13. A simple boundary on a metric graph and its contraction.

included in u, then $\widetilde{\nu}(u') \subset \widetilde{\nu}(u)$. Since $\widetilde{\nu}(u)$ is homeomorphic to the circle, and since u' is a cycle, $\widetilde{\nu}(u')$ is either empty or equal to $\widetilde{\nu}(u)$. Thus, u' has to be a multiple of u.

In general, unlike the case of the graphs, there exist other kinds of simple boundaries (see the red boundary on Figure 14).

Now we complete Examples 3.14 and 3.19 adding parameters.

Example 3.42. Let $\mathfrak{G}$ be a metric graph with an enumeration of vertices and another one of edges. Let $\mathcal{G}$ be the associated graph and Δ be the associated simplicial complex. Let v_1, v_2 be two points of $\mathfrak{G}$. We can assume that these points are vertices. Then, if we also denote by v_1 and v_2 the corresponding vertices in Δ, $u := \{v_2\} - \{v_1\}$ is a simple boundary of Δ. Let $\mathfrak{G}'$ be the metric graph $\mathfrak{G}/\{v_1, v_2\}$, i.e., $\mathfrak{G}'$ is the metric graph $\mathfrak{G}$ where we have glued v_1 and v_2. Put the corresponding enumeration of edges of $\mathfrak{G}'$, and set $\mathcal{G}'$ and Δ' the corresponding graph and simplicial complex. If $\mathfrak{G}$ is the metric graph on the left of Figure 13, then $\mathfrak{G}'$ is the metric graph on the right. Let Ψ be the natural isomorphism from $A\langle\Delta_d\rangle$ to $A\langle\Delta'_d\rangle$. Let $v \in A\langle\Delta_d\rangle$ be such that $\partial_\Delta(v) = u$, and $\mathbf{v} \in \mathcal{M}_{n,1}(A)$ be the column matrix corresponding to $\lambda(v)$. We will see that

$$\mathrm{Sym}_k(\Delta, u; \underline{x}) = \mathrm{Sym}_k(\Delta'; \underline{x})$$

modulo A^{*k}.

It suffices to prove that we can find a normal kernel matrix S of R, with some basis of $\mathrm{Im}(R^\intercal)$, such that $S \star \mathbf{v}$ is a normal kernel matrix of Δ'. This is clearly possible if $\Psi(\mathrm{Z}_d(\Delta) + Av) = \mathrm{Z}_d(\Delta')$. But, clearly, an element of $A\langle\Delta_d\rangle$ is in $\Psi^{-1}(\mathrm{Z}_d(\Delta'))$ iff its boundary is in $A(\{v_2\} - \{v_1\})$. Moreover, since $\partial_\Delta(v) = (\{v_2\} - \{v_1\})$ and since $\mathrm{Z}_d(\Delta) = \ker(\partial_\Delta)$, $\mathrm{Z}_d(\Delta) + Av = \partial_\Delta^{-1}(A\{v_2\} - \{v_1\})$. Thus, $\Psi(\mathrm{Z}_d(\Delta) + Av) = \mathrm{Z}_d(\Delta')$. One can conclude that

$$\mathrm{Sym}_k(\Delta, u; \underline{x}) = \mathrm{Sym}_k(\Delta'; \underline{x}).$$

Remark 3.43. In last example, we see that adding a parameter which is a simple boundary is equivalent to contracting the corresponding set. This is true in general. We will not state it rigorously, however, let us sketch the proof. It begins as in the previous example. Then, it is not difficult to show the inclusion $\Psi(\mathrm{Z}_d(\Delta) + Au) \subset \mathrm{Z}_d(\Delta')$. The opposite inclusion is given by the following argument. All new cycles obtained after the contraction has a preimage by Ψ which is a nonempty boundary included in u. Thus, this boundary is a multiple of u. That prooves the inclusion $\mathrm{Z}_d(\Delta') \subset (\mathrm{Z}_d(\Delta) + Au)$. Here is the second example.

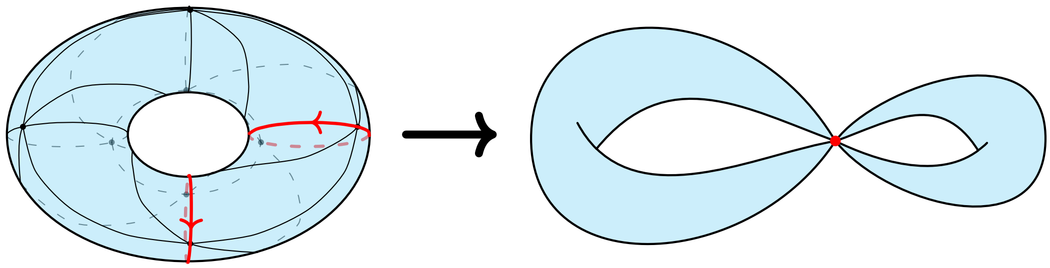


FIGURE 14. A simple boundary on a triangulated torus, and its contraction.

Example 3.44. We use the same notations as in Example 3.41. In Figure 14, $\Phi(\widetilde{\nu}(u))$ is represented in red. Let $\mathcal{S}'$ be the topological space obtained by contracting $\Phi(\widetilde{\nu}(u))$ (see the right hand-side of Figure 14). $\mathcal{S}'$ is composed of two compact orientable surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$ glued in one point. The triangulation (Δ, ν, Φ) induced a triangulation (Δ', ν', Φ') of $\mathcal{S}'$ which contains two particular subcomplexes, Δ^1 and Δ^2, which correspond to the preimages of S_1 and S_2 by $\Phi' \circ \nu'$. Let $(\Delta^i, \nu_i, \Phi_i)$, $i \in \{1, 2\}$, be the induced triangulation of $\mathcal{S}_i$. A 0-forest of Δ' is clearly a union of a 0-forest of Δ^1 and of a 0-forest of Δ^2. As in Example 3.19 about Symanzik polynomials on compact orientable surfaces, a 0-forest of Δ^i, $i \in \{1, 2\}$, is obtained removing any facet. Moreover, every coefficients of Symanzik polynomials of even order on Δ_i are equal to one. This is still true on Δ'. Thus, if k is a nonnegative even integer,

$$\mathrm{Sym}_k(\Delta^i; \underline{x}_{\mathrm{Fac}(\Delta^i)}) = \sum_{j \in \mathrm{Fac}(\Delta^i)} x^{\mathrm{Fac}(\Delta^i_2) - j},$$

where the j-th variable of $\underline{x}_{\mathrm{Fac}(\Delta^i)}$ is x_l, l being the number in Δ' of the j-th facet of Δ^i. Moreover,

$$\begin{aligned}\mathrm{Sym}_k(\Delta, u; \underline{x}) &= \mathrm{Sym}_k(\Delta'; \underline{x}) \\ &= \sum_{\substack{j_1 \in \mathrm{Fac}(\Delta^1_2) \\ j_2 \in \mathrm{Fac}(\Delta^2_2)}} x^{[1\,..\,n] \setminus \{j_1, j_2\}}.\end{aligned}$$

Thus,

$$\mathrm{Sym}_k(\Delta, u; \underline{x}) = \mathrm{Sym}_k(\Delta^1; \underline{x}_{\mathrm{Fac}(\Delta^1)})\, \mathrm{Sym}_k(\Delta^2; \underline{x}_{\mathrm{Fac}(\Delta^2)}).$$

Set a good measure π on $\mathcal{S}$ in a similar way as in Example 3.19. This measure can naturally be extended to $\mathcal{S}'$. Replacing x_js by the measure of the corresponding facets, we obtain that

$$\mathrm{Sym}_k(\Delta, u; \underline{x}) = \pi(S_1)\pi(S_2).$$

This can be generalized to larger numbers of parameters. Roughly speaking, on a compact oriented surface endowed with a fitting measure, the Symanzik polynomials of even order with parameters which are disjoint simple boundaries, and with variables corresponding to the measure of facets, is equal to the product of the measures of the different part of the surface obtained cutting along the parameters.

Before stating the last theorems of this paper about Symanzik polynomials (Theorem 5.1 and 5.2), we have to prove some interesting combinatorial results.

4. Exchange graph for matroids

This section could seem out of context: we will not talk about Symanzik polynomials. However, we need Corollary 4.13 below in the next section. Theorem 4.9 and its corollaries are interesting combinatorial results about connected components of what we call the exchange graph of a matroid. These results generalize Theorem 2.12 of [1] to the matroids, and they go further in the study of the exchange graph. The name "exchange graph" has been chosen because of the similarity with the exchange property of the bases of a matroid stated in Claim 2.32.

In this section, we fix a (nonoriented) matroid $\mathfrak{M} = (E, \mathcal{I})$. We set $r := \mathrm{rk}(\mathfrak{M})$. If $I \subset E$, $\mathrm{Fr}(I)$ will denote the complement of $\mathrm{cl}(I)$ in E, where we recall that $\mathrm{cl}(I)$ is the closure of I (see Subsection 2.4).

We are interested in finding the different connected components of some interesting subgraphs of the exchange graph of a matroid we define right below.

Definition 4.1. The *exchange graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ associated to $\mathfrak{M}$ is the graph with vertex set $\mathcal{V} := \mathcal{I} \times \mathcal{I}$ and edge set $\mathcal{E}$ such that two vertices (I_1, I_2) and (I'_1, I'_2) are adjacent if there exists $i \in E$ such that either, $I'_1 = I_1 + i$ and $I'_2 = I_2 - i$, or, $I'_1 = I_1 - i$ and $I'_2 = I_2 + i$.

We fix $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ the exchange graph of $\mathfrak{M}$. If $\mathcal{U}$ is a subset of $\mathcal{V}$, then $\mathcal{G}[\mathcal{U}]$ denotes the induced subgraph of $\mathcal{G}$ with vertex set $\mathcal{U}$: $\mathcal{G}[\mathcal{U}] = (\mathcal{U}, \mathcal{E}[\mathcal{U}])$ where $\mathcal{E}[\mathcal{U}]$ contains all the edgess of $\mathcal{E}$ connecting two vertices in $\mathcal{U}$. If $p, q \in [0 \cdot\cdot r]$, then we set $\mathcal{V}_{p,q} := \mathcal{I}_p \times \mathcal{I}_q$, where $\mathcal{I}_l$, $l \in [0 \cdot\cdot r]$, is the set of independents of rank l in $\mathfrak{M}$. Moreover, if $p \neq 0$ and $q \neq r$, we define the bipartite graph $\mathcal{G}_{p,q} := \mathcal{G}[\mathcal{V}_{p,q} \sqcup \mathcal{V}_{p-1,q+1}]$, whose edge set is denoted $\mathcal{E}_{p,q}$.

Remark 4.2. If $p \in [1 \cdot\cdot r]$ and $q \in [0 \cdot\cdot r-1]$, then we have a natural graph isomorphism:

$$\begin{aligned} \Phi_{p,q} :\quad & \mathcal{G}_{p,q} \xrightarrow{\sim} \mathcal{G}_{q+1,p-1}, \\ & (I_1, I_2) \in \mathcal{V}_{p,q} \sqcup \mathcal{V}_{p-1,q+1} \longmapsto (I_2, I_1) \in \mathcal{V}_{q,p} \sqcup \mathcal{V}_{q+1,p-1}. \end{aligned}$$

There are two important invariants in connected components of $\mathcal{G}$. They correspond to Definitions 4.3 and 4.6 below.

Definition 4.3. If I, J are two non necessarily disjoint set, we write $I \uplus J$ for the multiset containing elements of I and, disjointly, elements of J (such that elements in $I \cap J$ appear in $I \uplus J$ with multiplicity 2).

Abusing notation, if (U, V) and (I, J) are two ordered pairs of sets, then we write $(U, V) \subset (I, J)$ if $U \subset I$ and $V \subset J$. That defines a partial order on ordered pairs of sets.

Definition 4.4. If $(I, J) \in \mathcal{I} \times \mathcal{I}$ is an ordered pair of independent sets and if (U, V) is another one, we say that (U, V) is a *codependent ordered pair of* (I, J) if $(U, V) \subset (I, J)$ and $\mathrm{cl}(U) = \mathrm{cl}(V)$.

Notice that both members of a codependent ordered pair have the same size (because they are independent sets of the same rank, see Claim 2.31).

Claim 4.5. *Let $(I, J) \in \mathcal{I} \times \mathcal{I}$ be an ordered pair of independent sets. Let $(U, V), (U', V') \in \mathcal{I} \times \mathcal{I}$ be two codependent ordered pairs of (I, J). Then $(U \cup U', V \cup V')$ is a codependent ordered pair of (I, J).*

In the proofs of this section, we will often use the basic properties of the closure operator listed in Claim 2.31.

Proof. Clearly $(U \cup U', V \cup V') \subset (I, J)$. Moreover

$$U \cup U' \subset \mathrm{cl}(U) \cup \mathrm{cl}(U') \subset \mathrm{cl}(U \cup U').$$

Thus,

$$\mathrm{cl}(U \cup U') \subset \mathrm{cl}(\mathrm{cl}(U) \cup \mathrm{cl}(U')) \subset \mathrm{cl}(\mathrm{cl}(U \cup U')) = \mathrm{cl}(U \cup U'),$$

and so $\mathrm{cl}(\mathrm{cl}(U) \cup \mathrm{cl}(U')) = \mathrm{cl}(U \cup U')$. But, using Definition 4.4, $\mathrm{cl}(\mathrm{cl}(U) \cup \mathrm{cl}(U')) = \mathrm{cl}(\mathrm{cl}(V) \cup \mathrm{cl}(V'))$. Finally, $\mathrm{cl}(U \cup U') = \mathrm{cl}(V \cup V')$, which ends the proof. □

Thanks to the previous claim, and noticing that $(\emptyset, \emptyset)$ is always a codependent ordered pair, one can state the following definition.

Definition 4.6. If $(I, J) \in \mathcal{I} \times \mathcal{I}$ is an ordered pair of independent sets, we call *maximal codependent ordered pair (or MCP)* of I and J, denoted by $\mathrm{MCP}(I, J)$, the unique maximal codependent ordered pair of (I, J) for the inclusion.

Example 4.7. For example, let $\mathfrak{M}$ be a graphic matroid represented by a graph G with vertex set V and edge set E. Then, an ordered pair (I, J) of independent sets corresponds to an ordered pair of non-necessarily disjoint subforests (F_1, F_2) of G. Let U be a subset of V and let $H := G[U]$ be the induced subgraph of G of vertex set U. If $F_1[U]$ and $F_2[U]$ are subtrees of H, then the edge set of $F_1[U]$ and the edge set of $F_2[U]$ form a codependent ordered pair of (I, J). The maximal dependent ordered pair of (I, J) is the union of all ordered pairs of this kind. See Definition 2.5 of [1] about *saturated components* of a graph for more details.

It is easy to see that, if $(I, J) \in \mathcal{V}$, then $I \uplus J$ and $\mathrm{MCP}(I, J)$ are invariant in the connected component of (I, J) in $\mathcal{G}$ (we will prove it later). The nice result is that, in many interesting subgraphs, they form a complete set of invariants among the subgraph of non-isolated vertices. That is why we first study which vertices are isolated.

Proposition 4.8. *Let $p \in [1 \cdot\cdot r]$ and $q \in [0 \cdot\cdot r-1]$ be two integers. Let (I, J) be a vertex of $\mathcal{G}$. Then:*

1. *(I, J) is an isolated vertex of $\mathcal{G}$ if and only if $\mathrm{cl}(I) = \mathrm{cl}(J)$, i.e., $\mathrm{MCP}(I, J) = (I, J)$;*
2. *if $(I, J) \in \mathcal{G}_{p,q}$, then (I, J) is an isolated vertex of $\mathcal{G}_{p,q}$ if and only if: $(I, J) \in \mathcal{V}_{p,q}$ and $I \subset \mathrm{cl}(J)$, or $(I, J) \in \mathcal{V}_{p-1,q+1}$ and $J \subset \mathrm{cl}(I)$;*
3. *if $p = q + 1$, $\mathcal{G}_{p,q}$ has no isolated vertex.*

Proof. We will prove the different points of the theorem in another order.

(2) Suppose that $(I, J) \in \mathcal{V}_{p,q}$. If (I, J) is not isolated in $\mathcal{G}_{p,q}$, there exist $(I_1, J_1) \in \mathcal{V}_{p-1,q+1}$ and $i \in E$ such that $I_1 = I - i$ and $J_1 = J + i$. Thus, $i \in I \setminus \mathrm{cl}(J)$, and so $I \not\subset \mathrm{cl}(J)$. Reciprocally, if $I \not\subset \mathrm{cl}(J)$, let $i \in I \setminus \mathrm{cl}(J)$. Therefore $i \notin J$. The ordered pair $(I - i, J + i)$ is an element of $\mathcal{V}_{p-1,q+1}$ and it is adjacent to (I, J). Thus, (I, J) is not isolated. To summarize, $(I, J) \in \mathcal{V}_{p,q}$ is not isolated in $\mathcal{G}_{p,q}$ if and only if $I \not\subset \mathrm{cl}(J)$. The case of vertices of $\mathcal{V}_{p-1,q+1}$ is symmetric: it suffices to use the isomorphism $\Phi_{p,q}$ of Remark 4.2. We finally obtain the Point (2).

(1) A similar argument shows that (I, J) is isolated in $\mathcal{G}$ if and only if $I \subset \mathrm{cl}(J)$ and $J \subset \mathrm{cl}(I)$. But $I \subset \mathrm{cl}(J)$ is equivalent to $\mathrm{cl}(I) \subset \mathrm{cl}(J)$. Thus, (I, J) is isolated in $\mathcal{G}$ iff $\mathrm{cl}(I) \subset \mathrm{cl}(J)$ and $\mathrm{cl}(J) \subset \mathrm{cl}(I)$, iff $\mathrm{cl}(I) = \mathrm{cl}(J)$.

(3) If $p > q$ and if $(I,J) \in \mathcal{V}_{p,q}$, by definition of a matroid, there exists $i \in I$ such that $J+i$ is independent. Thus, $I \not\subset \mathrm{cl}(J)$. Therefore, if $p = q+1$ and $(I,J) \in \mathcal{V}_{p,q}$, then (I,J) is not isolated in $\mathcal{G}_{p,q}$. Thus there is no isolated vertex in the part $\mathcal{V}_{p,q}$ of the bipartite graph $\mathcal{G}_{p,q}$. But, since $p = q+1$, the isomorphism $\Phi_{p,q}$ is an automorphism on $\mathcal{G}_{p,q}$ which exchanges the two parts $\mathcal{V}_{p,q}$ and $\mathcal{V}_{p-1,q+1}$. Since automorphisms preserve isolated vertices, there could not be any isolated vertex in $\mathcal{G}_{p,q}$.

□

Theorem 4.9. *Let (I,J) and (I',J') be two non-isolated vertices of $\mathcal{G}_{p,q}$. Then (I,J) and (I',J') are in the same connected component of $\mathcal{G}_{p,q}$ if and only if*

$$I \uplus J = I' \uplus J' \qquad \textit{and} \qquad \mathrm{MCP}(I,J) = \mathrm{MCP}(I',J').$$

Proof. In this proof, the only graph we consider is $\mathcal{G}_{p,q}$.

Let us begin with the forward direction. Let $(I,J) \in \mathcal{V}_{p,q}$ and let (J_1,I_1) be one of its neighbor in $\mathcal{V}_{p-1,q+1}$. Let $i \in E$ be such that $I_1 = J+i$, and $I = J_1+i$. One has

$$I \uplus J = (J_1+i) \uplus J = J_1 \uplus (J+i) = J_1 \uplus I_1.$$

Moreover, if $(U,V) := \mathrm{MCP}(I,J)$, since $i \in \mathrm{Fr}(J)$, $i \notin \mathrm{cl}(V)$, and so $i \notin U$. Thus, $\mathrm{MCP}(I,J) = \mathrm{MCP}(I-i,J) = \mathrm{MCP}(J_1,J)$. Using the same argument on (J_1,I_1), one obtains that $\mathrm{MCP}(I,J) = \mathrm{MCP}(J_1,I_1)$, which concludes this first part.

Let (I_0,J_0) and (I_0',J_0') be two non-isolated vertices of $\mathcal{G}_{p,q}$ such that $I_0 \uplus J_0 = I_0' \uplus J_0'$. Denote by $\mathcal{W}$, resp. $\mathcal{W}'$, the set of vertices in the connected component of (I_0,J_0), resp. of (I_0',J_0'). Define $\mathcal{W}_{p,q} := \mathcal{W} \cap \mathcal{V}_{p,q}$, and define similarly $\mathcal{W}_{p-1,q+1}, \mathcal{W}'_{p,q}, \mathcal{W}'_{p-1,q+1}$. Since the two vertices are assumed to be non-isolated, the four previous sets are nonempty. If (I,J) and (I',J') are two elements of $\mathcal{V}_{p,q}$, we set

$$d((I,J),(I',J')) := |I \setminus I'| = p - |I \cap I'|.$$

Now, set $(I,J) \in \mathcal{W}_{p,q}$ and $(I',J') \in \mathcal{W}'_{p,q}$ such that $d((I,J),(I',J'))$ is minimal. Set $d := d((I,J),(I',J'))$. Assume that $d \neq 0$. In order to show the theorem, it suffices to prove that $\mathrm{MCP}(I,J) \neq \mathrm{MCP}(I',J')$. We will do it step by step.

Let us begin with two lemmas about matroids.

Lemma 4.10. *Let $U,U' \in \mathcal{I}$ be such that $|U| = |U'|$ and $\mathrm{cl}(U) \neq \mathrm{cl}(U')$. Then, $U \cap \mathrm{Fr}(U')$ and $U' \cap \mathrm{Fr}(U)$ are nonempty.*

Proof of the lemma. Suppose, without loss of generality, that $\mathrm{cl}(U) \setminus \mathrm{cl}(U')$ is non empty. Let $i \in \mathrm{cl}(U) \setminus \mathrm{cl}(U') = \mathrm{cl}(U) \cap \mathrm{Fr}(U')$. Let $j \in \mathrm{Fr}(U) \cap (U'+i)$ (it exists because $\mathrm{rk}(\mathrm{cl}(U)) = \mathrm{rk}(U) < \mathrm{rk}(U'+i)$, thus $(U'+i)$ cannot be included in $\mathrm{cl}(U)$). Since $i \in \mathrm{cl}(U)$ and $j \in \mathrm{Fr}(U)$, one has $i \neq j$, and so $j \in \mathrm{Fr}(U) \cap U' \neq \emptyset$. Let $j' \in \mathrm{Fr}(U') \cap (U+j)$. Once more, $j' \neq j$, thus $j' \in \mathrm{Fr}(U') \cap U \neq \emptyset$, which concludes the proof. □

Lemma 4.11. *If $U \in \mathcal{I}$ and $i \in \mathrm{cl}(U)$, then $\{C \subset U | i \in \mathrm{cl}(C)\}$ admits a least element for the inclusion.*

Proof of the lemma. Let C and D be two minimal elements of the set of the statement. It suffices to show that $C = D$. Let $j \in C$ (assume we are not in the trivial case $\{i\} \notin \mathcal{I}$). By minimality, $C - j + i \in \mathcal{I}$. Using several times the augmentation property (Definition 2.30, (3)), one obtains that $U - j + i \in \mathcal{I}$. Since $i \in \mathrm{cl}(D)$ but $i \notin \mathrm{cl}(U-j)$, necessarily $D \not\subset (U-j)$, and so $j \in D$. The end of the proof is now straightforward. □

Let us come back to the main proof. Let (I,J) and (I',J') be as defined above. Here are the first results.

$$I \cap J = I' \cap J'. \tag{42}$$

This can be easily deduced from $I \uplus J = I' \uplus J'$.

Next,

$$\mathrm{Fr}(J) = \mathrm{Fr}(J'). \tag{43}$$

To see this, suppose on the contray that this equation is false. By Lemma 4.10, there exists an $i \in J \cap \mathrm{Fr}(J')$. One can see that $i \in I'$, because $i \in (I \cup J) \setminus J'$. Moreover, let $j \in \mathrm{Fr}(I'-i) \cap I$. Similarly, $j \in J'+i$. One obtains that $(I'-i+j, J'+i-j) \in \mathcal{W}'_{p,q}$ and $(I'-i+j) \cap I = (I' \cap I)+j$, because $j \in I$ and $i \notin I$ (otherwise one would have $i \in I \cap J$, and so $i \in I' \cap J'$ by (42); but $i \in \mathrm{Fr}(J')$). The last equality contradicts minimality of d.

Now set $i \in \mathrm{Fr}(J) \cap I$. One has directly by (43) that $i \in \mathrm{Fr}(J')$, and so $i \in I \cap I'$. Then we prove that

$$\mathrm{Fr}(I-i) = \mathrm{Fr}(I'-i). \tag{44}$$

Otherwise, by Lemma 4.10, one could set $j \in (I'-i) \cap \mathrm{Fr}(I-i)$ and $j' \in (I-i) \cap \mathrm{Fr}(I'-i)$. But $(I-i+j, J+i-j) \in \mathcal{W}_{p,q}$ and $(I'-i+j', J'+i-j') \in \mathcal{W}'_{p,q}$. Moreover $(I-i+j) \cap (I'-i+j') = (I \cap I')-i+j+j'$. Once more, that contradicts minimality of d.

If (J_1,I_1) is a neighbor of (I,J), and if (I_2,J_2) is a neighbor of (J_1,I_1), then there exists an $i \in \mathrm{Fr}(J)$ such that $I_1 = J+i$, and there exists a $j \in \mathrm{Fr}(J_1) = \mathrm{Fr}(I-i)$ such that $I_2 = I-i+j$. By (43), $i \in \mathrm{Fr}(J')$, and, by (44), $j \in \mathrm{Fr}(I'-i)$. Setting $(I'_2,J'_2) := (I'-i+j, J'+i-j)$, one obtains that $(I'_2,J'_2) \in \mathcal{W}'_{p,q}$ and

$$J_2 \setminus J'_2 = (J+i-j) \setminus (J'+i-j) = J \setminus J'.$$

In particular, if one sets $R := J \setminus J'$, notice that R is nonempty and that $R \subset J_2$. Moreover, notice that $R \subset I_1$. Since the neighbors have been arbitrarily chosen, this property extends to the whole component $\mathcal{W}$. Thus, if we set

$$P_2 := \bigcap_{(U,V)\in\mathcal{W}} V,$$

which we will call *the set of fixed elements of* J, one has $R \subset P_2$. We similarly define

$$P_1 := \bigcap_{(U,V)\in\mathcal{W}} U,$$

as well as P'_1 and P'_2, where we replace $\mathcal{W}$ by $\mathcal{W}'$ in the definition.

The end of the demonstration will mainly consist in the proof of the following proposition.

Proposition 4.12. *With above notations,*

$$\mathrm{MCP}(I,J) = (P_1,P_2).$$

Let us assume the proposition for the moment, and let us end the proof of the theorem. If (I,J) and (I',J') are not in the same connected component, then R is nonempty. But $R \cap J' = \emptyset$, $P'_2 \subset J'$ and $R \subset P_2$. Finally, $P_2 \neq P'_2$ and, using the proposition, $\mathrm{MCP}(I,J) \neq \mathrm{MCP}(I',J')$. The contrapositive is: under the hypothesis $I \uplus J = I' \uplus J'$, if $\mathrm{MCP}(I,J) = \mathrm{MCP}(I',J')$ then (I,J) and (I',J') are in the same connected component. That is what we wanted to prove.

Proof of the proposition. It remains to show the proposition. Actually, we have already shown the inclusion: $\mathrm{MCP}(I,J) \subset (P_1,P_2)$. Indeed, at the beginning of the proof, we showed that if $(U,V) \in \mathcal{W}$, then $\mathrm{MCP}(U,V) = \mathrm{MCP}(I,J)$. Looking at the definitions of P_1 and of P_2, it is clear that $\mathrm{MCP}(I,J) \subset (P_1,P_2)$.

In order to prove the other inclusion, we have to introduce the two following sets

$$Q_1 := \bigcap_{(J_1,I_1)\in\mathcal{W}_{p-1,q+1}} \mathrm{cl}(J_1), \qquad Q_2 := \bigcap_{(I_2,J_2)\in\mathcal{W}_{p,q}} \mathrm{cl}(J_2).$$

Clearly,

$$P_1 \subset Q_1 \text{ and } P_2 \subset Q_2. \tag{45}$$

The interesting property of these sets is that

$$Q_1 = Q_2. \tag{46}$$

It suffices to proof that any element of E which is not in Q_2, is not in Q_1 either. Let i be an element of E which is not in Q_2. There exists $(I_2,J_2) \in \mathcal{W}_{p,q}$ such that $i \notin \mathrm{cl}(J_2)$. If $i \notin \mathrm{cl}(I_2)$, then $i \notin \mathrm{cl}(J_1)$ for any neighbor (J_1,I_1) of (I_2,J_2), and so $i \notin Q_1$. Otherwise, $i \in \mathrm{cl}(I_2)$. By Lemma 4.11, there exists a least element for the inclusion $C \subset I_2$ such that $i \in \mathrm{cl}(C)$. Since $i \notin \mathrm{cl}(J_2)$, we have $C \not\subset \mathrm{cl}(J_2)$. Let $j \in C \cap \mathrm{Fr}(J_2)$. One obtains that $(I_2 - j, J_2 + j) \in \mathcal{W}_{p-1,q+1}$, and that $i \notin \mathrm{cl}(I_2 - j)$ since $C \not\subset I_2 - j$. Thus, $i \notin Q_1$.

Now we show a last result, namely that

$$Q_2 \subset \mathrm{cl}(P_2). \tag{47}$$

To see this, let $i \in Q_2$. Let (J_1,I_1) be a neighbor of (I,J), and (I_2,J_2) be a neighbor of (J_1,I_1). One has $i \in \mathrm{cl}(J)$ and $i \in \mathrm{cl}(J_2)$, and so $i \in \mathrm{cl}(I_1)$. Thus, by Lemma 4.11, one can choose C, resp. C_1, C_2, minimal for the inclusion in J, resp. I_1, J_2, such that i is in the closure of C, resp. C_1, C_2. Since $C \subset I_1$, one has, by minimality, that $C_1 \subset C$. Then, $C_1 \subset J$, and so, by minimality, $C \subset C_1$. All this induces that $C = C_1$. Similarly, $C_2 = C_1$, and so $C = C_2$. By connectivity, for all $(U,V) \in \mathcal{W}$, $C \subset V$. Thus, we have $C \subset P_2$, and so $i \in \mathrm{cl}(P_2)$. The result is now straightforward.

Now we have all the needed intermediate results. Equations (45), (46) and (47) imply that $P_1 \subset Q_1 = Q_2 \subset \mathrm{cl}(P_2)$, and so $\mathrm{cl}(P_1) \subset \mathrm{cl}(P_2)$. Using a symmetric argument, we obtain that $\mathrm{cl}(P_1) = \mathrm{cl}(P_2)$. Thus, (P_1,P_2) is a dependent ordered pair of (I,J). In particular, $(P_1,P_2) \subset \mathrm{MCP}(I,J)$.

Finally, we obtain the second inclusion, and then the proposition follows. □

Thus, we have finished the proof of Theorem 4.9. To summarize, the proposition shows that fixed elements of $I \uplus J$ cannot be exchanged because they are in the MCP. The rest of the proof shows that if a configuration cannot be reached, that is because it does not have the fixed elements of (I,J). To conclude with, the MCPs are the unique nontrivial constraints preventing exchanges. □

This first corollary will be useful in Section 5.

Corollary 4.13. *Let $(I,J), (I',J')$ be two arbitrary vertices of $\mathcal{G}_{r,r-1}$. Then (I,J) and (I',J') are in the same connected component of $\mathcal{G}_{r,r-1}$ if and only if*

$$I \uplus J = I' \uplus J' \qquad \textit{and} \qquad \mathrm{MCP}(I,J) = \mathrm{MCP}(I',J').$$

Proof. It suffices to combine Theorem 4.9 with Point (3) of Proposition 4.8. □

The second corollary completes Theorem 4.9 with the case of the whole exchange graph.

Corollary 4.14. *Let (I,J), (I',J') be two arbitrary vertices of $\mathcal{G}$. Then (I,J) and (I',J') are in the same connected component of $\mathcal{G}$ if and only if*

$$I \uplus J = I' \uplus J' \qquad \text{and} \qquad \mathrm{MCP}(I,J) = \mathrm{MCP}(I',J').$$

Proof. The forward direction is identical as in the proof of Theorem 4.9. For the other direction, let (I,J) and (I',J') be two vertices of $\mathcal{G}$ such that $I \uplus J = I' \uplus J'$ and $\mathrm{MCP}(I,J) = \mathrm{MCP}(I',J')$. Let

$$p := \left\lceil \frac{|I|+|J|}{2} \right\rceil = \left\lceil \frac{|I'|+|J'|}{2} \right\rceil,$$
$$q := \left\lfloor \frac{|I|+|J|}{2} \right\rfloor = \left\lfloor \frac{|I'|+|J'|}{2} \right\rfloor.$$

If $|J| > |I|$, then $J \cap \mathrm{Fr}(I)$ is nonempty. Let j be an element of this set. Then $(I+j, J-i)$ is adjacent to (I,J). Iterating this process, it is clear that, if $|J| > |I|$, the connected component of (I,J) contains a vertex of $\mathcal{G}_{p,q}$. Actually, this is still true if $|J| \leqslant |I|$ putting elements of I in J and stopping at the right time. Let $(\widetilde{I}, \widetilde{J})$, resp. $(\widetilde{I}', \widetilde{J}')$, be an element of $\mathcal{G}_{p,q}$ in the connected component of (I,J), resp. of (I',J'). We have $\widetilde{I} \uplus \widetilde{J} = \widetilde{I}' \uplus \widetilde{J}'$ and $\mathrm{MCP}(\widetilde{I}, \widetilde{J}) = \mathrm{MCP}(\widetilde{I}', \widetilde{J}')$. Thus, $(\widetilde{I}, \widetilde{J})$ and $(\widetilde{I}', \widetilde{J}')$ are connected in $\mathcal{G}_{p,q}$, thus in $\mathcal{G}$, if they are not isolated in $\mathcal{G}_{p,q}$. There are two possibilities.

If $p = q+1$, then they cannot be isolated by Proposition 4.8 (3). Otherwise, $p = q$. In this case, suppose, for example, that $(\widetilde{I}, \widetilde{J})$ is isolated. Then, by Proposition 4.8 (2), $\widetilde{J} \subset \mathrm{cl}(\widetilde{I})$. But, since $|\widetilde{J}| = |\widetilde{I}|$, Lemma 4.10 implies $\mathrm{cl}(\widetilde{J}) = \mathrm{cl}(\widetilde{I})$, thus $\mathrm{MCP}(\widetilde{I}, \widetilde{J}) = (\widetilde{I}, \widetilde{J})$, and so $\mathrm{MCP}(\widetilde{I}', \widetilde{J}') = (\widetilde{I}, \widetilde{J})$. Looking at cardinalities, this last equality implies that $(\widetilde{I}, \widetilde{J}) = (\widetilde{I}', \widetilde{J}')$.

In every case, $(\widetilde{I}, \widetilde{J})$ and $(\widetilde{I}', \widetilde{J}')$ are in the same connected component, so are (I,J) and (I',J'). ☐

Now we arrive to the last section of this paper which generalizes Theorem 1.1 of [1].

5. Variation of Symanzik rational fractions

In this section we take $A = \mathbb{R}$. We set n, p, q three positive integers, $R \in \mathcal{M}_{n,p}(\mathbb{R})$ a matrix, r its rank, f a basis of $\mathrm{Im}(R^\intercal)$, $\mathfrak{M} := \mathfrak{M}_R$ the matroid associated to R, $s := n - r$, $S \in \mathcal{M}_{n,s}(\mathbb{R})$ a normal kernel matrix of R with basis f, $u \in \mathrm{Im}(R^\intercal)$ a nonzero vector, $v = (\mathbf{v}_1, \dots, \mathbf{v}_n) \in \mathbb{R}^n$ such that $R^\intercal \mathbf{v} = \mathbf{u}$, $T := S \star \mathbf{v}$ and $X \in \mathcal{M}_n(\mathbb{R}[\underline{x}])$ the diagonal matrix $\mathrm{diag}(x_1, \dots, x_n)$. Moreover, we set Δ a simplicial complex of dimension d, d being a positive integer, with n facets and p $(d-1)$-dimensional faces.

In this last section, we state a nice property of Symanzik rational fractions. One can roughly states it as "a bounding deformation of the metric of a simplicial complex only implies a uniformly bounded variation of the Symanzik rational fraction of even order with one parameter", where uniformly means that the bound does not depend on the chosen metric.

As seen in Proposition 2.6 and using Proposition 2.27 with standard bases, we have

$$\widetilde{\mathrm{Sym}}_2(R, u; \underline{x}) = \frac{\det(T^\intercal X T)}{\det(S^\intercal X S)}.$$

If R is the transpose of the d-th incidence matrix of the simplicial complex Δ of dimension d, we have seen in Examples 3.14, 3.19 and 3.44 that it is natural to replace x_i, for each

$i \in [1 \cdot\cdot n]$, by the measure of the i-th facet of Δ. That's why we will deform the metric of Δ by slightly perturbing X.

Let U be some space and $F : U \to \mathcal{M}_n(\mathbb{R})$ be a bounded map (i.e., such that there exists a positive constant C such that, for all $t \in U$, all entries of $F(t)$ are in $[-C, C]$). Let $y_1, \dots, y_n : U \to \mathbb{R}_+$ be n functions and let

$$\begin{array}{rccl} Y: & U & \longrightarrow & \mathcal{M}_n(\mathbb{R}), \\ & t & \longmapsto & \mathrm{diag}(y_1(t), \dots, y_n(t)). \end{array}$$

Suppose that $S^\intercal(F(t) + Y(t))S$ is invertible for all $t \in U$.

If ϕ and ψ are two functions from U to $\mathbb{R}$, then the notation $\phi = \mathcal{O}_{\underline{y}}(\psi)$ means that there exist two positive constants c and C such that, for all $t \in U$, $y_1(t), \dots, y_n(t) \geqslant C$ implies that $|\phi(t)| \leqslant c|\psi(t)|$. Similarly, the notation $\phi = \mathrm{o}_{\underline{y}}(\psi)$ means that, for all positive real ε, there exists a positive real C_ε such that, for all $t \in U$, $y_1(t), \dots, y_n(t) \geqslant C_\varepsilon$ implies that $|\phi(t)| \leqslant \varepsilon|\psi(t)|$.

We will show he following theorem.

Theorem 5.1. *With the above notations,*

$$\frac{\det(T^\intercal Y T)}{\det(S^\intercal Y S)} - \frac{\det(T^\intercal (Y + F)T)}{\det(S^\intercal (Y + F)S)} = \mathcal{O}_{\underline{y}}(1).$$

Before giving the proof, we remark that one can generalize this theorem to Symanzik polynomials of even positive order (the case of the order 0 is trivial), thanks to multidimensional matrices, in the following way.

Let k be any even positive integer. Let $F : U \to \mathfrak{C}_n^k(\mathbb{R})$ be a bounded map, let $y_1, \dots, y_n : U \to \mathbb{R}_+$ be n functions and let

$$\begin{array}{rccl} Y: & U & \longrightarrow & \mathfrak{C}_n^k(\mathbb{R}), \\ & t & \longmapsto & \mathrm{diag}^k(y_1(t), \dots, y_n(t)), \end{array}$$

where diag^k is defined in Proposition 2.7. Suppose that $\det((Y(t) + F(t)) \cdot_1 S \cdots \cdot_k S)$ is nonzero for all $t \in U$.

Theorem 5.2. *With the above notations,*

$$\frac{\det(Y \cdot_1 T \cdots \cdot_k T)}{\det(Y \cdot_1 S \dots \cdot_k S)} - \frac{\det((Y + F) \cdot_1 T \cdots \cdot_k T)}{\det((Y + F) \cdot_1 S \cdots \cdot_k S)} = \mathcal{O}_{\underline{y}}(1).$$

Proof of both theorems. Let k be any positive even integer. The proof essentially follow the proof of Theorem 1.1 in [1]. Let us set the following functions

$$\begin{aligned} f_1 &:= \det(Y \cdot_1 S \cdots \cdot_k S), \\ f_2 &:= \det(Y \cdot_1 T \cdots \cdot_k T), \\ g_1 &:= \det((Y + F) \cdot_1 S \cdots \cdot_k S), \\ g_2 &:= \det((Y + F) \cdot_1 T \cdots \cdot_k T). \end{aligned}$$

We have already seen (for example in Claim 2.36) that $\det(S_{I^c}) \neq 0$ if and only if $I \in \mathcal{B}(\mathfrak{M}) = \mathcal{I}_r$. Moreover, $\det(T_{J^c}) \neq 0$ only if $J \in \mathcal{I}_{r-1}$ (see Proposition 2.38). Thus, using Proposition 2.7,

$$f_1 = \sum_{I \in \mathcal{I}_r} \det(S_{I^c})^k \underline{y}^{I^c},$$

$$f_2 = \sum_{J \in \mathcal{I}_{r-1}} \det(T_{J^c})^k \underline{y}^{J^c}.$$

Notice that f_1 and f_2 are homogeneous polynomials of $\mathbb{R}[\underline{y}]$ of respective degrees r and $r-1$, and, more importantly, that all their coefficients are positive.

By the generalized Cauchy-Binet formula (Proposition A.5) applied, for example, to g_1, one obtains

$$\begin{aligned} g_1 &= \det((Y+F) \cdot_1 S \cdots \cdot_k S) \\ &= \sum_{\substack{I_k \subset [1..n] \\ |I_k| = s}} \det\Big(((Y+F) \cdot_1 S \cdots \cdot_{k-1} S)_{k:I_k}\Big) \det(S_{I_k}) \\ &= \sum_{\substack{I_k \subset [1..n] \\ |I|_k = s}} \sum_{\substack{I_{k-1} \subset [1..n] \\ |I|_{k-1} = s}} \det\Big(((Y+F) \cdot_1 S \cdots \cdot_{k-2} S)_{k-1:I_{k-1}, k:I_k}\Big) \det(S_{I_{k-1}}) \det(S_{I_k}) \\ &= \qquad \vdots \\ &= \sum_{\substack{I_1,\dots,I_k \subset [1..n] \\ |I_1| = \cdots = |I_k| = s}} \det\Big((Y+F)_{1:I_1,\dots,k:I_k}\Big) \det(S_{I_1}) \cdots \det(S_{I_k}). \end{aligned}$$

Moreover, as we have seen, if we take the complementaries of the I_is, then we can restrict them in the following way.

$$g_1 = \sum_{I_1,\dots,I_k \in \mathcal{I}_r} \det\Big((Y+F)_{1:I_1^c,\dots,k:I_k^c}\Big) \det(S_{I_1^c}) \cdots \det(S_{I_k^c}). \tag{48}$$

In the same way,

$$g_2 = \sum_{J_1,\dots,J_k \in \mathcal{I}_{r-1}} \det\Big((Y+F)_{1:J_1^c,\dots,k:J_k^c}\Big) \det(T_{J_1^c}) \cdots \det(T_{J_k^c}). \tag{49}$$

Admitting coefficients to be functions, g_1 and g_2 are still polynomials of respective degrees r and $r-1$, but they are no more homogeneous. Moreover all coefficients of g_1 and g_2 are bounded.

If I is a subset of $[1..n]$ and if $h \in \mathbb{R}[\underline{y}]$, let us denote by $[\underline{y}^I]h$ the coefficient of the monomial $\underline{y}^I$ in h. For example, if $I \in \mathcal{I}_r$, the monomial $\underline{y}^{I^c}$ of g_1 is only present in the term where all I_is are equal to I. Thus

$$\begin{aligned} [\underline{y}^{I^c}]g_1 &= [\underline{y}^{I^c}]\Big(\det\Big((Y+F)_{1:I^c,\dots,k:I^c}\Big)\det(S_{I^c})^k\Big) \\ &= \det(S_{I^c})^k. \end{aligned}$$

We deduce that these coefficients are constant, and that for all $I \in \mathcal{I}$,

$$[\underline{y}^{I^c}]g_1 = [\underline{y}^{I^c}]f_1. \tag{50}$$

Similarly, if $J \in \mathcal{I}_{r-1}$, $[\underline{y}^{J^c}]g_2 = [\underline{y}^{J^c}]f_2$.

The statements of the theorems are that $f_2/f_1 - g_2/g_1 = \mathcal{O}_{\underline{y}}(1)$. Let us simplify this statement thanks to the following claim.

Claim 5.3. *We have* $g_1 - f_1 = \mathrm{o}_{\underline{y}}(f_1)$.

Proof of the claim. By (50), $g_1 - f_1$ is a polynomial of degree at most $r-1$, whose coefficients are bounded functions. Moreover, if $J \subset [1 \cdot\cdot n]$ is an arbitrary subset such that $[\underline{y}^J](g_1 - f_1)$ is nonzero, then, looking at (48), it is clear that J is strictly included in some basis $I \in \mathcal{I}_r$ of $\mathfrak{M}$. It happens that $[\underline{y}^I]f_1 = \det(S_I)^k$ is a positive integer. Since $[\underline{y}^J](g_1 - f_1)$ is bounded,

$$\Big([\underline{y}^J](g_1 - f_1)\Big)\underline{y}^J = \mathrm{o}_{\underline{y}}(\underline{y}^I).$$

Since all coefficients of f_1 are positive, we can sum all terms of $(g_1 - f_1)$, and then conclude the proof. □

Multiplying $f_2/f_1 - g_2/g_1$ by $f_1 g_1$, and using the claim, it remains to show that

$$g_1 f_2 - f_1 g_2 = \mathcal{O}_{\underline{y}}(f_1^2).$$

Notice that the monomials with nonzero coefficients in f_1^2 are exactly the monomials of the form $\underline{y}^{I^c}\underline{y}^{I'^c}$ where $(I, I') \in \mathcal{I}_r \times \mathcal{I}_r$.

Let us rewrite

$$g_1(t)f_2(t) = \sum_{I_1,\dots,I_k \in \mathcal{I}_r} \sum_{J \in \mathcal{I}_{r-1}} a(I_1,\dots,I_k,J)h(I_1,\dots,I_k,J;t),$$

where

$$h(I_1,\dots,I_k,J;t) := \det\Big((Y+F)_{1:I_1^c,\dots,k:I_k^c}\Big)\underline{y}^{J^c}$$

is a polynomial whose coefficients are functions, and

$$a(I_1,\dots,I_k,J) = \det(S_{I_1^c})\cdots\det(S_{I_k^c})\det(T_{J^c})^k$$

is a real number. Similarly,

$$f_1(t)g_2(t) = \sum_{J_1,\dots,J_k \in \mathcal{I}_{r-1}} \sum_{I \in \mathcal{I}_r} a(J_1,\dots,J_k,I)h(J_1,\dots,J_k,I;t),$$

where

$$h(J_1,\dots,J_k,I;t) := \det\Big((Y+F)_{1:J_1^c,\dots,k:J_k^c}\Big)\underline{y}^{I^c},$$

$$a(I_1,\dots,I_k,J) := \det(T_{J_1^c})\cdots\det(T_{J_k^c})\det(S_{I^c})^k.$$

It is clear that

$$\begin{aligned} h(K_1,\dots,K_k,L;t) &= \mathcal{O}_{\underline{y}}(\underline{y}^{K_1^c\cap\cdots\cap K_n^c}\underline{y}^{L^c}) \\ &= \mathcal{O}_{\underline{y}}(\underline{y}^{(K_1\cup\cdots\cup K_n)^c}\underline{y}^{L^c}). \end{aligned} \tag{51}$$

Let us define a new graph which is slightly similar to the exchange graph $\mathcal{G}_{r,r-1}$ of $\mathfrak{M}$. Let $\mathfrak{G}$ be a bipartite graph with vertex set $\mathfrak{V} = \mathfrak{V}_{r-1,r} \sqcup \mathfrak{V}_{r,r-1}$ and edge set $\mathfrak{E}$ where

$$\mathfrak{V}_{r-1,r} := (\mathcal{I}_{r-1})^k \times \mathcal{I}_r,$$

$$\mathfrak{V}_{r,r-1} := (\mathcal{I}_r)^k \times \mathcal{I}_{r-1},$$

and where two vertices $(J_1,\dots,J_k,I) \in \mathfrak{V}_{r-1,r}$ and $(I_1,\dots,I_k,J) \in \mathfrak{V}_{r,r-1}$ are connected by an edge if and only if there exists $i \in \mathrm{Fr}(J_1) \cap \cdots \cap \mathrm{Fr}(J_k) \cap I$ such that $I = J + i$ and $I_l = J_l + i$, for all $l \in [1 \cdot\cdot k]$.

Definition 5.4. A vertex $(J_1,\dots,J_k,I)$ in $\mathfrak{V}_{r-1,r}$ is said *ordinary* if $\mathrm{Fr}(J_1) = \cdots = \mathrm{Fr}(J_k)$. A vertex $(I_1,\dots,I_k,J)$ in $\mathfrak{V}_{r,r-1}$ is said *ordinary* if $I_1 \cap \mathrm{Fr}(J) = \cdots = I_k \cap \mathrm{Fr}(J)$. A vertex of $\mathfrak{V}$ which is not *ordinary* is called *special*.

Now one can see h and a as some functions on $\mathfrak{G}$. Moreover

$$g_1(t)f_2(t) = \sum_{\mathfrak{u}\in\mathfrak{V}_{r,r-1}} a(\mathfrak{u})h(\mathfrak{u};t), \tag{52}$$

$$g_2(t)f_1(t) = \sum_{\mathfrak{u}\in\mathfrak{V}_{r-,r}} a(\mathfrak{u})h(\mathfrak{u};t). \tag{53}$$

That is why, we will need the following claim.

Claim 5.5. *Here are some properties of h and a.*

(1) *If $\mathfrak{u}$ and $\mathfrak{v}$ are two adjacent vertices of $\mathfrak{G}$, then $h(\mathfrak{u}) - h(\mathfrak{v}) = \mathcal{O}_{\underline{y}}(f_1^2)$.*
(2) *If $\mathfrak{u}$ is a special vertex of $\mathfrak{V}$, then $h(\mathfrak{u};t) = \mathcal{O}_{\underline{y}}(f_1^2)$.*
(3) *If $(J_1,\dots,J_k,I) \in \mathfrak{V}_{r-1,r}$ is an ordinary vertex, and if $(I_1,\dots,I_k,J) \in \mathfrak{V}_{r,r-1}$ is one of its neighbors, then, for all $l \in [1 \,..\, k]$,*

$$\frac{\det(S_{I_l^c})}{\det(S_{I_1^c})} = \frac{\det(T_{J_l^c})}{\det(T_{J_1^c})}.$$

Proof. The three points are proven independently.

(1) Let $\mathfrak{u} = (J_1,\dots,J_k,I) \in \mathfrak{V}_{k-1,k}$ and $\mathfrak{v} = (I_1,\dots,I_k,J)$ be two adjacent vertices. Let $i \in [1 \,..\, n]$ be such that $I = J + i$. Let us extract y_i from $h(\mathfrak{u};t)$ and from $h(\mathfrak{v};t)$. One has

$$\begin{aligned} h(I_1,\dots,I_k,J;t) &= \det\Big((Y+F)_{1:I_1^c,\dots,k:I_k^c}\Big)\underline{y}^{J^c} \\ &= \Big(\det\Big((Y+F)_{1:I_1^c,\dots,k:I_k^c}\Big)\underline{y}^{I^c}\Big)y_i, \end{aligned}$$

and, using the Laplace decomposition along the column of the determinant which contains y_i,

$$\begin{aligned} h(J_1,\dots,J_k,I;t) &= \det\Big((Y+F)_{1:J_1^c,\dots,k:J_k^c}\Big)\underline{y}^{I^c} \\ &= \Big(\det\Big((Y+F)_{1:I_1^c,\dots,k:I_k^c}\Big)y_i + \mathcal{O}_{\underline{y}}(y^{I_1^c\cap\dots\cap I_k^c})\Big)\underline{y}^{I^c}. \end{aligned}$$

Thus,

$$\begin{aligned} h(\mathfrak{u};t) - h(\mathfrak{v};t) &= \mathcal{O}_{\underline{y}}(\underline{y}^{I_1^c\cap\dots\cap I_k^c}\underline{y}^{I^c}) \\ &= \mathcal{O}_{\underline{y}}(\underline{y}^{I_1^c}\underline{y}^{I^c}). \end{aligned}$$

But the monomial $\underline{y}^{I_1^c}\underline{y}^{I^c}$ is present in f_1^2 with a positive coefficient. Thus,

$$h(\mathfrak{u};t) - h(\mathfrak{v};t) = \mathcal{O}_{\underline{y}}(f_1^2).$$

(2) Since they are two kinds of special vertices, we will make two cases. Let $\mathfrak{u} = (J_1,\dots,J_k,I)$ be a special vertex of $\mathfrak{V}_{r-1,r}$. Assume, without loss of generality, that $\mathrm{Fr}(J_1) \neq \mathrm{Fr}(J_2)$. We have seen in (51) that

$$h(\mathfrak{u};t) = \mathcal{O}_{\underline{y}}(\underline{y}^{(J_1\cup\dots\cup J_k)^c}\underline{y}^{I^c}).$$

By Lemma 4.10, $\mathrm{Fr}(J_1) \neq \mathrm{Fr}(J_2)$ implies that there exists $j \in J_1 \cap \mathrm{Fr}(J_2)$. Since $\mathrm{rk}(J_2) = r - 1$, $I' := J_2 + j$ is in $\mathcal{I}_r$. But $I' \subset J_1 \cup \cdots \cup J_k$, and so

$$\underline{y}^{(J_1 \cup \cdots \cup J_k)^c} = \mathcal{O}_{\underline{y}}(I'^c).$$

Then,

$$h(\mathfrak{u}; t) = \mathcal{O}_{\underline{y}}(\underline{y}^{I'^c} \underline{y}^{I^c}),$$

but the monomial $\underline{y}^{I'^c} \underline{y}^{I^c}$ is present in f_1^2, and so

$$h(\mathfrak{u}; t) = \mathcal{O}_{\underline{y}}(f_1^2).$$

In the same way, if $\mathfrak{u} = (I_1, \ldots, I_k, J)$ is a special vertex of $\mathfrak{V}_{r,r-1}$ we have

$$h(\mathfrak{u}; t) = \mathcal{O}_{\underline{y}}(\underline{y}^{(I_1 \cup \cdots \cup I_k)^c} \underline{y}^{J^c}).$$

Assume, without loss of generality, that there exists an element i in $(I_1 \cap \mathrm{Fr}(J)) \setminus (I_2 \cap \mathrm{Fr}(J))$. One has

$$i \notin (I_1 \cup \cdots \cup I_k)^c, \qquad (I_1 \cup \cdots \cup I_k)^c \subset I_2^c, \qquad i \in I_2^c.$$

This implies $(I_1 \cup \cdots \cup I_k)^c + i \subset I_2^c$. Moreover, $I := J + i$ is in $\mathcal{I}_r$. One obtains

$$\begin{aligned} \underline{y}^{(I_1 \cup \cdots \cup I_k)^c} \underline{y}^{J^c} &= \underline{y}^{(I_1 \cup \cdots \cup I_k)^c + i} \underline{y}^{J^c - i} \\ &= \mathcal{O}_{\underline{y}}(\underline{y}^{I_2^c} \underline{y}^{I^c}). \end{aligned}$$

Finally, the monomial $\underline{y}^{I_2^c} \underline{y}^{I^c}$ has a positive coefficient in f_1^2. Thus

$$h(\mathfrak{u}; t) = \mathcal{O}_{\underline{y}}(f_1^2).$$

(3) We use the notations of the statement. Let i be such that $I = J + i$. The partial result (37) in the proof of Theorem 3.32 can be restated here as

$$\det(T_{J_1^c}) = \sum_{j \in \mathrm{Fr}(J_1)} \zeta_{J_1^c - j, \{j\}} \sigma(J_1 + j) \det{}_f(R^{\intercal}_{J_1 + j}) \mathbf{v}_j,$$

where

$$\zeta_{J_1^c - j, \{j\}} = \frac{\det_f(R^{\intercal}_{J_1} \star R^{\intercal}_{\{j\}})}{\det_f(R^{\intercal}_{J_1 + j})}.$$

Since the vertex is ordinary, $\mathrm{cl}(J_1) = \mathrm{cl}(J_l)$. Thus, there exists an invertible matrix $P_l \in \mathcal{M}_n(\mathbb{R})$ such $R_{J_l} = P_l R_{J_1}$. We set

$$\widetilde{P}_l := \left(\begin{array}{c|c} P_l & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array} \right).$$

We also have $\mathrm{Fr}(J_l) = \mathrm{Fr}(J_1)$, and

$$\begin{aligned} \det(T_{J_l^c}) &= \sum_{j \in \mathrm{Fr}(J_l)} \zeta_{J_l^c - j, \{j\}} \, \sigma(J_l + j) \det{}_f(R^{\intercal}_{J_l + j}) \mathbf{v}_j \\ &= \sum_{j \in \mathrm{Fr}(J_1)} \frac{\det_f(R^{\intercal}_{J_l} \star R^{\intercal}_{\{j\}})}{\det_f(R^{\intercal}_{J_l + j})} \, \sigma(J_l + j) \det{}_f(R^{\intercal}_{J_l + j}) \mathbf{v}_j \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j\in \mathrm{Fr}(J_1)} \sigma(J_l+j) \det{}_f(R^\intercal_{J_l}\star R^\intercal_{\{j\}})\mathbf{v}_j \\
&= \sum_{j\in \mathrm{Fr}(J_1)} \sigma(J_l+j) \det{}_f(\widetilde{P}_l(R^\intercal_{J_1}\star R^\intercal_{\{j\}}))\mathbf{v}_j \\
&= \det(\widetilde{P}_l) \sum_{j\in \mathrm{Fr}(J_1)} \frac{\sigma(J_l+j)}{\sigma(J_1+j)}\, \sigma(J_1+j) \det{}_f(R^\intercal_{J_1}\star R^\intercal_{\{j\}})\mathbf{v}_j \\
&= \det(P_l) \sum_{j\in \mathrm{Fr}(J_1)} \frac{\sigma(J_l)}{\sigma(J_1)} \zeta_{J_1^c-j,\{j\}}\, \sigma(J_1+j) \det{}_f(R^\intercal_{J_1+j})\mathbf{v}_j \\
&= \det(P_l)\frac{\sigma(J_l)}{\sigma(J_1)} \det(T_{J_1^c}).
\end{aligned}
$$

We have to make a second, much easier, computation. We use Equation (4), in the proof of Theorem 2.15, and the fact that S is a normal kernel matrix of R with basis f.

$$
\begin{aligned}
\det(S_{I_l^c}) &= \sigma(I_l^c)\det(R^\intercal_{I_l}) \\
&= \frac{\sigma(I_l^c)}{\sigma(I_1^c)}\, \sigma(I_1^c)\det(P_l R^\intercal_{I_1}) \\
&= \frac{\sigma(J_l+i^c)}{\sigma(J_1+i^c)} \det(P_l)\, \sigma(I_1^c)\det(R^\intercal_{I_1}) \\
&= \frac{\sigma(J_l^c)}{\sigma(J_1^c)} \det(P_l)\det(S_{I_1^c}),
\end{aligned}
$$

and

$$
\frac{\sigma(J_l^c)}{\sigma(J_1^c)} = \frac{\sigma([1\,..\,n])}{\sigma([1\,..\,n])}\frac{\sigma(J_l^c)}{\sigma(J_1^c)} = \frac{\sigma(J_l)}{\sigma(J_1)}.
$$

Looking both last equations of both computations, we can conclude that

$$
\frac{\det(S_{I_l^c})}{\det(S_{I_1^c})} = \frac{\det(T_{J_l^c})}{\det(T_{J_1^c})}.
$$

□

Let $\mathrm{CC}(\mathfrak{G})$ be the set of connected components of $\mathfrak{G}$. If $\mathfrak{H}\in\mathfrak{G}$, we set

$$
\begin{aligned}
\mathfrak{H}_{r-1,r} &:= \mathfrak{H}\cap\mathfrak{G}_{r-1,r} \text{ and} \\
\mathfrak{H}_{r,r-1} &:= \mathfrak{H}\cap\mathfrak{G}_{r,r-1}.
\end{aligned}
$$

Moreover, we denote by $\mathrm{SCC}(\mathfrak{G})$ the set of *special connected components* of $\mathfrak{G}$, i.e., of connected components of $\mathfrak{G}$ containing a special vertex, and by $\mathrm{NCC}(\mathfrak{G}) := \mathrm{CC}(\mathfrak{G})\setminus\mathrm{SCC}(\mathfrak{G})$ the set of *ordinary connected components* of $\mathfrak{G}$.

The equation we wanted to show,

$$
g_1 f_2 - g_2 f_1 = \mathcal{O}_{\underline{y}}(f_1^2),
$$

is equivalent to

$$
\sum_{\mathfrak{H}\in\mathrm{CC}(\mathfrak{G})}\Bigg(\sum_{\mathfrak{u}\in\mathfrak{H}_{r,r-1}} a(\mathfrak{u})h(\mathfrak{u};t) - \sum_{\mathfrak{u}\in\mathfrak{H}_{r-1,r}} a(\mathfrak{u})h(\mathfrak{u};t)\Bigg) = \mathcal{O}_{\underline{y}}(f_1^2).
$$

In the above sum, we can remove the special vertices because of Point (2) of Claim 5.5. But we can also remove any neighbor of a special vertex, thanks to the Point (1), and even all the vertices which are connected to a special vertex by a path in $\mathfrak{G}$. Thus, it remains to show

$$\sum_{\mathfrak{H}\in \mathrm{NCC}(\mathfrak{G})} \left(\sum_{\mathfrak{u}\in\mathfrak{H}_{r,r-1}} a(\mathfrak{u})h(\mathfrak{u};t) - \sum_{\mathfrak{u}\in\mathfrak{H}_{r-1,r}} a(\mathfrak{u})h(\mathfrak{u};t)\right) = \mathcal{O}_{\underline{y}}(f_1^2).$$

Actually, we will prove that, for all $\mathfrak{H} \in \mathrm{NCC}(\mathfrak{G})$,

$$\sum_{\mathfrak{u}\in\mathfrak{H}_{r,r-1}} a(\mathfrak{u})h(\mathfrak{u};t) - \sum_{\mathfrak{u}\in\mathfrak{H}_{r-1,r}} a(\mathfrak{u})h(\mathfrak{u};t) = \mathcal{O}_{\underline{y}}(f_1^2).$$

Fix a connected component $\mathfrak{H}$ in $\mathrm{NCC}(\mathfrak{G})$. Let $\mathfrak{V}'$ be its vertex set and $\mathfrak{E}'$ be its edge set. Let $\mathcal{G}$ be the exchange graph of $\mathfrak{M}$, as defined in Section 4, with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. We define the following projection:

$$\begin{array}{rccc} \pi : & \mathfrak{V} & \to & \mathcal{V}_{r,r-1} \sqcup \mathcal{V}_{r-1,r}, \\ & (K_1,\dots,K_k,L) & \mapsto & (K_1,L). \end{array}$$

Let $\mathcal{H} := \mathcal{G}[\pi(\mathfrak{V}')]$ be the induced subgraph of $\mathcal{G}$ with vertex set the image of $\mathfrak{V}'$ by π. Let $\mathcal{V}'$ be its vertex set and $\mathcal{E}'$ be its edge set. We have the following claim.

Claim 5.6. *$\mathcal{H}$ is a connected component of $\mathcal{G}$, and the map π induces an isomorphism of graphs between $\mathfrak{H}$ and $\mathcal{H}$.*

Proof. First we prove that π is injective on $\mathfrak{H}$. Notice that if $\mathfrak{u} = (K_1,\dots,K_k,L)$ and $\mathfrak{v} = (K'_1,\dots,K'_k,L')$ are two vertices in $\mathfrak{H}$, then $K_l \uplus L = K'_l \uplus L'$ for all $l \in [1\cdot\cdot k]$ (the proof is identical as the case of the exchange graph). Thus, if we know $\mathfrak{u}$, we can retrieves $\mathfrak{v}$ only knowing L'. But L' is encoded in $\pi(\mathfrak{v})$. That concludes the injectivity. Thus π induces a bijection between $\mathfrak{V}'$ and $\mathcal{V}'$.

Next, we prove that π induces a natural bijection between $\mathfrak{E}'$ and $\mathcal{E}'$. Let $\mathfrak{e} \in \mathfrak{E}'$. $\mathfrak{e}$ is incident to a vertex in $\mathfrak{V}'_{r-1,r}$, which is ordinary. Let $\mathfrak{u} = (J_1,\dots,J_k,I) \in \mathfrak{V}'_{r-1,r}$ be this vertex. Let $\mathfrak{v} = (I_1,\dots,I_k,J)$ be the other endpoint of $\mathfrak{e}$. Then there exists $i \in [1\cdot\cdot n]$ such that $I = J+i$ and $I_1 = J_1+i$. Thus, $\pi(\mathfrak{v}) = (I_1,J)$ is a neighbor of $\pi(\mathfrak{u}) = (J_1,I)$.

Reciprocally, let e be an edge of $\mathcal{G}_{r,r-1}$ which is incident to a vertex u of $\mathcal{V}'$. Let $\mathfrak{u} \in \mathfrak{V}$ such that $\pi(\mathfrak{u}) = u$. Let v be the other endpoint of e. We want to show that there exists $\mathfrak{v}$ in $\mathfrak{V}$ such that $\pi(\mathfrak{v}) = \mathfrak{V}$ (in order to show that $v \in \mathcal{V}'$, and so that $\mathcal{H}$ is a connected component), and that $\mathfrak{u}$ and $\mathfrak{v}$ are connected by an edge. There are two cases.

- If $\mathfrak{u} = (J_1,\dots,J_k,I) \in \mathfrak{V}_{r-1,r}$, then $u = (J_1,I)$. There exists $i \in \mathrm{Fr}(J_1)$ such that $v = (J_1+i, I-i)$. Since $\mathfrak{u}$ is an ordinary vertex, $i \in \mathrm{Fr}(J_1)$ implies that $i \in \mathrm{Fr}(J_l)$ for all $l \in [1\cdot\cdot k]$. Thus, $\mathfrak{v} := (J_1+i,\dots,J_r+i,I-i)$ is a neighbor of $\mathfrak{u}$ such that $\pi(\mathfrak{v}) = v$.
- If $\mathfrak{u} = (I_1,\dots,I_k,J) \in \mathfrak{V}_{r,r-1}$, then $u = (I_1,J)$. There exists $i \in \mathrm{Fr}(J) \cap I_1$ such that $v = (I_1-i, J+i)$. Since $\mathfrak{u}$ is a special vertex, $i \in \mathrm{Fr}(J) \cap I_l$ for all $l \in [1\cdot\cdot k]$. Finally, $\mathfrak{v} := (I_1-i,\dots,I_k-i,J+i)$ is a neighbor of $\mathfrak{u}$ in $\mathfrak{H}$, and $\pi(\mathfrak{v}) = v$.

π induces a bijection between vertices of $\mathfrak{H}$ and of $\mathcal{H}$, and between edges of $\mathfrak{H}$ and edges of $\mathcal{G}_{r,r-1}$ which are incident to a vertex of $\mathcal{H}$. Thus, the claim is true. $\square$

Let us denote by $\pi' : \mathfrak{H} \to \mathcal{H}$ the isomorphism induced by π.

Now we study a second bijection. The well-definiteness will be justified in Claim 5.7 below. Set

$$\begin{array}{rccl} \Phi: & \mathcal{V}'_{r,r-1} & \to & \mathcal{V}'_{r-1,r}, \\ & (I,J) & \mapsto & \Big(U \cup (J\setminus V), V \cup (I \setminus U)\Big), \end{array}$$

where (U,V) is the MCP of any vertex of $\mathcal{H}$. This map induces a map on $\mathfrak{H}$:

$$\begin{array}{rccl} \widetilde{\Phi}: & \mathfrak{V}'_{r,r-1} & \to & \mathfrak{V}'_{r-1,r}, \\ & \mathfrak{u} & \mapsto & \pi'^{-1} \circ \Phi \circ \pi'(\mathfrak{u}). \end{array}$$

Claim 5.7. *Φ and $\widetilde{\Phi}$ have the following properties.*

(1) *Φ and $\widetilde{\Phi}$ are well-defined, and both are bijections.*
(2) *If $\mathfrak{u} \in \mathfrak{V}'_{r,r-1}$, then $a(\widetilde{\Phi}(\mathfrak{u})) = a(\mathfrak{u})$.*

Proof. We prove the two points independently.

(1) It suffices to prove the first point for Φ. Let $(I,J) \in \mathcal{H}_{r,r-1}$. Let $(U,V) := \mathrm{MCP}(I,J)$. Let $(J',I') := \Big(U \cup (J\setminus V), V \cup (I\setminus U)\Big)$. We want to apply Corollary 4.13 in order to show that $(J',I') \in \mathcal{H}$. In the definition of (J',I'), the unions are disjoint because, if $i \in U \cap J$, then $(\{i\},\{i\})$ is a codependent ordered pair of (I,J), and so $i \in V$. Thus

$$\begin{aligned} (U \sqcup (J\setminus V)) \uplus (V \sqcup (I \setminus U)) &= U \uplus (J\setminus V) \uplus V \uplus (I\setminus U) \\ &= I \uplus J. \end{aligned}$$

Moreover, set $(U',V') := \mathrm{MCP}(J',I')$. Clearly $(U,V) \subset (U',V')$. And

$$\begin{aligned} \mathrm{cl}(U') &= \mathrm{cl}(U \sqcup (U'\setminus U)) \\ &= \mathrm{cl}(\mathrm{cl}(U) \sqcup (U'\setminus U)) \\ &= \mathrm{cl}(\mathrm{cl}(V) \sqcup (U'\setminus U)) \\ &= \mathrm{cl}(V \sqcup (U'\setminus U)). \end{aligned}$$

Similarly,

$$\mathrm{cl}(V') = \mathrm{cl}(U \sqcup (V'\setminus V)).$$

But $V \sqcup (U'\setminus U) \subset V \sqcup (J\setminus V) = J$ and $U \sqcup (V'\setminus V) \subset I$. Since

$$\mathrm{cl}(U \sqcup (V'\setminus V)) = \mathrm{cl}(V') = \mathrm{cl}(U') = \mathrm{cl}(V \sqcup (U'\setminus U)),$$

$\Big(U \sqcup (V'\setminus V), V \sqcup (U'\setminus U)\Big)$ is a codependent ordered pair of (I,J). Thus, it is included in (U,V), and so $U'\setminus U$ and $V'\setminus V$ are empty. Finally,

$$I \uplus J = J' \uplus I' \text{ and } \mathrm{MCP}(I,J) = \mathrm{MCP}(J',I').$$

We can apply Theorem 4.9, and we obtain $(J',I') \in \mathcal{H}_{r-1,r}$. Thus, Φ is well-defined.

One can easily retrieve (I,J) from (J',I') by $(I,J) := (U \cup (I'\setminus V), V \cup (J'\setminus U))$. Thus, Φ is a bijection. So is $\widetilde{\Phi}$.

(2) Let $\mathfrak{u} = (I, I_2, \dots, I_k, J) \in \mathfrak{V}_{r,r-1}$ be a vertex and let $\mathfrak{v} = (J', J_2, \dots, J_k, I') \in \mathfrak{V}_{r-1,r}$ be the image of $\mathfrak{u}$ by $\widetilde{\Phi}$. By connectivity, Property (3) of Claim 5.5 extends to the whole component $\mathfrak{H}$. Thus, for all $l \in [2 \cdot\cdot k]$,

$$\frac{\det(S_{I_l^c})}{\det(S_{I^c})} = \frac{\det(T_{J_l^c})}{\det(T_{J'^c})}.$$

Setting $a_l := \det(S_{I_l^c})/\det(S_{I^c})$ for all $l \in [2\,..\,k]$, we obtain

$$a(\mathfrak{u}) = \det(S_{I^c})^k 1 \cdot a_2 \cdots a_k \det(T_{J^c})^k,$$
$$a(\mathfrak{v}) = \det(T_{J'^c})^k 1 \cdot a_2 \cdots a_k \det(S_{I'^c})^k.$$

It remains to show that $\det(S_{I^c})^k \det(T_{J^c})^k = \det(S_{I'^c})^k \det(T_{J'^c})^k$. The proof will be very similar to the proof of (3) in Claim 5.5. We have

$$\begin{aligned}\mathrm{cl}(J') &= \mathrm{cl}(U \cup (J \setminus V)) \\ &= \mathrm{cl}(\mathrm{cl}(U) \cup (J \setminus V)) \\ &= \mathrm{cl}(\mathrm{cl}(V) \cup (J \setminus V)) \\ &= \mathrm{cl}(J).\end{aligned}$$

In particular, a consequence of Example 3.31 is that, if ε_J and $\varepsilon_{J'}$ are orientions relative to J and to J' on $\mathfrak{M}$, then $\varepsilon_J = \pm\varepsilon_{J'}$. Moreover, since $\mathrm{cl}(U) = \mathrm{cl}(V)$, there exists an invertible matrix $P \in \mathcal{M}_{|U|}(\mathbb{R})$ such that $R_V^t = R_U^t P$. A consequence of Theorem 3.32 is that there exists ε_J and $\varepsilon_{J'}$ two orientations relative to J and to J' on $\mathfrak{M}$ such that

$$\begin{aligned}\det(T_{J'^c}) &= \sum_{j \in \mathrm{Fr}(J')} \varepsilon_{J'}(j) |\det{}_f(R^\intercal_{J'+j})| \mathbf{v}_j \\ &= \sum_{j \in \mathrm{Fr}(J)} \pm\varepsilon_J(j) |\det{}_f(R^\intercal_U \star R^\intercal_{(J\setminus V)+j})| \mathbf{v}_j \\ &= \pm \sum_{j \in \mathrm{Fr}(J)} \varepsilon_J(j) \left| \det{}_f \left(R^\intercal_V \star R^\intercal_{(J\setminus V)+j} \begin{pmatrix} P^{-1} & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ 0 \cdots 0 & \mathrm{Id} \end{pmatrix} \right) \right| \mathbf{v}_j \\ &= \pm \frac{1}{\det(P)} \sum_{j \in \mathrm{Fr}(J)} \varepsilon_J(j) \left| \det{}_f(R^\intercal_{J+j}) \right| \mathbf{v}_j \\ &= \pm \frac{\det(T_{J^c})}{\det(P)}.\end{aligned}$$

And for S, thanks to (4) and using the fact that S is a normal kernel matrix of R with basis f,

$$\begin{aligned}|\det(S_{I'^c})| &= |\det{}_f(R^\intercal_{I'})| \\ &= |\det{}_f(R^\intercal_V \star R^\intercal_{I\setminus U})| \\ &= \left| \det{}_f \left(R^\intercal_U \star R^\intercal_{I\setminus U} \begin{pmatrix} P & \begin{matrix}0\\ \vdots \\ 0\end{matrix} \\ 0 \cdots 0 & \mathrm{Id} \end{pmatrix} \right) \right| \\ &= |\det(P)| \cdot |\det{}_f(R^\intercal_I)|.\end{aligned}$$

Since k is even, we obtain that

$$\det(S_{I'^c})^k \det(T_{J'^c})^k = \det(S_{I^c})^k \det(T_{J^c})^k.$$

Finally,
$$a(\mathfrak{u}) = a(\Phi(\mathfrak{u})).$$
□

We recall that we wanted to show
$$\sum_{\mathfrak{u}\in\mathfrak{H}_{r,r-1}} a(\mathfrak{u})h(\mathfrak{u};t) - \sum_{\mathfrak{u}\in\mathfrak{H}_{r-1,r}} a(\mathfrak{u})h(\mathfrak{u};t) = \mathcal{O}_{\underline{y}}(f_1^2).$$
By Claim 5.7, this equation is equivalent to
$$\sum_{\mathfrak{u}\in\mathfrak{H}_{r,r-1}} a(\mathfrak{u})h(\mathfrak{u};t) - a(\widetilde{\Phi}(\mathfrak{u}))h(\widetilde{\Phi}(\mathfrak{u};t)) = \mathcal{O}_{\underline{y}}(f_1^2),$$
and even to
$$\sum_{\mathfrak{u}\in\mathfrak{H}_{r,r-1}} a(\mathfrak{u})(h(\mathfrak{u};t) - h(\widetilde{\Phi}(\mathfrak{u};t))) = \mathcal{O}_{\underline{y}}(f_1^2).$$

As we have already seen, Point (1) of Claim 5.5, which states that $h(\mathfrak{u})-h(\mathfrak{v}) = \mathcal{O}_{\underline{y}}(f_1^2)$ if $\mathfrak{u}$ and $\mathfrak{v}$ are adjacent, can be extended to the whole connected component. Thus, if $\mathfrak{u} \in \mathfrak{V}_{r,r-1}$,
$$h(\mathfrak{u};t) - h(\widetilde{\Phi}(\mathfrak{u};t)) = \mathcal{O}_{\underline{y}}(f_1^2).$$
We obtain a sum of $\mathcal{O}_{\underline{y}}(f_1^2)$, which is a $\mathcal{O}_{\underline{y}}(f_1^2)$. Finally,
$$g_1 f_2 - f_1 g_2 = \mathcal{O}_{\underline{y}}(f_1^2),$$
which concludes the proof of the theorem. □

One can easily obtain from Theorem 5.1 the following corollary.

Corollary 5.8. *Let l be a positive integer, $u_1, \dots, u_l$ be l independent vectors in $\mathrm{Im}(R^\intercal)$ and $T_l := S \star \mathbf{v}_1 \star \dots \star \mathbf{v}_l$ where, for each $i \in [1 \,.\,.\, l]$, $v_i \in \mathbb{R}^n$ and verifies $R^\intercal \mathbf{v}_i = \mathbf{u}_i$. Let F and Y be two functions as in Theorem 5.1. Then*
$$\frac{\det(T_l^\intercal Y T_l)}{\det(S^\intercal Y S)} - \frac{\det(T_l^\intercal (Y+F) T_l)}{\det(S^\intercal (Y+F) S)} = \mathcal{O}_{\underline{y}}\Big(\max_{i\in[1\,.\,.\,n]}(y_i^{l-1})\Big).$$

Similarly, from generalized Theorem 5.2 we have the following generalization.

Corollary 5.9. *Let l be a positive integer, $u_1, \dots, u_l$ be l independent vectors in $\mathrm{Im}(R^\intercal)$ and $T_l := S \star \mathbf{v}_1 \star \dots \star \mathbf{v}_l$ where, for each $i \in [1 \,.\,.\, l]$, $v_i \in \mathbb{R}^n$ and verifies $R^\intercal \mathbf{v}_i = \mathbf{u}_i$. Let F and Y be two functions as in Theorem 5.2. Then*
$$\frac{\det(Y \cdot_1 T_l \cdots \cdot_k T_l)}{\det(Y \cdot_1 S \cdots \cdot_k S)} - \frac{\det((Y+F) \cdot_1 T_l \cdots \cdot_k T_l)}{\det((Y+F) \cdot_1 S \dots \cdot_k S)} = \mathcal{O}_{\underline{y}}\Big(\max_{i\in[1\,.\,.\,n]}(y_i^{l-1})\Big).$$

Proof of both corollaries. We use the notations of Corollary 5.9. First, if $l' \in [0 \,.\,.\, l]$, $T_{l'}$ will denote $S \star \mathbf{v}_1 \star \dots \star \mathbf{v}_{l'}$. If $l' \neq l$, set $R_{l'}$ a normal kernel matrix of $T_{l'}$ for the standard basis. It happens that, symmetrically, $T_{l'}$ will be a normal kernel matrix of $R_{l'}$ for the standard basis. Thus, applying Theorem 5.1, one obtains that
$$\frac{\det(Y \cdot_1 T_{l'+1} \cdots \cdot_k T_{l'+1})}{\det(Y \cdot_1 T_{l'} \cdots \cdot_k T_{l'})} - \frac{\det((Y+F) \cdot_1 T_{l'+1} \cdots \cdot_k T_{l'+1})}{\det((Y+F) \cdot_1 T_{l'} \dots \cdot_k T_{l'})} = \mathcal{O}_{\underline{y}}\Big(\max_{i\in[1\,.\,.\,n]}(y_i^{l-1})\Big).$$

Next, we will prove that
$$\frac{g_2}{g_1} = \mathcal{O}_{\underline{y}}(\max_{i\in[1\,.\,.\,n]} y_i),$$

where

$$g_1 := \det((Y+F)\cdot_1 T_1 \dots \cdot_k T_1),$$
$$g_2 := \det((Y+F)\cdot_1 S \cdots \cdot_k S).$$

By Claim 5.3, it suffices to prove that

$$\frac{g_2}{f_1} = \mathcal{O}_{\underline{y}}(\max_{i\in[1\,..\,n]} y_i),$$

where

$$f_1 := \sum_{I\in\mathcal{I}_r} \det(S_{I^c})^k \underline{y}^{I^c}.$$

Since f_1 has only positive coefficients, and since coefficients of g_2 are bounded, it remains to show that, if $J \subset [1\,..\,n]$ is such that $[y^J]g_2$ is nonzero, then there exists $I \in \mathcal{I}_r$ and $i \in I$ such that $J \subset I^c + i$. But we have Formula (49):

$$g_2 = \sum_{J_1,\dots,J_k\in\mathcal{I}_{r-1}} \det\Big((Y+F)_{1:J_1^c,\dots,k:J_k^c}\Big)\det(T_{J_1^c})\cdots\det(T_{J_k^c}).$$

It implies that, if $[y^J]g_2$ is nonzero, then $J \subset J'^c$ for some $J' \in \mathcal{I}_{r-1}$. Thus, one can find $i \in \mathrm{Fr}(J')$, and set $I = J' + i$. Then, $J \subset J'^c = I^c + i$, and so

$$\underline{y}^J = \mathcal{O}_{\underline{y}}(f_1 \max_{i\in[1\,..\,n]}(y_i)).$$

Summing all monomials, we obtain

$$\frac{g_2}{f_1} = \mathcal{O}_{\underline{y}}(\max_{i\in[1\,..\,n]}(y_i)).$$

This last formula can be applied to all $T_{l'}$, $l' \in [0\,..\,l-1]$: replace T by $T_{l'+1}$ and S by $T_{l'}$. If $l' \in [0\,..\,l]$, we define the functions

$$a_{l'} := \frac{\det(Y\cdot_1 T_{l'+1}\cdots\cdot_k T_{l'+1})}{\det(Y\cdot_1 T_{l'}\cdots\cdot_k ST_{l'})},$$
$$b_{l'} := \frac{\det((Y+F)\cdot_1 T_{l'+1}\cdots\cdot_k T_{l'+1})}{\det((Y+F)\cdot_1 T_{l'}\cdots\cdot_k T_{l'})}.$$

The previous results can be summarized by the formulæ

$$a_{l'} - b_{l'} = \mathcal{O}_{\underline{y}}(1),$$
$$b_{l'} = \mathcal{O}_{\underline{y}}(\max_{i\in[1\,..\,n]}(y_i)).$$

Actually, we also have

$$a_{l'} = \mathcal{O}_{\underline{y}}(\max_{i\in[1\,..\,n]}(y_i)).$$

Now the last computation:

$$\begin{aligned}\frac{\det(Y\cdot_1 T_l\cdots\cdot_k T_l)}{\det(Y\cdot_1 S\cdots\cdot_k S)} &- \frac{\det((Y+F)\cdot_1 T_l\cdots\cdot_k T_l)}{\det((Y+F)\cdot_1 S\dots\cdot_k S)}\\ &= a_1\cdots a_l - b_1\cdots b_l\\ &= (a_1-b_1)a_2\cdots a_l + b_1(a_2-b_2)a_3\cdots a_l + \cdots + b_1\cdots b_{l-1}(a_l-b_l).\end{aligned}$$

Each term is a $\mathcal{O}_{\underline{y}}(1)\big(\mathcal{O}_{\underline{y}}(\max_{i\in[1\,..\,n]} y_i)\big)^{l-1}$, which concludes the proof. □

One cannot expect a better asymptotic for Corollary 5.8 because of the following counter-example.

Example 5.10. Let $n = 3$ be an integer We set

$$R = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

And F will be constant equal to

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \det(T_3^\intercal Y T_3) &= \det(Y) = y_1 y_2 y_3 y_4, \\ \det(S^\intercal Y S) &= y_1, \\ \det(T_3^\intercal (Y+F) T_3) &= \det(Y+F) = y_1 y_2 y_3 (y_4 - 1), \\ \det(S^\intercal (Y+F) S) &= y_1. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{\det(T_l^\intercal Y T_l)}{\det(S^\intercal Y S)} - \frac{\det(T_l^\intercal (Y+F) T_l)}{\det(S^\intercal (Y+F) S)} &= \frac{y_1 y_2 y_3 y_4}{y_1} - \frac{y_1 y_2 y_3 (y_4 - 1)}{y_1} \\ &= y_2 y_3 \\ &= \mathcal{O}_{\underline{y}}\Big(\max_{i \in [1\,..\,4]} (y_i^2) \Big). \end{aligned}$$

Conclusion

We have seen several mathematical results where Symanzik polynomials appear. Moreover, our generalization of Symanzik polynomials induces generalization of some of these results. However, there are still many ways to explore. For example, since Theorem 1.1 of [1] are generalized in Section 5, this could lead to a natural generalization of Theorem 1.2 of [1]. We also think that Example 3.14 about Jacobian torus could be extend to greater dimensions using Poincaré duality.

In conclusion, Symanzik polynomials has many interesting properties in many different domains: quantic field theory, combinatorics, geometry, algebraic geometry. The author will pursue its research on these interesting objects, and he hopes he will find many more interesting results.

Appendix. Multidimensional matrices

In this appendix, we will defined what we called multidimensional matrices, and some basic operations on it. We will almost only discuss here what we need in the rest of this article; that is to say: define a multiplication, state some properties of it, define a generalized determinant called hyperdeterminant, state the multiplicativity of the hyperdeterminant, and state a generalized Cauchy-Binet formula. We will make no proof. They all can be done, eventually adapting the corresponding proof on usual matrices. In fact, hyperdeterminants were first discovered by Arthur Cayley in 1843 (see [5]), and all below results are known for some time.

Let A be a PID. A *k-dimensional matrix of size $n_1 \times \cdots \times n_k$ on A*, where k is a positive integer, and where the *size* denoted by $n_1 \times \cdots \times n_k$ is a k-tuple of positive integers, is a family of elements of A indexed by $[1 \,..\, n_1] \times \cdots \times [1 \,..\, n_k]$. If C is a multidimensional matrix, we denote by $\dim(C)$ its dimension, by $\mathrm{size}(C)$ its size, and by $\mathrm{size}_i(C)$, $i \in [1 \,..\, k]$, the i-th element of $\mathrm{size}(C)$. Moreover, in order to not make too many indices in what follows, if r is a nonnegative integer, if $l_1, \dots, l_r$ are different integers of $[1 \,..\, k]$, if $i_1, \dots, i_r$ are positive integers, and if j_l for $l \in [1 \,..\, k] \setminus \{i_1, \dots, i_r\}$ are positive integers too, then

$$[l_1 : i_1, \dots, l_r : i_r, l : j_l \text{ otherwise}]$$

is the k-tuple $(\alpha_1, \dots, \alpha_k)$ where

$$\alpha_l = \begin{cases} i_m & \text{if } m \in [1 \,..\, r] \text{ and } l = l_m, \\ j_l & \text{otherwise.} \end{cases}$$

If $[u]$ denotes a k-tuple $(u_1, \dots, u_k)$ of positive integers such that $[u] \leqslant \mathrm{size}(C)$ (i.e., such that $u_l \leqslant \mathrm{size}_l(C)$ for all $l \in [1 \,..\, k]$), then $\mathbf{c}[u]$ is the element of C of index u. In a similar way, if n, s are positive integers and if P is a (usual) matrix of $\mathcal{M}_{n,s}(A)$, then $\mathbf{p}[i, j]$, with $i \in [1 \,..\, n]$ and $j \in [1 \,..\, s]$, will denote the entry of P indexed by (i, j).

Now we fix k a positive integer, and C a k-dimensional matrix on A.

If B is another k-dimensional matrix on A with $\mathrm{size}(B) = \mathrm{size}(C)$, then $D := B + C$ is naturally defined by: for all $[u] \leqslant \mathrm{size}(B)$,

$$\mathbf{d}[u] = \mathbf{b}[u] + \mathbf{c}[u].$$

Set s a positive integer, $m \in [1 \,..\, k]$, $n := \mathrm{size}_m(C)$, and P a (usual) matrix of $\mathcal{M}_{n,s}(A)$. Then the *(right) multiplication of C by P along the m-th direction*, denoted by $C \cdot_m P$, is the k-dimensional matrix B of size

$$\mathrm{size}(B) = [m : s, l : \mathrm{size}_l(C) \text{ otherwise}]$$

verifying, for all k-tuple $u \leqslant \mathrm{size}(B)$,

$$\mathbf{b}[u] = \sum_{i=1}^{n} \mathbf{a}[m : i, l : u_l \text{ otherwise}] \cdot \mathbf{p}[i, u_m].$$

In some cases, this multiplication verifies some simple properties similar to the associativity and the commutativity.

Claim A.1. *Let k be a positive integer and C be a k-dimensional matrix.*

- *Let $l \in [1 \,..\, k]$ be an integer, $p := \mathrm{size}_l(C)$, r and s be two positive integers, and $P \in \mathcal{M}_{p,r}(A)$ and $Q \in \mathcal{M}_{r,s}(A)$ be two matrices, then*

$$C \cdot_l P \cdot_l Q = C \cdot_l (PQ).$$

- *Let $l, l' \in [1 \,..\, k]$ be two different integers, $p := \mathrm{size}_l(C)$, $q := \mathrm{size}_{l'}(C)$, r and s be two positive integers, $P \in \mathcal{M}_{p,r}(A)$, and $Q \in \mathcal{M}_{q,s}(A)$, then*

$$C \cdot_l P \cdot_{l'} Q = C \cdot_{l'} Q \cdot_l P.$$

Now we will define the hyperdeterminant which gives a meaning to Proposition 2.7.

We say that C is *hypercubic of size* n if $\mathrm{size}_i(C) = n$ for every i in $[1 \cdot\cdot k]$. Let n be a positive integer. Then, $\mathfrak{C}_n^k(A)$ will denote the set of k-dimensional hypercubic matrices of size n on A. If $C \in \mathfrak{C}_n^k(A)$, the *hyperdeterminant* $\det(C)$ *of* C is defined by

$$\det(C) := \frac{1}{n!} \sum_{\tau_1,\dots,\tau_k \in \mathfrak{S}_n} \prod_{i=1}^{k} \sigma(\tau_i) \prod_{i=1}^{n} \mathbf{c}[l : \tau_l(i)].$$

With usual matrices, we are more accustomed to a definition with a sum where τ_1 is always the identity permutation, making disappear the multiplicative constant $1/n!$:

$$\widetilde{\det}(C) := \sum_{\tau_2,\dots,\tau_k \in \mathfrak{S}_n} \prod_{i=2^k} \sigma(\tau_i) \prod_{i=1}^{n} \mathbf{c}[1 : i, l : \tau_l(i)].$$

Our first definition is more symmetric and, more importantly, it has the expected properties of the determinant. The other one has not. The link between both definitions is the following claim.

Claim A.2.

$$\det(C) = \begin{cases} \widetilde{\det}(C) & \textit{if } k \textit{ is even,} \\ 0 & \textit{otherwise.} \end{cases}$$

The following claim states that, if we see a k-dimensional matrix as a superposition of n $(k-1)$-dimensional matrix, then the determinant is alternating and n-linear. It also states that the determinant is invariant by permutation of the directions.

Claim A.3. *Let $k \geqslant 2$ be an integer, l, l' be two integers of $[1 \cdot\cdot k]$ with $l < l'$, n be a positive integer, $C_1, \dots, C_n, C_l' \in \mathfrak{C}_n^{k-1}(A)$ be multidimensional matrices, and a be an element of A. If we denote by*

$$\star(C_1, \dots, C_n)$$

the element of $\mathfrak{C}_n^k(A)$ being the superposition along the k-th direction of the C_is, i.e., verifying, for every k-tuple $[u] \leqslant (n, \dots, n)$,

$$\star(C_1, \dots, C_n)[u] = C_{u_k}[l : u_l],$$

then we have

$$\det(\star(C_1, \dots, C_l + aC_l', \dots, C_n)) = \det(\star(C_1, \dots, C_n)) + a \star(C_1, \dots, C_l', \dots, C_n)$$

and

$$\det(\star(C_1, \dots, C_{l-1}, C_{l'}, C_{l+1}, \dots, C_{l'-1}, C_l, C_{l'+1}, \dots, C_n)) = -\det(\star(C_1, \dots, C_n)).$$

Moreover, if k and n are positive integers, $C \in \mathfrak{C}_n^k(A)$, $\tau \in \mathfrak{S}_k$, and if we denote by $C^\tau \in \mathfrak{C}_n^k(A)$ the matrix verifying, for every k-tuple $[u] \leqslant \mathrm{size}(C)$,

$$C^\tau[u] = C[l : u_{\tau^{-1}(l)}],$$

then

$$\det(C) = \det(C^\tau).$$

Now we state the multiplicativity of the determinant relatively to the multiplication by usual matrices.

Proposition A.4. *If k and n are positive integers, if $l \in [1 \cdot\cdot k]$, if $C \in \mathfrak{C}_n^k(A)$, and if $P \in \mathcal{M}_n(A)$, then*

$$\det(C \cdot_l P) = \det(C)\det(P).$$

And finally, the generalization of the Cauchy-Binet formula. Let n, s, k be positive integers, $m \in [1 \cdot\cdot k]$ be an integer, and $\Gamma = \{\gamma_1, \dots, \gamma_s\}$ be a set of size s of elements in $[1 \cdot\cdot n]^s$ such that $\gamma_1 < \dots < \gamma_s$. If C is a k-dimensional matrix with $\mathrm{size}_m(C) = n$, then $C_{m:\Gamma}$ denote the k-dimensional matrix which verifies

$$\mathrm{size}(C_{m:\Gamma}) = [m : s, l : \mathrm{size}_l(C) \text{ otherwise}],$$

and, for any k-tuple $[u] \leqslant \mathrm{size}(C_{m:\Gamma})$,

$$C_{m:\Gamma}[u] = C[m : \gamma_{u_m}, l : u_l \text{ otherwise}].$$

Of course, if $m' \in [1 \cdot\cdot k]$ is different from m and if $\Gamma' \subset [1 \cdot\cdot \mathrm{size}_{m'}(C)]$, then

$$C_{m:\Gamma, m':\Gamma'} := \left(C_{m:\Gamma}\right)_{m':\Gamma'}.$$

Proposition A.5 (Generalized Cauchy-Binet formula)**.** *Let k, n, s be positive integers, $m \in [1 \cdot\cdot k]$, C a k-dimensional matrix on A of size*

$$\mathrm{size}(C) = [m : n, l : s \textit{ otherwise}]$$

and $P \in \mathcal{M}_{n,s}(A)$. Then

$$\det(C \cdot_m P) = \sum_{\substack{I \subset [1 \cdot\cdot n] \\ |I| = s}} \det(C_{m:I}) \det(P_I).$$

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